



Simplicial localization of monoidal structures, and a non-linear version of Deligne’s conjecture

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ABSTRACT

We show that if (M, \otimes, I) is a monoidal model category then $\mathbb{R}\text{End}_M(I)$ is a (weak) 2-monoid in $s\text{Set}$. This applies in particular when M is the category of A -bimodules over a simplicial monoid A : the derived endomorphisms of A then form its Hochschild cohomology, which therefore becomes a simplicial 2-monoid.

Introduction

Deligne’s conjecture. Deligne’s conjecture (stated informally in a letter in 1993) states that the Hochschild cohomology $\text{HH}(A)$ of an associative algebra A is a 2-algebra; this means that up to homotopy it has two compatible multiplication laws.

Various versions of this conjecture have been proved; cf. e.g. [GJ94, GV95, Tam98, Kon99, Vor00, MS02, KS00, HKV03, BF01]. All these proofs are technical, and a more conceptual proof would certainly be desirable (we refer for example to [Bat02] for a conceptual point of view on Deligne’s conjecture based on higher category theory). In the present work we state a non-linear version of this conjecture, and provide an elementary proof of it based on model category theory and simplicial localization techniques *à la* Dwyer and Kan.

The main result of this work is the following. It can reasonably be considered as a model category version of the well-known fact that the endomorphisms of the unit of a monoidal category form a commutative monoid.

THEOREM. *Let (M, \otimes, I) be a monoidal model category. The simplicial set of derived endomorphisms of the unit, $\mathbb{R}\text{End}_M(I)$, is a simplicial 2-monoid (cf. § 1.2).*

The theorem applies in particular when M is the category of A -bimodules over a simplicial monoid A . Then the Hochschild cohomology $\text{HH}(A)$ is naturally identified with $\mathbb{R}\text{End}_{\text{Bimod}_A}(A)$, and hence becomes a 2-monoid in $s\text{Set}$. This is what we refer to as the *non-linear analogue of Deligne’s conjecture*.

The proof of our theorem relies heavily on ideas of Segal [Seg74] and of Dwyer and Kan [DK80a]. Once the statements have been formulated in terms of Segal categories, the theorem follows from two easy observations and an application of a theorem of Dwyer and Kan.

First it is observed that, if a monoidal structure on a category is strictly compatible with a notion of equivalence, the Dwyer–Kan localization is a *monoid in the category of simplicial categories* (see § 1.2), and by taking the endomorphism space of the unit object we get instead a monoid in the category of simplicial monoids, i.e. what we call a simplicial 2-monoid.

Remark. We warn the reader that the word *monoid* is used in this work in a much weaker sense than usual, and always refers to an underlying notion of *equimorphisms*. See § 1.2 for details.

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Second, in the case of a monoidal model category (in the sense of Hovey [Hov99]), the monoidal operation does not preserve equivalences, but we observe that Hovey’s ‘unit axiom’ expresses exactly that a suitable equivalent full subcategory has the strict compatibility and hence we have reduced to the first case.

The theorem of Dwyer and Kan [DK80a] describes the derived homomorphisms of a model category (the simplicial function complexes) in terms of its simplicial localization.

It is fair to point out that our viewpoint and proof do not seem to work for the original Deligne conjecture, since currently the theory of Segal categories does not work well in linear contexts (like chain complexes), but only in Cartesian monoidal contexts. Also, we did not investigate the relations between 2-monoids and simplicial sets with an action of the little 2-cube operad, and therefore our version of Deligne’s conjecture might be considered as a bit far from the original one. However, our original motivation was not to give an additional proof of Deligne’s conjecture, but rather to try to understand it from a more conceptual point of view. The new insight provided by our approach may also shed light on related subjects. We also think it is an interesting application of simplicial localization techniques.

