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ON THE SINGULARITY OF GREEN FUNCTIONS IN MARKOV PROCESSES II

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To Professor Katuzi Ono on the occasion of his 60th birthday

§1. Introduction.

In the previous paper [3] we have studied Green functions with singularity $\varphi \in \Phi$, especially in connection with regular points. In this article we shall show that the converse of Theorem 5 in [3] within our scheme holds under a certain contition, that is,

THEOREM.[1] *Let $X^0 = (X^0_t, \zeta^0, M^0_t, P^0_x)$ (resp. $X = (X_t, \zeta, M_t, P_x)$) be a Feller process on a domain $\Omega \subset R^d$, $d \geqq 3$, which has a Green function $G^0(x, y)$ with singularity $\varphi^0 \in \Phi_{d-\alpha}$ where $\alpha > 1$ (resp. $G(x, y)$ with singularity $\varphi \in \Phi$). Further let us suppose that both the processes X^0 and X satisfy Hunt's condition (H). Then, if $K^{reg}_X = K^{reg}_{X^0}$ for each compact subset K of Ω, we can choose φ^0 as a singularity function of $G(x, y)$.*

By using this Theorem, we get the following

COROLLARY. *Let X_α be a stable process in R^d, $d \geqq 3$, corresponding to the Riesz potential of order α, where $2 \geqq \alpha > 1$. Let X be a Lévy process in R^d, $d \geqq 3$, with a bounded continuous density*[2] *and with exponent $\Psi(z)$ of the form $\Psi(z) = F(2\pi^2|z|^2)$, where F is a Bernstein's function of class C^∞ on $(0, +\infty)$ (i.e. $F \geqq 0$, $(-1)^p F^{(p)} \leqq 0$, $p \geqq 1$).*[3] *Then $K^{reg}_{X_\alpha} = K^{reg}_X$ holds for each compact set K, if and only if X has a Green function with singularity $r^{\alpha-d}$.*

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1) For the notations in the Theorem see the next section.

2) In case a transtion probability density $p(t, x)$ exists and is a bounded continuous function of x for $t>0$, we say that the Lévy process has a bounded continuous density.

3) This process is a Markov process subordinate to the Brownian motion, but we do not use this fact in this article.

§2. Preliminaries.

In this article we say that $X=(X_t, \zeta, M_t, P_x)$ is a *Feller process* on a domain $\Omega \subset R^d$, if it is a Hunt process on Ω and its semi-group operators $\{T_t\}$ map $C_0(\Omega)$ into $C_0(\Omega)$ and strongly continuous on $C_0(\Omega)$, where $C_0(\Omega)$ is the space of continuous functions on Ω vanishing at infinity. If a Green function $G(x, y)$ of a Feller process on Ω (in the sense of [3][4]) satisfies

$$C_2\varphi(|x-y|) \leqq G(x, y) \leqq C_1\varphi(|x-y|)$$

for each $x_0 \in \Omega$ and each $x, y \in O_{x_0}$, where O_{x_0} is a ball in Ω centering at x_0, $C_1 \geqq C_2 > 0$ are constants depending only on O_{x_0} and the function $\varphi(r)$ is chosen from the following class;

$$\Phi = \Big\{\varphi;\ \varphi(r) \textit{ is a positive, continuous and monotonically nonincreasing function on } (0, \delta) \textit{ for some } \delta > 0 \textit{ such that}$$
$$\lim_{r\to 0} \varphi(r) = +\infty \quad \textit{and} \quad \int_0^\delta t^{d-1}\varphi(t)dt < +\infty\Big\},$$

then we say that $G(x, y)$ is a *Green function with singularity* $\varphi \in \Phi$ and $\varphi(r)$ is a *singularity function of* $G(x, y)$. A singularity function φ is said to belong to Φ_p for some real number $p > 1$, if $\varphi \in \Phi$ and $r^p\varphi(p)$ is a monotonically increasing function of r on $(0, \delta)$. For a Feller process $X=(x_t, \zeta, M_t, P_x)$ on Ω we set $\sigma_K(\omega) = inf(t > 0,\ x_t(\omega) \in K)$, $= \zeta$ if the set $(t > 0,\ x_t(\omega) \in K)$ is empty and we denote by K_X^{reg} the set of all regular points of K for X, where K is a Borel subset of Ω. Then we have the following

