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The Gradient of a Solution of the Poisson Equation in the Unit Ball and Related Operators

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Abstract. In this paper we determine the $L^1 \to L^1$ and $L^\infty \to L^\infty$ norms of an integral operator $\mathcal{N}$ related to the gradient of the solution of Poisson equation in the unit ball with vanishing boundary data in sense of distributions.

1 Introduction and Notation

We denote by $\mathbf{B} = B^n$ and $\mathbf{S} = S^{n-1}$ the unit ball and the unit sphere in $\mathbf{R}^n$, respectively. We will assume that $n > 2$ (the case $n = 2$ has already been treated [10, 11]). By the vector norm $|\cdot|$ we consider the standard Euclidean distance $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.

The norm of an operator $T: X \to Y$ defined on the normed space X with image in the normed space Y is defined as $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$.

Let G be the Green function, *i.e.*, the function

$$G(x, y) = c_n\Big(\frac{1}{|x-y|^{n-2}} - \frac{1}{[x, y]^{n-2}}\Big),$$

where $c_n = \frac{1}{(n-2)\omega_{n-1}}$, where ω_{n-1} is the Hausdorff measure of S^{n-1} and

$$[x, y] := \big|x|y| - y/|y|\big| = \big|y|x| - x/|x|\big|.$$

The Poisson kernel P is defined

$$P(x, \eta) = \frac{1-|x|^2}{|x-\eta|^n}, \quad |x| < 1, \eta \in S^{n-1}.$$

We are going to consider the Poisson equation

$$\triangle u(x) = g, x \in \Omega, \quad u|_{\partial\Omega} = f,$$

where $f: S^{n-1} \to \mathbf{R}$ is a bounded integrable function on the unit sphere S^{n-1}, and $g: B^n \to \mathbf{R}$ is a continuous function.

The solution of the equation in the sense of distributions is given by

$$u(x) = P[f](x) - \mathcal{G}[g](x) := \int_{S^{n-1}} P(x,\eta) f(\eta)\, d\sigma(\eta) - \int_{B^n} G(x,y) g(y)\, dy,$$

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$|x| < 1$. Here $d\sigma$ is the normalized Lebesgue $n-1$ dimensional measure of the unit sphere $\mathbf{S} = S^{n-1}$.

Our main focus of observation is related to the special case of a Poisson equation with the Dirichlet boundary condition $\triangle u(x) = g$, for $x \in \Omega$, $u|_{\partial\Omega} = 0$, where $g \in L^\infty(B^n)$. The weak solution is then given by

$$u(x) = -\mathcal{G}[g](x) = -\int_{\mathbf{B}} G(x,y)g(y)\,dy, \quad |x| < 1.$$

The problem of estimating the norm of the operator $\mathcal{G}$ in case of various L^p-spaces was established by both authors in [13].

Since

$$\nabla_x G(x,y) = c_n(2-n)\Big(\frac{x-y}{|x-y|^n} - \frac{|y|^2x - y}{[x,y]^n}\Big),$$

this naturally induces the differential operator related to the Poisson equation

$$\mathcal{D}[g](x) := \nabla u(x) = \frac{1}{\omega_{n-1}}\int_{\mathbf{B}}\Big(\frac{x-y}{|x-y|^n} - \frac{|y|^2x-y}{[x,y]^n}\Big)g(y)\,dy. \tag{1.1}$$

Related to the problem of estimating the norm of the operator $\mathcal{D}$, we are going to observe the operator $\mathcal{N}\colon L^\infty(B^n) \to L^\infty(B^n)$ defined by

$$\mathcal{N}[f](x) = \frac{1}{\omega_{n-1}}\int_{\mathbf{B}}\Big|\frac{x-y}{|x-y|^n} - \frac{|y|^2x-y}{[x,y]^n}\Big|f(y)\,dy. \tag{1.2}$$

The main goal of our paper is related to estimating various norms of the integral operator and $\mathcal{N}$. Then we use those results to obtain some norm estimates of the operator $\mathcal{D}$. The compressive study of this problem for $n = 2$ has been done by Kalaj [10, 11] and by Dostanić [6, 7]. For related results we refer [3, 4].