1. Localization of monoidal coloured categories

1.1 *Coloured categories and simplicial localization.* By a *coloured category* we mean a pair (C, W) where C is a category and W is a subclass of arrows, called *equimorphisms* (or *coloured arrows*), closed under composition, and comprising all isomorphisms. Key examples are *Top*, *sSet*, and *Cat* with the usual notions of (weak) equivalences as equimorphisms. For the present purposes, an equally important example is *sCat*, the category of simplicial categories (cf. [DK80b]): a simplicial functor $F : A \rightarrow B$ is coloured if $\pi_0 F : \pi_0 A \rightarrow \pi_0 B$ is an equivalence of categories and for each pair of objects $x, y \in A$, the map $A(x, y) \rightarrow B(Fx, Fy)$ is a weak equivalence of simplicial sets.

The importance of coloured categories is that they can be localized and thus serve as context for expressing weak structures. Let $CCat$ denote the category whose objects are coloured categories and whose arrows are functors that preserve equimorphisms. The classical notion of localization [GZ67] is the functor $\text{Ho} : CCat \rightarrow Cat$ defined by formally inverting all equimorphisms. A much more sophisticated construction is the simplicial localization introduced by Dwyer and Kan [DK80b], which can be seen as a derived version of Ho . It is a functor $L : CCat \rightarrow sCat$. It reflects much more homotopy theoretic information than the classical localization, and in many respects it seems to be the ‘correct’ localization, of which the classical localization is just a truncation. (Indeed, the category of connected components of LC is equivalent to $\text{Ho}(C)$.)

There are several possible ways of turning $CCat$ into a coloured category itself; the crucial desired property is that L should be colour-preserving. For simplicity we take this as the definition: a (colour-preserving) functor $F : (C, W) \rightarrow (C', W')$ between coloured categories is called an *equifunctor* if LF is an equivalence of simplicial categories.

1.2 *Monoids and 2-monoids.* Let (S, W) be any of the coloured categories mentioned above; in particular S is monoidal with Cartesian product as multiplication and the singleton object $*$ as unit. A *monoid in* (S, W) is a functor $X : \Delta^{\text{op}} \rightarrow S$ satisfying the following conditions:

- S0) $X_0 = *$.
- S1) The natural maps $X_k \rightarrow X_1 \times \cdots \times X_1$ are equimorphisms ($k \geq 1$).

This last axiom is the *Segal condition*. It played a crucial role in Segal’s work [Seg74] and was subsequently named after him by Tamsamani [Tam99]. A *monoid homomorphism* is a natural

transformation of such functors. The category $Mon(S)$ of monoids and monoid homomorphisms is coloured via the forgetful functor to S . A 2-monoid in (S, W) is by definition a monoid in $Mon(S)$.

In the case $S = sSet$, a monoid is just a Segal category with a single object, and a 2-monoid is a Segal category with a single 0-cell and a single 1-cell. For the basic theory of Segal categories, see [HS98] or [TV02]. Note that this notion of monoid makes sense only in Cartesian monoidal categories (in the usual sense), since it depends on the universal property of the product.

1.3 Monoidal categories as weak monoids. A monoidal category can be described as a sort of weak monoid object in Cat . The weakness is usually described in terms of 2-cells subject to coherence constraints (e.g. as a bicategory with a single object). Here, we will adopt instead the simplicial viewpoint, and define a monoidal category as a monoid in Cat , in the sense of § 1.2, conveniently hiding all questions of coherence from the user interface.

This notion is not the same as the usual one defined in terms of coherence, but since monoidal categories in either sense are equivalent to strict monoidal categories, the two notions lead to the same homotopy theory. It is not trivial to make specific translation between the two languages (cf. [Lei00]; see also [Seg74]).

1.4 Monoidal coloured categories. A monoidal coloured category is a monoidal structure on a coloured category (C, W) whose structure functors are colour-preserving. Precisely, we define a *monoidal coloured category* to be a functor $\Delta^{\text{op}} \rightarrow CCat$ satisfying conditions S0 and S1.