LEMMA 1.[5] *Let K be a bounded open or compact subset of Ω. Then there exists a measure $\mu_K(dy)$ supported in $\bar{K}$ such that*

$$P_x(\sigma_K < \zeta) = \int_\Omega G(x, y)\mu_K(dy), \quad x \in \Omega.$$

Proof.. In case K is bounded open, the statement holds in the same way as in the proof of Theorem 1 in [3], because $P_x(\sigma_K(\omega^+_{\sigma_a}) < \zeta) = P_x(\sigma_K < \zeta)$

[4] We say that a function $G(x,y)$ on $\Omega \times \Omega$ is a Green function of a Hunt process $X=(x_t, \zeta, M_t, P_x)$, if $G(x,y)$ is nonnegative, continuous on $\Omega \times \Omega$ except at the diagonal, $E_x\left(\int_0^\zeta f(x_t)dt\right) = \int_\Omega G(x,y)f(y)dy$ for each $f \in C_K(\Omega)$ (=the space of continuous functions of compact support in Ω) and G maps $C_K(\Omega)$ into $C_0(\Omega)$.

[5] Without special mentioning the process we shall investigate is a Feller process having a Green function with singularity $\varphi \in \Phi$.

for each open set G such that $G \supset \bar{K}$. In case K is compact, let us choose a monotone sequence of open sets $\{G_n\}$ such that $G_n \downarrow K$ and $P_x(\sigma_{G_n} < \zeta) \downarrow P_x(\sigma_K < \zeta)$ for all $x \notin K \cap K_X^{\text{irreg}}$. The existence of such a sequence is proved as follows. For each fixed $x \notin K \cap K_X^{\text{irreg}}$ we can find a monotone sequence of open sets $\{G_n\}$ such that $P_x(\sigma_{G_n} \uparrow \sigma_K) = 1$ by Hunt's Theorem. Noting that K is compact, the above equality holds for all points $x \notin K \cap K_X^{\text{irreg}}$. As $G(x, y)$ is positive, locally integrable[6] and X is a Feller process, it holds $P_x({}^{\exists}\delta_A(\omega) < +\infty,\ x_t(\omega) \notin A \textit{ for any } t > \delta_A(\omega)) = 1$ for each compact set $A \subset \Omega$ by Hunt's proposition 12.5 in [2]. Hence, by choosing a compact set A such that its interior includes K, we have $P_x(\bigcap_n \{\sigma_{G_n} < \zeta\},\ \sigma_K = \zeta) = P_x(\bigcap_n \{\sigma_{G_n} < \delta_A\},\ \sigma_K = \zeta)$ for $x \in \Omega$. The right-hand side equals zero. Indeed, by using Blumenthal's theorem, $P_x(\lim_{n\to\infty} x_{\sigma_{G_n}} = x_{\lim_{n\to\infty}\sigma_{G_n}} / \lim_{n\to\infty} \sigma_{G_n} < +\infty) = 1$, therefore, noting that $\lim_{n\to+\infty} x_{\sigma_{G_n}} \in K$, we have $P_x(\zeta > \lim_{n\to+\infty}\sigma_{G_n} \geqq \sigma_K / \lim_{n\to+\infty} \sigma_{G_n} < +\infty) = 1$. Hence $P_x(\bigcap_n \{\sigma_{G_n} < \delta_A\},\ \sigma_K = \zeta) = P_x(\bigcap_n \{\sigma_{G_n} < \delta_A\},\ \lim_{n\to+\infty}\sigma_{G_n} = \zeta) = 0$, $x \notin K \bigcap_n K_X^{\text{irreg}}$. Consequently it holds $\lim_{n\to+\infty} P_x(\sigma_{G_n} < \zeta) = P_x(\bigcap_n \{\sigma_{G_n} < \zeta\},\ \sigma_K < \zeta) + P_x(\bigcap_n \{\sigma_{G_n} < \zeta\},\ \sigma_K = \zeta) = P_x(\sigma_K < \zeta)$ for $x \notin K \bigcap_n K_X^{\text{irreg}}$. Now, let $\{\mu_{G_n}(dy)\}$ be a sequence of measures such that $P_x(\sigma_{G_n} < \zeta) = \int_\Omega G(x, y)\mu_{G_n}(dy)$ then $\{\mu_{G_n}(dy)\}$ is uniformly bounded, and so we can choose a subsequence which coverges weakly to some measure $\mu_K(dy)$ supported in K. Hence, noting that $K \cap K_X^{\text{irreg}}$ has Lebesgue measure zero, we can show that $P_x(\sigma_K < \zeta) = \int_\Omega G(x, y)\mu_K(dy)$ in the same way as in the proof of Theorem 1 in [3].