1.1 Gauss Hypergeometric Function

Throughout the paper we will often use the properties of the hypergeometric functions. First of all, the hypergeometric function $F(a,b;c,t) = {}_2F_1(a,b;c,t)$ is defined by the series expansion

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}t^n, \quad \text{for } |t| < 1,$$

and by the continuation elsewhere. Here $(a)_n$ denotes a shifted factorial, *i.e.*, $(a)_n = a(a+1)\cdots(a+n-1)$ and a is any real number. The following identity will be used in proving the main results of this paper (see [14, 2.5.16(43)]):

$$\int_0^\pi \frac{\sin^{\mu-1} t}{(1+r^2-2r\cos t)^\nu}\,dt = \mathrm{B}\Big(\frac{\mu}{2},\frac{1}{2}\Big)F\Big(\nu, \nu + \frac{1-\mu}{2}; \frac{1+\mu}{2}, r^2\Big), \tag{1.3}$$

where B is the beta function.

By using Chebychev's inequality, one can easily obtain the following inequality for the Gamma function (see [8]).

Proposition 1.1 *Let m, p, and k be real numbers with $m, p > 0$ and $p > k > -m$: If*

$$k(p-m-k) \geqslant 0 \quad \textit{respectively} \leqslant,$$

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then we have

$$\Gamma(p)\Gamma(m) \geqslant \Gamma(p-k)\Gamma(m+k) \quad \textit{respectively } \leqslant . \tag{1.4}$$

1.2 Möbius Transformations of the Unit Ball

The set of isometries of the hyperbolic unit ball B^n is a Kleinian subgroup of all Möbius transformations of the extended space $\overline{\mathbf{R}}^n$ onto itself denoted by $\mathbf{Conf}(\mathbf{B}) = \mathbf{Isom}(\mathbf{B})$. We refer to Ahlfors [2] for a detailed survey to this class of important mappings. The Möbius transformation $z = T_x y$ is defined by

$$T_x y = \frac{(1-|x|^2)(y-x) - |y-x|^2 x}{[x,y]^2},$$

and satisfies

$$|T_x y| = \Big|\frac{x-y}{[x,y]}\Big|, \quad \text{and} \quad dy = \Big(\frac{1-|x|^2}{[z,-x]^2}\Big)^n dz.$$

2 The L^∞ Norm of the Operator $\mathcal{N}$

In this section we are going to find the norm of the operator $\mathcal{N}$, defined in (1.2), and, by using this, we estimate the norm of operator $\mathcal{D}$.

Theorem 2.1 *Let $\mathcal{N}: L^\infty(\mathbf{B}) \to L^\infty(\mathbf{B})$ be the operator defined in (1.2). Then*

$$\|\mathcal{N}\|_{L^\infty \to L^\infty} = \frac{2n\pi^{n/2}}{(n+1)\Gamma(n/2)}.$$

Proof First let us note that

$$\|\mathcal{N}\|_{L^\infty \to L^\infty} = \sup_{x\in\mathbf{B}} \int_{\mathbf{B}} \Big|\frac{x-y}{|x-y|^n} - \frac{|y|^2 x - y}{[x,y]^n}\Big|\, dy.$$

So we need to find $\sup_{x\in\mathbf{B}} K(x)$, where

$$K(x) = \int_{\mathbf{B}} \Big|\frac{x-y}{|x-y|^n} - \frac{|y|^2 x - y}{[x,y]^n}\Big|\, dy.$$

Now we are going to use the change of variables $y = T_{-x} z (T_x y = z)$, where $T_{-x}: \mathbf{B} \to \mathbf{B}$ is the Möbius transform defined by

$$T_{-x}(z) = \frac{(1-|x|^2)(y+x) + x|z+x|^2}{[z,-x]^2}.$$

Now we use the following relations $|T_x(y)| = \big|\frac{x-y}{[x,y]}\big|$ and

$$x - T_{-x}(z) = \frac{(1-|x|^2)(-x|z|^2 - z)}{[z,-x]^2}, \quad |x - T_{-x}(z)| = |z|\frac{1-|x|^2}{[z,-x]}.$$