1.5 Localization of monoidal coloured categories. The way we have set things up, it is immediate that the localization of a monoidal coloured category is a monoidal simplicial category. Indeed, it is just the composite

$$\Delta^{\text{op}} \rightarrow CCat \xrightarrow{L} sCat.$$

Since L preserves the terminal object, preserves products up to equivalence (see [TV02, Corollary 4.1.2]), and preserves equivalences, this composite will again satisfy conditions S0 and S1. (Similar remarks hold of course for the classical localization.)

1.6 Endomorphisms of the unit. In fact the simplicial categories appearing in the image are all pointed; the base point is simply the image of $[0]$. Thus we can in a canonical way compose with the endomorphism functor $sCat_* \rightarrow Mon(sSet)$, associating to each pointed simplicial category the endomorphism monoid of the base point. This is a strict simplicial monoid, and this functor preserves products, terminal object, and equimorphisms. The whole composite is therefore a monoid object in $Mon(sSet)$, i.e. a simplicial 2-monoid. Hence we obtain the following:

THEOREM 1. *Let $((C, W), \otimes, I)$ be a monoidal coloured category. Then $LC(I, I)$ is a simplicial 2-monoid.*

2. Localization of monoidal model categories

Model categories are prominent examples of coloured categories, and their richer structure allows for important variations on the localization theme.

2.1 The pushout product axiom. Localization of monoidal structure in a model category M was considered from the very beginning of model category theory: Quillen [Qui67] observed that, in order to induce a monoidal structure on $\text{Ho}(M)$, it is not necessary for a monoidal structure on M to preserve equivalences on the nose, as in the general coloured case of § 1.4. It is enough that the

unit is cofibrant and that \otimes satisfies the *pushout product axiom*: given cofibrations $A_1 \rightarrow A_2$ and $B_1 \rightarrow B_2$, the induced map

$$(A_1 \otimes B_2) \coprod_{A_1 \otimes B_1} (A_2 \otimes B_1) \longrightarrow A_2 \otimes B_2$$

is again a cofibration, and if furthermore one of the two original maps is a *trivial* cofibration then the induced map is too. Indeed, in this case it follows easily from Ken Brown’s Lemma that the full subcategory of cofibrant objects M^c is a monoidal coloured category, and in any case M^c and M have the same homotopy type, so one can induce a monoidal structure on $\text{Ho}(M)$ by taking it from $\text{Ho}(M^c)$.

2.2 *The unit axiom.* Later, Hovey [Hov99] remarked that the requirement that the unit be cofibrant can be relaxed. Assuming the pushout product axiom holds, it is enough that M satisfies the *unit axiom*: for a given cofibrant replacement functor $Q : M \rightarrow M^c$, and for every cofibrant X , the composite $QI \otimes X \rightarrow I \otimes X \rightarrow X$ is an equivalence (and similarly from the right). In this situation, even though the multiplication law on M^c is not unital, the induced multiplication on $\text{Ho}(M) \simeq \text{Ho}(M^c)$ does in fact acquire a unit. This justifies the terminology of [Hov99], which has become standard:

DEFINITION. A *monoidal model category* is a model category with a monoidal structure satisfying the pushout product axiom and the unit axiom.

The following simple observation seems not to have been made before. Assume that the pushout property axiom holds in (M, \otimes, I) , and let M^\heartsuit denote the full subcategory of all cofibrant objects together with the unit.

LEMMA 2.3. *The unit axiom holds in (M, \otimes, I) if and only if $(M^\heartsuit, \otimes, I)$ is a monoidal coloured category (i.e. the monoidal operation preserves equivalences).*

Proof. Simply note that the unit axiom holds for $QI \simeq I$ if and only if for *any* cofibrant Z with an equivalence $Z \simeq I$ the conclusion of the unit axiom holds: for cofibrant X , the map $Z \otimes X \rightarrow I \otimes X \rightarrow X$ is an equivalence.