In the following we shall call the measure $\mu_K(dy)$ a *capacitary measure* of K. Let $\varphi(r)$ be a positive continuous, and monotonically nonincreasing function on $(0, +\infty)$ such that $\lim_{r\to\infty}\varphi(r) = 0$ and K be compact set, and let us set $V^\varphi(K) = \mathit{inf}\left\{ \mathit{sup}_{x\in R^d} \int_K \varphi(|x-y|)\mu(dy)\right\}$, where infimum is taken over the measures supported in K whose total masses are 1.

Remark 1. In order to define $V^\varphi(K)$ for a singularity function φ, it is necessary to extend $\varphi(r)$ so that it is positive, continuous and monotonically nonincreasing function on $(0, +\infty)$ such that $\lim_{r\to+\infty}\varphi(r) = 0$. However this does not cause a confucion, because we have only to know whether

[6] If $\int_K G(x, y)dy$ is bounded for each compact subset K, we say that $G(x, y)$ is locally integrable.

$V^{\varphi}(K)$ is infinite or finite for a sufficiently small compact set K.

Now we shall extend Lemma 3 of §7 in [3] to a wider class of Green functions.

LEMMA 2. *Let $X=(x_t, \zeta, M_t, P_x)$ be a Feller process on $\Omega\subset R^d$ which has a Green function with singularity $\varphi\in\Phi$, then, for each sufficiently small compact set K, it holds*

$$\mu_K(K)=0 \Longleftrightarrow V^{\varphi}(K)=+\infty.$$

Proof. The proof of $V^{\varphi}(K)=+\infty \Longrightarrow \mu_K(K)=0$ is esay (see the proof of Lemma 3 of §7 in [3]), therefore we shall show $\mu_K(K)=0 \Longrightarrow V^{\varphi}(K)=+\infty$. If $V^{\varphi}(K)<+\infty$, there exists a measure $\nu(dx)$ supported in K such that $\nu(K)>0$ and $\int\varphi(|x-y|)\nu(dy)\leqq 1$. Let $\{G_n\}$ be a sequence of open sets such that $G_n\downarrow K$ as $n\to+\infty$ and $P_x(\sigma_{G_n}<\zeta)\downarrow P_x(\sigma_K<\zeta)$ for $x\notin K\cap K_X^{\text{irreg}}$ (such a sequence exists by the proof of Lemma 1), and let $\mu_{G_n}(dy)$ be a capacitary measure of G_n, then we have

$$\begin{aligned}\mu_{G_n}(\bar{G}_n) &\geqq \iint\varphi(|x-y|)\nu(dy)\mu_{G_n}(dx)\geqq \frac{C_2}{C_1}\iint G(y,x)\mu_{G_n}(dx)\nu(dy)\\ &= \frac{C_2}{C_1}\int P_y(\sigma_{G_n}<\zeta)\,\nu(dy)=\frac{C_2}{C_1}\cdot\nu(K).\end{aligned}$$