We have that

$$
\begin{aligned}
(2.1)\quad \Big|\frac{x-y}{|x-y|^n}-\frac{|y|^2x-y}{[x,y]^n}\Big|
&=\frac{1}{|x-y|^n}\Big|(x-y)-(|y|^2x-y)\Big|\frac{x-y}{[x,y]}\Big|^n\Big|\\
&=\frac{1}{|x-y|^n}\big|(x-y)-(|y|^2x-y)|z|^n\big|\\
&=\frac{1}{|x-T_{-x}z|^n}\big|(x-T_{-x}z)-(|T_{-x}z|^2x-T_{-x}z)|z|^n\big|\\
&=\frac{[z,-x]^n}{(1-|x|^2)^n|z|^n}\Big|\frac{(1-|x|^2)(-x|z|^2-z)}{[z,-x]^2}+\frac{(1-|x|^2)(z+x)}{[z,-x]^2}|z|^n\Big|\\
&=\frac{[z,-x]^n}{(1-|x|^2)^n|z|^n}\frac{(1-|x|^2)}{[z,-x]^2}|z|\big|(-x|z|-z/|z|)+|z|^{n-1}(z+x)\big|\\
&=\frac{[z,-x]^{(n-2)}}{(1-|x|^2)^{(n-1)}|z|^{n-1}}\big||z|^{n-1}(z+x)-(x|z|+z/|z|)\big|.
\end{aligned}
$$

According to the identity (2.1), we have

$$I=\sup_{x\in B^n}K(x)=\sup_{x\in B^n}(1-|x|^2)\int_0^1 dr\int_{\mathbf{S}}\frac{|rx(r^{n-2}-1)+\xi(r^n-1)|}{|rx+\xi|^{n+2}}\,d\xi.$$

Furthermore, we have the following simple inequality

$$|rx(r^{n-2}-1)+\xi(r^n-1)\|\leqslant(1-r^{n-2})|rx+\xi|+r^{n-2}-r^n.$$

Thus,

$$\frac{|rx(r^{n-2}-1)+\xi(r^n-1)|}{|rx+\xi|^{n+2}}\leqslant\frac{(1-r^{n-2})|rx+\xi|}{|rx+\xi|^{n+2}}+r^{n-2}-r^n|rx+\xi|^{n+2}.$$

So

$$I\leqslant\max_{x\in B^n}(1-|x|^2)\int_0^1 dr\int_{\mathbf{S}}\frac{((1-r^{n-2})|rx+\xi|+r^{n-2}-r^n)}{|rx+\xi|^{n+2}}\,d\xi.$$

Then we have

$$\int_S\frac{d\xi}{|rx+\xi|^a}=\frac{\omega_{n-1}}{\int_0^\pi\sin^{n-2}t\,dt}\int_0^\pi\frac{\sin^{n-2}t\,dt}{(1+r^2|x|^2+2r|x|\cos t)^{a/2}}.$$

By (1.3) we obtain

$$\int_0^\pi\frac{\sin^{n-2}t\,dt}{(1+r^2|x|^2+2r|x|\cos t)^{a/2}}=\frac{\sqrt{\pi}\Gamma\big(\frac12(-1+n)\big)}{\Gamma\big(\frac n2\big)}F\Big(a/2,1-\frac n2+a/2,\frac n2,r^2x^2\Big).$$

In view of

$$\frac{\omega_{n-1}}{\int_0^\pi\sin^{n-2}t\,dt}=\frac{\frac{2\pi^{n/2}}{\Gamma[n/2]}}{\frac{\sqrt{\pi}\Gamma[1/2(-1+n)]}{\Gamma[n/2]}}=\frac{2\pi^{n/2}}{\sqrt{\pi}\Gamma[1/2(-1+n)]},$$

we then infer

$$\int_S \frac{d\xi}{|rx+\xi|^a} = 2\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} F\Big(a/2, 1-\frac{n}{2}+a/2, \frac{n}{2}, r^2x^2\Big).$$

Hence $I \leqslant C_n \sup_{x\in\mathbf{B}} J(x)$ with

$$J(x) = (1-|x|^2)\int_0^1 (1-r^{n-2}) F\Big(\frac{3}{2}, \frac{1+n}{2}, \frac{n}{2}, r^2x^2\Big)\,dr \\ + (1-|x|^2)\int_0^1 (r^{n-2}-r^n)\frac{n-(n-4)r^2|x|^2}{n(1-r^2|x|^2)^3}\,dr = \int_0^1 K_r(x)\,dr.$$