Now an equivalence in M^\heartsuit is either one between cofibrant objects (which case is covered by the pushout product axiom), or of the type $Z \simeq I$ (the situation just analysed), or $I \simeq Z$. But this last type of equivalence is preserved under \otimes , provided the unit axioms holds, as it readily follows by taking a cofibrant replacement of the map and invoking the 2-out-of-3 axiom for a model category. □

It is easy to see that the monoidal structure induced on $\text{Ho}(M)$ by Hovey’s arguments (respectively on LM) is merely the one coming from $\text{Ho}(M^\heartsuit)$ (respectively from LM^\heartsuit) via the direct construction of § 1.5. One observation is due for this to make sense:

LEMMA 2.4. *The full embedding $F : M^\heartsuit \hookrightarrow M$ induces an equivalence $LM^\heartsuit \simeq LM$ of simplicial categories.*

Proof. In fact this is true for any full subcategory sandwiched between M^c and M . A cofibrant replacement functor $Q : M \rightarrow M^c \hookrightarrow M^\heartsuit$ comes with natural transformations $Q \circ F \Rightarrow \text{id}_{M^\heartsuit}$ and $F \circ Q \Rightarrow \text{id}_M$, whose components are equivalences. By standard arguments (see [HS98, Lemma 8.1] for all details), this induces an equivalence after simplicial localization. □

2.5 *Derived endomorphisms.* The derived hom set (simplicial function complex) of a pair of objects in a model category is usually defined in terms of fibrant–cofibrant resolution functors

(see e.g. [DK80a]). We will denote them by $\mathbb{R}\underline{\text{Hom}}_M(-, -)$. For two objects A and B in M , $\mathbb{R}\underline{\text{Hom}}_M(A, B)$ is an object in $s\text{Set}$ defined up to equivalence. Of course, $\mathbb{R}\underline{\text{Hom}}_M(A, A)$ is denoted by $\mathbb{R}\underline{\text{End}}_M(A)$.

A deep result of Dwyer and Kan [DK80a] states that this simplicial set is equivalent to the simplicial hom sets of the simplicial localization:

$$\mathbb{R}\underline{\text{Hom}}_M(A, B) \simeq LM(A, B) \simeq LM^\natural(A, B).$$

(This was actually the original motivation for introducing simplicial localization.) In particular, by Lemma 2.4 we have $\mathbb{R}\underline{\text{End}}_M(I) \simeq LM^\natural(I, I)$, and in combination with Theorem 1 we get the following theorem.

THEOREM 2. *Let (M, \otimes, I) be a monoidal model category. Then $\mathbb{R}\underline{\text{End}}_M(I)$ is a simplicial 2-monoid.*

Of course, the expression ‘is’ in the above theorem really means ‘is equivalent to the underlying simplicial set of a 2-monoid in $s\text{Set}$ ’.

Remark 2.6. In some cases the trick of just adding the non-cofibrant unit by hand is not appropriate: for example in K -theory one studies Waldhausen categories, which are subcategories of the category of cofibrant objects, and one cannot just add the unit. In a similar vein, Spitzweck [Spi01] works with a notion of monoidal model category with pseudo-unit: this pseudo-unit does not act as a unit, but its cofibrant replacements do, up to homotopy. In these cases the important structure is not the ‘unit’ itself but rather the space of cofibrant replacements. These cases are accounted for by the theory of monoidal categories with weak units, and more generally higher categories with weak identity arrows, where instead of a strict identity each object has a contractible space of up-to-homotopy identity arrows. The basics of this theory are worked out elsewhere; see [Koc04] for an introduction. In fact our original approach to the theorem was with weak units, but for the present purpose the M^\natural trick seems simpler.