Nextly let us fix a point $x\notin K$ and choose a sufficiently large number n such that $P_x(\sigma_K<\zeta)>P_x(\sigma_{G_n}<\zeta)-\varepsilon$, where

$$\varepsilon=\frac{1}{2}\cdot\left(\inf_{y\in G_1}G(x,y)\right)\frac{C_2}{C_1}\nu(K).$$

Then $sup_{y\in K}\{G(x,y)\}\mu_K(K)\geqq inf_{y\in\bar{G}}\{G(x,y)\mu_{G_n}(\bar{G}_n)-\varepsilon$, and hence $\mu_K(K)>0$ by the above estimate. This completes the proof of Lemma 2.

The following two Lemmas play essential roles in the proof of the Theorem. The first one is due to S. J. Taylor [5] and the second one is nothing but Lemma 1 of §5 in [3], if only we note the Lemma 1 of this section.

LEMMA 3. *(S. J. Taylor) Let φ_1 and φ_2 be singularity functions $\in\Phi$ such that $\varphi_1\in\Phi_{d-1}$, $\underline{\lim}_{r\to 0}\dfrac{\varphi_1(r)}{\varphi_2(r)}=0$, then there exists a sufficiently small compact set K such that $V^{\varphi_1}(K)<+\infty$ and $V^{\varphi_2}(K)=+\infty$.*

LEMMA 4. *Let us set*

$$L_\varphi^r = \frac{1}{r^{d-1}} \int_0^r \varphi(2s) s^{d-2} \left(1 - \left(\frac{s}{r}\right)^2\right)^{\frac{d-3}{2}} ds$$

and suppose L_φ^r is finite. Then it holds each sufficiently small $r > 0$

$$M_1 \cdot \frac{1}{L_\varphi^r} \geqq \mu_{\partial Q_r}(\partial Q_r) \geqq M_2 \cdot \frac{1}{L_\varphi^r},$$

where ∂Q_r is the sphere of radius r and $\mu_{\partial Q_r}(dy)$ is a capacitary measure of ∂Q_r for a Feller process having a Green function with singularity $\varphi(r) \in \Phi$ and $M_1 \geqq M_2 > 0$ are constants independent of r.

§3. Proofs of the Theorem and the Corollary.

Without confusions we shall use the symbol "*const.*" for all the constants appeared in the culculations.

Proof of the theorem. We shall prove this step by step.

1). It is clear that $\varphi(r) \leqq \text{const.}\, \varphi^0(r)$ for all sufficiently small $r > 0$. Indeed, if $\underline{\lim}_{r\to 0} \dfrac{\varphi^0(r)}{\varphi(r)} = 0$, there exists a compact set K such that $V^{\varphi_0}(K) < +\infty$ and $V^\varphi(K) = +\infty$ by Lemma 3. Therefore, by Lemma 2, $\mu_K^{X^0}(K) > 0$ and $\mu_K^X(K) = 0$, where $\mu_K^{X^0}(dy)$ and $\mu_K^X(dy)$ are capacitary measures of K for the processes X^0 and X respectively. By Hunt's condition (H), $K_X^{\text{reg}} = \phi$ and $K_{X^0}^{\text{reg}} \neq \phi$, which contradicts to the assumption. Hence $\underline{\lim}_{r\to 0} \dfrac{\varphi^0(r)}{\varphi(r)} > 0$. $\varphi^0(r) \leqq \text{const.}\, \varphi(r)$ is not obvious, because it is unknown whether $\varphi(r)$ belongs to Φ_{d-1} or not