Here $K_r(x) = \sum_{m=0}^{\infty} A_m(r)|x|^{2m}$ where $A_0(r) = 1-r^n$, $r=|x|$, and, for $m \geqslant 1$,

$$A_m(r) = \frac{r^{2m-4}}{2n} r^n(1-r^2)\big(-2m(-2+n+2m)+2(1+m)(n+2m)r^2\big) \\ + r^{2m-4}(r^2-r^n)\big(-2m(-2+2m+n)+(1+2m)(-1+2m+n)r^2\big) \\ \times \frac{\Gamma[\frac{n}{2}]\Gamma[\frac{1+2m}{2}]\Gamma[\frac{n+2m-1}{2}]}{2\sqrt{\pi}m!\Gamma(\frac{1+n}{2})\Gamma(\frac{n}{2}+m)}.$$

Thus $I \leqslant a_0 + \sum_{m=1}^{\infty} a_m|x|^{2m}$, where

$$a_0 = \frac{2n\pi^{n/2}}{(n+1)\Gamma(n/2)},$$
$$a_m = \frac{2(-3+n)n+4(-2+n)m}{n(-3+n+2m)(-1+n+2m)(1+n+2m)} \\ - \frac{(-2+n)(-3+n+4m)\Gamma(\frac{n}{2})\Gamma(-\frac{1}{2}+m)\Gamma(\frac{1}{2}(-3+n)+m)}{8\sqrt{\pi}\Gamma(\frac{1+n}{2})\Gamma(1+m)\Gamma(\frac{n}{2}+m)}.$$

Then $a_m < 0$ if and only if

$$b_m := \frac{2\big((-3+n)n+2(-2+n)m\big)\sqrt{\pi}m!\Gamma(\frac{1+n}{2})\Gamma(\frac{n}{2}+m)}{(-2+n)n(-3+n+4m)\Gamma(\frac{n}{2})\Gamma(-\frac{1}{2}+m)\Gamma(\frac{3+n}{2}+m)} < 1.$$

Then by (1.4) we have

$$\Gamma\Big(-\frac{1}{2}+m\Big)\Gamma\Big(\frac{3+n}{2}+m\Big) \geqslant \Gamma\Big(m\Big)\Gamma\Big(\frac{2+n}{2}+m\Big),$$

and so

$$b_m \leqslant c(m) := \frac{2m((n-3)n+2(n-2)m)\sqrt{\pi}\Gamma(\frac{1+n}{2})}{(-2+n)(n+2m)(-3+n+4m)\Gamma(1+\frac{n}{2})}.$$

The last expression increases in m because

$$c'(m) = \frac{2\big((n-3)^2n^2+4(n-3)(n-2)nm+4(6+(n-3)n)m^2\big)\sqrt{\pi}\Gamma(\frac{1+n}{2})}{(n-2)(n+2m)^2(n+4m-3)^2\Gamma(1+\frac{n}{2})} \geqslant 0,$$

so we have

$$b_m \leqslant \lim_{m\to\infty} c(m) = \frac{\sqrt{\pi}\Gamma(\frac{1+n}{2})}{2\Gamma(1+\frac{n}{2})} < 1.$$

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Then

$$\sup K(x) = K(0) = \frac{2n\pi^{n/2}}{(n+1)\Gamma(n/2)},$$

as required. ∎

Corollary 2.2 *Let $\mathcal{D}$ be the mapping defined in (1.1) and $v = \nabla u = \mathcal{D}g$, $g \in L^\infty(B^n)$. Then*

$$\|v\|_\infty \le \frac{2n\pi^{n/2}}{(n+1)\Gamma(n/2)}\|g\|_\infty.$$

Proof First let us note that

$$\nabla_x G(x,y) = c_n(2-n)\Big(\frac{x-y}{|x-y|^n} - \frac{|y|^2x-y}{[x,y]^n}\Big).$$

For $x \in \mathbf{B}$ we have

$$\begin{aligned}\|\nabla u(x)\| &= \sup_{|\xi|=1}\Big|\Big\langle \int_{\mathbf{B}} \nabla G(x,y)g(y)\,dy, \xi\Big\rangle\Big| = \sup_{|\xi|=1}\Big|\int_{\mathbf{B}}\Big\langle \nabla G(x,y),\xi\Big\rangle g(y)\,dy\Big| \\ &= (n-2)c_n \sup_{|\xi|=1}\Big|\int_{\mathbf{B}}\Big\langle \frac{x-y}{|x-y|^n} - \frac{|y|^2x-y}{[x,y]^n},\xi\Big\rangle g(y)\,dy\Big| \\ &\le (n-2)c_n \int_{\mathbf{B}}\sup_{|\xi|=1}\Big|\Big\langle \frac{x-y}{|x-y|^n} - \frac{|y|^2x-y}{[x,y]^n},\xi\Big\rangle\Big||g(y)|\,dy \\ &= (n-2)c_n \int_{\mathbf{B}}\Big|\frac{x-y}{|x-y|^n} - \frac{|y|^2x-y}{[x,y]^n}\Big||g(y)|\,dy.\end{aligned}$$