3. A simplicial version of Deligne’s conjecture

3.1 Bimodules. Let A be a simplicial monoid (in the strict sense, i.e. a simplicial object in the category of monoids), then $A \times A^{\text{op}}$ is again a simplicial monoid, and we can consider the category of $(A \times A^{\text{op}})$ -modules (i.e. simplicial sets with an $(A \times A^{\text{op}})$ -action). $(A \times A^{\text{op}})$ -modules will be called A -bimodules, and the category of A -bimodules is denoted by Bimod_A . This category carries a natural model structure whose fibrations and equivalences are induced via the forgetful functor $\text{Bimod}_A \rightarrow s\text{Set}$ (this is standard, see e.g. Schwede and Shipley [SS00]). There is a tensor product defined on Bimod_A as the coequalizer $M \times A \times N \rightrightarrows M \times N \rightarrow M \otimes_A N$. The bimodule A itself is the unit for \otimes_A .

LEMMA 3.2. *$(\text{Bimod}_A, \otimes_A, A)$ is a monoidal model category.*

Proof. The proof of the lemma relies on a small object argument, using the standard generating sets of cofibrations and trivial cofibrations (described in [SS00]), as explained in [Hov99, § 4.3].

Let us recall that the forgetful functor $\text{Bimod}_A \rightarrow s\text{Set}$ possesses a left-adjoint $F : s\text{Set} \rightarrow \text{Bimod}_A$, sending a simplicial set X to the free A -bimodule $F(X) = A \times X \times A$. If I_0 (respectively J_0) is a set of generating cofibrations (respectively trivial cofibrations) in $s\text{Set}$ then $I = F(I_0)$ (respectively $J = F(J_0)$) is a set of generating cofibrations (respectively trivial cofibrations) in Bimod_A .

To prove the pushout product axiom in Bimod_A it is enough by [Hov99, § 4.3] to notice that for two simplicial sets X and Y one has a natural isomorphism of A -bimodules

$$F(X) \otimes_A F(Y) \simeq F(X \times A \times Y).$$

The pushout product axiom in $Bimod_A$ is then a direct consequence of the well-known facts that the functor F is left-Quillen and that the pushout product axiom holds in $sSet$.

It remains to prove the unit axiom in $Bimod_A$. For this we use the standard free resolution associated to the forgetful functor $Bimod_A \rightarrow sSet$ (see e.g. [Ill71]). Let us recall that for any A -bimodule M , one constructs a simplicial object $P_*(M)$ in $Bimod_A$, together with an augmentation $P_0(M) \rightarrow M$ such that the natural morphism

$$\operatorname{hocolim}_{n \in \Delta^{\text{op}}} P_n(M) \longrightarrow M$$

is an equivalence in $Bimod_A$. Furthermore, each A -bimodule $P_n(M)$ is free and given by $P_n(M) := F(P_{n-1}(M))$, and the various face and degeneracy morphisms are given by using the adjunction between the forgetful functor and F . Since each $P_n(M)$ is a cofibrant object in $Bimod_A$ (as it is free), we can use Hirschhorn [Hir03, Theorem 19.4.2] to see that $\operatorname{hocolim}_n P_n(M)$ is a cofibrant model for M . To check the unit axiom it is therefore enough by [Hov99, § 4.3] to prove that for any simplicial set X the natural morphism

$$(\operatorname{hocolim}_n P_n(A)) \otimes_A F(X) \longrightarrow A \otimes_A F(X) \simeq F(X)$$

is an equivalence. Clearly this morphism is isomorphic to

$$\operatorname{hocolim}_n (P_n(A) \otimes_A F(X)) \longrightarrow A \otimes_A F(X) \simeq F(X).$$

But, $P_n(A) \otimes_A F(X) \simeq P_n(A) \times X \times A$, at least in $sSet$, and therefore the morphism is in fact isomorphic, as a morphism in $sSet$, to

$$\operatorname{hocolim}_n (P_n(A) \times X \times A) \longrightarrow A \times X \times A = F(X).$$

The fact that this last morphism is an equivalence follows simply from the fact that $P_*(A)$ is a simplicial resolution of A and that homotopy colimits commute, up to equivalences, with products. \square

Note that the unit object A of the model category $Bimod_A$ is not cofibrant. Indeed, cofibrant means roughly ‘free’, i.e. direct sum of copies of $A \times A^{\text{op}}$, but A is rather a quotient.