2). For each $x \in \Omega$, let ∂Q_r be a sphere of radius r centering at x, then $\underline{\lim}_{r\to 0} P_x(\sigma_{\partial Q_r} < \zeta) = \gamma_x > 0$. At first we shall show $\underline{\lim}_{r\to 0} P_x^0(\sigma_{\partial Q_r} < \zeta) > 0$. Indeed, by using Lemma 4, we have

$$P_x^0(\sigma_{\partial Q_r} < \zeta) \geqq \text{const.}\, \varphi^0(r) \mu_{\partial Q_r}^0(\partial Q_r) \geqq \text{const.}\, \frac{\varphi^0(r)}{L_{\varphi^0}^r}$$

$$\geqq \text{const.}\, \frac{\varphi^0(r)}{r^{1-d} \int_0^r \varphi^0(2s) s^{d-2} ds},$$

and, on the otherhand, by the assumption $\varphi^0 \in \Phi_{d-\alpha}$, we have $r^{1-d}\int_0^r \varphi^0(2s) s^{d-2}\, ds$

$\leqq r^{1-d}\varphi^0(2r)r^{d-\alpha}\dfrac{r^{\alpha-1}}{\alpha-1}=\dfrac{1}{\alpha-1}\varphi^0(2r)$, therefore $P_x^0(\sigma_{\partial Q_r}<\zeta)\geqq \text{const.}>0$. If we assume $\underline{\lim}_{r\to 0}P_x(\sigma_{\partial Q_r}<\zeta)=0$, we can choose a sequence $\{r_n\}_{n=1,2,\dots}$ such that $r_n\downarrow 0$ and $\sum_{n=1}^{+\infty}P_x(\sigma_{\partial Q_{r_n}}<\zeta)<+\infty$, which means that x is a irregular point of $K(\equiv\bigcup_{n=1}^{+\infty}\partial Q_{r_n}\cup\{x\})$ for X by Borel-Cantelli's lemma. On the otherhand, as $\varphi^0\in\Phi_{d-\alpha}$, Wiener's test holds in the sense of the Proposition[7] of §7 in [3] and $\sum_{n=1}^{+\infty}P_x^0(\sigma_{\partial Q_{r_n}}<\zeta)\geqq\sum_{n=1}^{+\infty}\text{const.}=+\infty$, therefore the point x belongs to $K_{X^0}^{\text{reg}}$ and hence x belongs to K_X^{reg} by the assumption. Consequently $\underline{\lim}_{r\to 0}P_x(\sigma_{\partial Q_r}<\zeta)=\gamma_x$ must be positive.

3). Let us set $\tilde\varphi(r)=r^{1-d}\int_0^r\varphi(2s)s^{d-2}ds$, then $\tilde\varphi$ belongs to Φ_{d-1} and we can choose $\tilde\varphi$ as a singularity function of $G(x,y)$. As $\tilde\varphi\in\Phi_{d-1}$ can be proved by a simple culculation, we shall show that $\tilde\varphi$ is a singularity function of $G(x,y)$. Let us fix a point $x\in\Omega$, then $1\geqq P_x(\sigma_{\partial Q_r}<\zeta)\geqq\gamma_x-\varepsilon>0$, by the result of **2**). Hence we have

$$const.\ r^{1-d}\int_0^r\varphi(2s)s^{d-2}\left(1-\left(\frac{s}{r}\right)^2\right)^{\frac{d-3}{2}}ds\leqq\varphi(r)$$

$$\leqq const.\ r^{1-d}\int_0^r\varphi(2s)s^{d-2}\left(1-\left(\frac{s}{r}\right)^2\right)^{\frac{d-3}{2}}ds.$$

On the otherhand we have

$$r^{1-d}\int_0^r\varphi(2s)s^{d-2}\left(1-\left(\frac{s}{r}\right)^2\right)^{\frac{d-3}{2}}ds\geqq const.\ r^{1-d}\int_0^{\frac{r}{2}}\varphi(2s)s^{d-2}ds$$

$$\geqq const.\ \tilde\varphi(r)\left\{1-\frac{r^{1-d}\int_{\frac{r}{2}}^r\varphi(2s)s^{d-2}ds}{\tilde\varphi(r)}\right\}$$

$$\geqq const.\ \tilde\varphi(r)\left\{1-\frac{r^{1-d}\int_{\frac{r}{2}}^r\varphi(2s)s^{d-2}ds}{r^{1-d}\int_0^{\frac{r}{2}}\varphi(2s)s^{d-2}ds+r^{1-d}\int_{\frac{r}{2}}^r\varphi(2s)s^{d-2}ds}\right\}$$