So we obtain the upper estimate for the gradient of u, *i.e.*, $\|\nabla u\|_\infty \le \|\mathcal{N}\|\|g\|_\infty$. ∎

3 The L^1 Norm of the Operator $\mathcal{N}$

In the sequel let us state a well-known result related to the Riesz potential. Let Ω be a domain of R^n, and let $|\Omega|$ be its volume. For $\mu \in (0,1]$ define the operator V_μ on the space $L^1(\Omega)$ by the Riesz potential $(V_\mu f)(x) = \int_\Omega |x-y|^{n(\mu-1)}f(y)\,dy$. The operator V_μ is defined for any $f \in L^1(\Omega)$, and V_μ is bounded on $L^1(\Omega)$, or more generally we have the next lemma.

Lemma 3.1 ([9, pp. 156–159]) *Let V_μ be defined on the $L^p(\Omega)$ with $p > 0$. Then V_μ is continuous as a mapping $V_\mu: L^p(\Omega) \to L^q(\Omega)$, where $1 \le q \le \infty$, and*

$$0 \le \delta = \delta(p,q) = \frac{1}{p} - \frac{1}{q} < \mu.$$

Moreover, for any $f \in L^p(\Omega)$

$$\|V_\mu f\|_q \le \Big(\frac{1}{\mu-\delta}\Big)^{1-\delta}\Big(\frac{\omega_{n-1}}{n}\Big)^{1-\mu}|\Omega|^{\mu-\delta}\|f\|_p.$$

Theorem 3.2 *The norm of the operator $\mathcal{N}: L^1 \to L^1$ is $\frac{1}{n-2}$.*

Corollary 3.3 *Let $g \in L^1(\mathbf{B})$ and $v = \nabla u = \mathcal{D}[g]$. Then $\|v\|_1 \le \frac{1}{n-2}\|g\|_1$.*

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In order to prove Theorem 3.2, we need the following lemma.

Lemma 3.4 *Let*

$$H(x,y)=\frac{y-x}{|y-x|^n}-\frac{|x|^2y-x}{[y,x]^n},$$

and let

$$\mathcal{H}[g]=\frac{1}{(n-2)\omega_{n-1}}\int_{\mathbf{B}}|H(x,y)|g(y)\,dy.$$

Then

$$\|\mathcal{H}\|_{L^\infty\to L^\infty}=\frac{1}{(n-2)\omega_{n-1}}\int_{\mathbf{B}}|H(0,y)|\,dy=\frac{1}{n-2}. \tag{3.1}$$

Proof We need to find $\sup_x\int_{\mathbf{B}}|H(x,y)|\,dy$. We will show that its supremum is achieved for $x=0$. We first have

$$|H(x,y)|\leqslant K(x,y)+L(x,y)=|x-y|\Big(\frac{1}{|x-y|^n}-\frac{1}{[x,y]^n}\Big)+\frac{|y|(1-|x|^2)}{[x,y]^n}.$$

Further we have

$$\sup_{x\in B^n}\int_{B^n}|K(x,y)|\,dy=\sup_{x\in B^n}\int_{B^n}\frac{1}{|x-y|^{n-1}}\Big|1-\Big|\frac{x-y}{[x,y]}\Big|^n\Big|\,dy.$$

We use the change of variables $z=T_xy$, *i.e.*, $T_{-x}z=y$, where T_xy is the Möbius transform

$$T_xy=\frac{(1-|x|^2)(y-x)-|y-x|^2x}{[x,y]^2},\quad |T_xy|=\Big|\frac{x-y}{[x,y]}\Big|.$$