3.3 The Hochschild cohomology. The Hochschild cohomology of a simplicial monoid A can naturally be defined as

$$\operatorname{HH}(A) := \mathbb{R}\underline{\operatorname{End}}_{Bimod_A}(A).$$

More explicit descriptions can be given in the style of the Hochschild complex of an associative algebra; these can be obtained by taking explicit cofibrant replacement of A as an A -bimodule.

The following non-linear version of Deligne’s conjecture is now an immediate consequence of Theorem 2.

THEOREM 3. *Let A be a simplicial monoid. Then the Hochschild cohomology $\operatorname{HH}(A)$ is a simplicial 2-monoid.*

Remark 3.4. The Hochschild cohomology of a simplicial monoid is a homotopy version of its centre. Indeed, if M is a monoid (in the category of sets), the endomorphisms of M as an M -bimodule is naturally isomorphic to the centre of M . In this sense, Deligne’s conjecture can be seen as a homotopy analogue of the tautology that the centre of a monoid in Set is a commutative monoid (i.e. a 2-monoid in Set). It is also a homotopical version of the fact that the centre of a monoidal category is a braided monoidal category (i.e. a weak 2-monoid in Cat); cf. [JS93, Example 2.3].

4. Higher-dimensional generalization

There is also a higher version of Deligne's conjecture which states that the Hochschild complex $\mathrm{HH}(A)$ of a d -algebra A is itself a $(d + 1)$ -algebra (see for example [Tam00, KS00, HKV03]). Such d -algebras are complexes endowed with an action of the little d -disc operad, and are usually interpreted as complexes with d multiplication laws, compatible with each other up to coherent homotopies.

The higher version of Deligne's conjecture also has a simplicial analogue. Indeed, our Theorem 3 can be generalized in the following way in terms of Segal categories (starting from the observation that a simplicial monoid is a Segal 1-monoid). First of all, the definition of monoids in a coloured category as described in § 1.2 can be iterated. Starting by letting $0\text{-}SeMon$ be the coloured category of simplicial sets, one defines (for $d \geq 1$) the coloured category $d\text{-}SeMon$ of Segal d -monoids as

$$d\text{-}SeMon := Mon((d - 1)\text{-}SeMon),$$

in the sense of § 1.2.

Any Segal d -monoid M has an underlying Segal 1-monoid, which up to equivalence can be chosen to be a simplicial monoid in the usual sense (i.e. a simplicial object in the category of monoids); cf. e.g. [HS98, § 8]. We define *the Hochschild cohomology of a Segal d -monoid* to be the Hochschild cohomology of its underlying simplicial monoid, as defined in § 3.3. Theorem 3 now has the following generalization.

THEOREM 4. *Let A be a Segal d -monoid. Then the Hochschild cohomology $\mathrm{HH}(A)$ is a Segal $(d + 1)$ -monoid.*

We will not include the proof of this theorem as it uses the theory of Segal categories and the so-called *strictification theorem* stated in [TV02] (some more details can be found in [Toë03, Theorem 2.13]). It would be interesting however to have a model category proof of Theorem 4. A possible approach would be through a suitable notion of *iterated model category*, which roughly would be an iterated monoidal category in the sense of [BFSV03], together with a compatible model category structure. Our Theorem 2 should then generalize as follows: if M is a d -times iterated monoidal model category then $\mathbb{R}\underline{\mathrm{End}}_M(I)$ is a $(d + 1)$ -monoid.

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