[7] The Proposition of §7 in [3] holds without the condition B by Lemma 1 of **§2**.

$$\geqq const.\ \tilde{\varphi}(r)\left\{1-\frac{1}{\dfrac{\varphi(r)(d-1)^{-1}\left(\frac{r}{2}\right)^{d-1}}{\varphi(r)\,(d-1)^{-1}\left\{r^{d-1}-\left(\frac{r}{2}\right)^{d-1}\right\}}+1}\right\}\geqq const.\ \tilde{\varphi}(r)$$

and $r^{1-d}\int_0^r \varphi(2s)s^{d-2}\left(1-\left(\frac{s}{r}\right)^2\right)^{\frac{d-3}{2}} ds \leqq \tilde{\varphi}(r)$. Consequently it holds *const.* $\varphi(r) \leqq \tilde{\varphi}(r) \leqq const.\ \varphi(r)$, which means $\tilde{\varphi}$ is a singularity function of $G(x, y)$.

4). In the same way as in **1**). we can prove $\underline{\lim}_{r\to 0}\frac{\varphi^0(r)}{\tilde{\varphi}(r)} > 0$. This completes the proof of the Theorem.

PROOF OF THE COROLLARY. As "*if*" part follows from Theorem 5[8] in [3], we prove "*only if*" part. Firstly we shall note that the lower index β defined by $\beta = \sup\{\gamma \geqq 0;\ |z|^{-\gamma}\Psi(z) \to +\infty\ as\ |z| \to +\infty\}$ is positive. Indeed, if $\beta = 0$, using S. Orey's lemma 2. 1 in [4] we see that any polar set[9] for a stable process $X_{\alpha'}$ corresponding to the Riesz potential of order α' for $0 < \alpha' \leqq 2$ is also a polar set for X. On the otherhand, if $0 < \alpha_{\prime} < 1$, there exists a compact set K such that K is a polar set for $X_{\alpha'}$ but not a polar set for X_α by Lemma 3 and Lemma 2 in §**2**. Consequently K is a polar set for X but not a polar set for X_α, which means $K^{\text{reg}}_{X_\alpha} \neq K^{\text{reg}}_X$, because Hunt's condition (H) holds for X_α. Secondly we shall prove that there exists a Green function $G(x, y)$ of X such that $G(x, y) = \varphi(|x-y|)$, where $\varphi \in \Phi$. As the lower index is positive and hence there exists a constant $a > 0$ such that $\Psi(z) \neq 0$ for all z, $a < |z| < 2a$, $\frac{1}{\Psi(z)}$ is locally summable from the proof of Proposition 9 in C.S. Herz [1] in case $d \geqq 3$. Therefore, noting $\Psi(z)$ is negative type[10], $\frac{1}{\Psi(z)}$ is a positive definite function on R^d. Hence $K = \mathscr{F}_d^{-1}\left(\frac{1}{\Psi(z)}\right)(x)$ is a slowly increasing measure by Bochner-Schwartz theorem, where $\mathscr{F}_d^{-1}$ denotes the inverse Fourier trans-

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8) By the Lemma 1 in §**2** Theorem 5 in [3] holds without the condition B in [3].

9) We say that a Borel subset A of Ω is a polar set for $X=(x_t, \zeta, M_t, P_x)$ if $P_x(\sigma_A<\zeta)=0$ for all x.