We obtain

$$dy=\Big(\frac{1-|x|^2}{[z,-x]^2}\Big)^n dz.$$

Assume, without loss of generality, that $x=|x|e_1$. Furthermore, for $\xi=(\xi_1,\dots,\xi_n)$,

$$\begin{aligned}
&\sup_{x\in\mathbf{B}}\int_{\mathbf{B}}|H(x,y)|\,dy\\
&=\sup_{x\in\mathbf{B}}\int_{\mathbf{B}}\frac{1}{|x-T_{-x}z|^{n-1}}|1-|z|^n|\frac{(1-|x|^2)^n}{[z,-x]^{2n}}\,dz\\
&=\sup_{x\in\mathbf{B}}(1-|x|^2)^n\int_{\mathbf{B}}\frac{(1-|z|^{n-2})}{\big|\frac{x[z,-x]^2-(1-|x|^2)(x+z)-|x+z|^2x}{[z,-x]^2}\big|^{n-1}}\frac{dz}{[z,-x]^{2n}}\\
&=\sup_{x\in\mathbf{B}}(1-|x|^2)^n\int_{\mathbf{B}}\frac{(1-|z|^n)}{|z|^{n-1}\big|\frac{1-|x|^2}{[z,-x]}\big|^{n-1}}\frac{dz}{[z,-x]^{2n}}\\
&=\sup_{x\in\mathbf{B}}(1-|x|^2)^{n-(n-1)}\int_{\mathbf{B}}\Big(\frac{1-|z|^n}{|z|^{n-1}}\Big)[z,-x]^{(n-1)-2n}\,dz\\
&=\sup_{x\in\mathbf{B}}(1-|x|^2)\int_0^1(1-r^n)r^{n-(n-1)-1}\,dr\int_{\mathbf{S}}\frac{d\xi}{|rx+\xi|^{2n-(n-1)}}\\
&=\sup_{x\in\mathbf{B}}(1-|x|^2)\int_0^1(1-r^n)\,dr\int_{\mathbf{S}}\frac{d\xi}{(r^2|x|^2+2r|x|\xi_1+1)^{\frac{n+1}{2}}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2\pi^{\frac{1}{2}(-1+n)}}{\Gamma[\frac{1}{2}(-1+n)]}\sup_{x\in\mathbf{B}}(1-|x|^2)\int_0^1(1-r^n)\,dr\int_0^\pi \frac{\sin^{n-2}t}{(r^2|x|^2+2r|x|\cos t+1)^{\frac{n+1}{2}}}\,dt,\\
&= \frac{2\pi^{n/2}}{\Gamma[\frac{n}{2}]}\sup_{x\in\mathbf{B}}(1-|x|^2)\int_0^1(1-r^n)F\Big(\frac{3}{2},\frac{1+n}{2},\frac{n}{2},r^2|x|^2\Big)\,dr\\
&= \frac{2\pi^{n/2}}{\Gamma[\frac{n}{2}]}\sup_{x\in\mathbf{B}}J(x) = \omega_{n-1}\sup_{x\in\mathbf{B}}J(x),
\end{aligned}$$

where

$$J(x) = \frac{n}{1+n} - \sum_{m=1}^{\infty} e_m|x|^{2m} \leqslant J(0),$$

with

$$e_m = \frac{n\big(8m^2+(n-1)^2+2m(3n-5)\big)\Gamma(m-\frac{1}{2})\Gamma(\frac{3}{2}+m+\frac{n}{2})\Gamma(\frac{n}{2})}{(2m+n-1)^2(2m+n+1)^2\sqrt{\pi}\Gamma[1+m]\Gamma(m+\frac{n}{2})\Gamma(\frac{1+n}{2})}.$$

In the first appearance of a hypergeometric function we used (1.3). On the other hand, similarly we prove that

$$\begin{aligned}
L(x) = \int_{\mathbf{B}} L(x,y)\,dy &= C'_n(1-|x|^2)F\Big(1,\frac{1+n}{2},\frac{3+n}{2},|x|^2\Big)\\
&= C'_n\Big(1-\sum_{m=1}^{\infty}\frac{2(1+n)}{-1+4m^2+4mn+n^2}|x|^{2m}\Big)\\
&\leqslant L(0),
\end{aligned}$$

where

$$C'_n = L(0) = \int_{\mathbf{B}}|y|\,dy = \frac{n}{n+1}\frac{\pi^{n/2}}{\Gamma[1+n/2]} = \frac{\omega_{n-1}}{n+1}.$$

Hence

$$\begin{aligned}
\sup_x\int_{\mathbf{B}}|H(x,y)|\,dy \leqslant \int_{\mathbf{B}}|H(0,y)|\,dy &= \omega_{n-1}J(0)+L(0)\\
&= \Big(\frac{n}{n+1}+\frac{1}{n+1}\Big)\omega_{n-1} = \omega_{n-1}.
\end{aligned}$$

This implies (3.1). ■

Proof of Theorem 3.3 and Corollary 3.2 Since $\|\mathcal{N}\|_{L^1\to L^1} = \|\mathcal{N}^*\|_{L^\infty\to L^\infty}$, where $\mathcal{N}^*$ is the appropriate adjoint operator and

$$\mathcal{N}^*f(x) = \int_{\mathbf{B}}\overline{\mathcal{N}(y,x)}f(y)\,dy = \int_{\mathbf{B}}|H(x,y)|f(y)\,dy,\quad f\in L^\infty(B),$$

we have $\|\mathcal{N}^*\|_{L^1\to L^1} = \|\mathcal{H}\|_{L^\infty\to L^\infty}$. So Theorem 3.3 follows from Lemma 3.4. On the other hand, Corollary 3.2 follows from the inequality

$$\|\mathcal{D}[g]\|_{L^1\to L^1} \le \|\mathcal{N}\|_{L^1\to L^1}.$$ ■

Now let us point out the fact that $\mathcal{D}: L^p(\mathbf{B},\mathbf{R}) \to L^p(\mathbf{B},\mathbf{R}^n)$, where $L^p(\mathbf{B},\mathbf{R}^n)$ is the appropriate Lebesgue space of vector functions. By $\|\mathcal{D}\|_p$ we denote the norm of the operator $\mathcal{D}$.

By using the Ries–Thorin interpolation theorem, we obtain the next estimates of the norm for the operators $\mathcal{N}$ and $\mathcal{D}$

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Corollary 3.5 *Let us denote by* $\|\mathcal{N}\|_i := \|\mathcal{N}\|_{L^i\to L^i}$, $i \in \{1,\infty\}$. *Then*

$$\|\mathcal{D}\|_p < \|\mathcal{N}\|_p \le \|\mathcal{N}\|_1^{\frac{1}{p}} \|\mathcal{N}\|_\infty^{\frac{p-1}{p}}, \quad 1<p<\infty.$$

Conjecture 3.6 We know that $\mathcal{D}$ and $\mathcal{N}$ map $L^p(\mathbf{B})$ into $L^\infty(\mathbf{B})$ for $p>n$. We have that $\|\mathcal{N}g\|_\infty \le A_p\|g\|_p$ and $\|\mathcal{D}g\|_\infty \le B_p\|g\|_p$, where

$$A_p = \frac{1}{\omega_{n-1}} \sup_{x\in\mathbf{B}} \Big(\int_{\mathbf{B}} \Big|\frac{x-y}{|x-y|^n} - \frac{|y|^2x-y}{[x,y]^n}\Big|^q\,dy\Big)^{1/q},$$

$$B_p = \frac{1}{\omega_{n-1}} \sup_{x\in\mathbf{B},|\eta|=1} \Big(\int_{\mathbf{B}} \Big|\Big\langle \frac{x-y}{|x-y|^n} - \frac{|y|^2x-y}{[x,y]^n},\eta\Big\rangle\Big|^q\,dy\Big)^{1/q}.$$

Then we conjecture that

$$A_p = \frac{1}{\omega_{n-1}}\Big(\int_{\mathbf{B}}\Big(\frac{1}{|y|^{n-1}} - |y|\Big)^q\,dy\Big)^{1/q} = \omega_{n-1}^{-\frac{1}{p}}\Big(\frac{\Gamma[1+q]\Gamma[1+(-1+\frac{1}{n})q]}{n\Gamma[2+\frac{q}{n}]}\Big)^{1/q},$$

$$B_p = \frac{1}{\omega_{n-1}}\sup_{|\eta|=1}\Big(\int_{\mathbf{B}}|\langle y,\eta\rangle|^q\Big(\frac{1}{|y|^n}-1\Big)^q\,dy\Big)^{1/q}$$

$$= \omega_{n-1}^{-\frac{1}{p}}\Big(\frac{\Gamma[\frac{n}{2}]\Gamma[1+q]\Gamma[\frac{1}{2}(-1+n+q)]\Gamma[1+(-1+\frac{1}{n})q]}{n\Gamma[\frac{1}{2}(-1+n)]\Gamma[\frac{n+q}{2}]\Gamma[2+\frac{q}{n}]}\Big)^{1/q}.$$

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