10) We say that a continuous function $\Psi(z)$ on R^d is negative type, if for each natural number n, each $x_1, x_2, \cdots, x_n \in R^d$ and complex numbers $\xi_1, \xi_2, \cdots, \xi_n$

$$\sum_{i\cdot j=1}^{n}[\Psi(x_i)+\overline{\Psi(x_j)}-\Psi(x_i-x_j)]\xi_i\bar{\xi}_j \geqq 0.$$

In case F is a Bernstein's function of C^∞ on $(0, +\infty)$, $F(2\pi^2|z|^2)$ is negative type on R^d for each d

form in R^d in distribution sense. Now let us set $p_d(t,x)=\mathscr{F}_d^{-1}(\exp(-t\Psi(z))(x)$, then $p_d(t,x)$ is a transition probability density of X and we have, for each infinitely differentiable function f of compact support,

$$\int_0^{+\infty}\int_{R^d} p_d(t,x)f(x)\,dx\,dt=\int_0^{+\infty}\int_{R^d}\mathscr{F}_d^{-1}(\exp(-t\Psi(z))(x)f(x)\,dx\,dt$$

$$=\int_0^{+\infty}\int_{R^d}\exp(-t\Psi(x))\mathscr{F}_d^{-1}f(x)\,dx\,dt=\int_{R^d}\int_0^{+\infty}\exp(-t\Psi(x))\,dt\,\mathscr{F}_d^{-1}f(x)\,dx$$

$$=\int_{R^d}\frac{1}{\Psi(x)}\mathscr{F}_d^{-1}f(x)\,dx=K^*f(0).$$

Hence, by Lebesgue-Fubini's theorem, $\int_0^{+\infty}p_d(t,x)dt$ exists for almost all x and locally summable with respect to x. As $p_d(t,x)$ is a function of $|x|$, we set $\tilde{p}_d(t,r)=p(t,x)$ for $|x|=r$. Then $\tilde{p}_d(t,r)$ is a monotonically non-increasing function of r. Indeed, let us define a function $\tilde{\Psi}(z)$ on R^{d+2} by $\tilde{\Psi}(z)=F(2\pi^2|z|^2)$, $z\in R^{d+2}$, then, using the well-known formula for the Bessel function $J_n(s)$;

$$\frac{1}{s}\frac{d}{ds}\left(\frac{J_n(\lambda s)}{s^n}\right)=-\lambda\frac{J_{n+1}(\lambda s)}{s^{n+1}},$$

and noting that $\exp(-t\Psi(z))$ is rapidly decreasing, we have

$$(-2\pi)^{-1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)p_d(t,r)=\mathscr{F}_{d+2}^{-1}(\exp(-t\tilde{\Psi}(z))(x),\quad |x|=r$$

and the right-hand side is nonnegative, because $\exp(-t\tilde{\Psi}(z))$ is a positive definite function on R^{d+2}. Hence $\frac{\partial}{\partial r}p_d(t,r)\leqq 0$. Therefore $\int_0^{+\infty}p_d(t,x)dt$ is finite for all $x\neq 0$ and so it is a Green function of X. Now let us set $\varphi(r)=\int_0^{+\infty}p_d(t,r)dt$, then it is a monotonically nonincreasing and $\varphi(|x-y|)=\int_0^{+\infty}p_d(t,x-y)dt$. If $\varphi(r)$ were bounded, a point is a regular for itself. Hence $\lim_{r\to 0}\varphi(r)=+\infty$. Consequently $\varphi\in\Phi$. Using the Theorem, the proof is completed.

Remark. Most of Markov processes subordinate to the diffusion processes with Hölder continuous coefficients are important examples of Feller processes having Green functions with singularities $\in\Phi$, which we shall show elsewhere.

REFERENCES

[1] C.S. Herz, Theorie elementaire des distributions de Beurling, lecture note.
[2] G.A. Hunt, Markoff processes and potentials. 2, Illinois J. Math. (1957), 316–369.
[3] M. Kanda, On the singularity of Green functions in Markov processes, Nagoya Math. J. Vol. **33,** (1968), 21–52.
[4] S. Orey, Polar sets for Processes with Stationary Independent Increments, Markov Processes and Potential Theory Edited by J. Chover, John Wiley & Sons, Inc., 1967.
[5] S.J. Taylor, On the connection between Hausdorff measures and generalized capacity. Proc. Cambridge Philos. Soc., **57** (1961), 524–531.

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