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# POSITIVE CORRELATIONS IN A THREE-NODE JACKSON QUEUEING NETWORK

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**Abstract**

We show for the three-node Jackson network studied in [3] that a customer's sojourn times in nodes 1 and 3 are positively correlated. We actually prove a stronger result, that the two sojourn times are associated random variables. Our proof uses a stochastic ordering argument similar to that in [4].

## 1. Introduction

In this paper, we revisit the sojourn time problem in three-node Jackson network studied in [3]. They analyzed the following network. Customers arrive according to a Poisson process with rate $\lambda$. All customers enter node 1. Service times at node $i$, $i = 1, 2, 3$, are independent, exponentially distributed with parameter $\mu_i$. Customers choose either the route $r_1 \equiv 1 \to 2 \to 3$ or $r_2 \equiv 1 \to 3$ according to a Bernoulli process with parameter $p$ of choosing $r_1$. This decision process, the arrival, and the service processes are mutually independent. In [3], it is shown that the sojourn times at nodes 1 and 3 are not independent for a customer taking the route $r_1$. Thus, the total sojourn time of a customer in the network is not the sum of independent random variables although this is a good approximation (cf. [2]). Whitt ([5], Conjecture 1.1, p. 412) conjectures that in more general networks 'the sojourn times of a customer at the different nodes on his route through the network are positively dependent, for example, associated.' On p. 418, Whitt mentions that it would be nice to show that the correlation between the sojourn times is non-negative in three-node network.

We give a short proof that the sojourn times in this three-node network are associated. Of course, all of the sojourn times are independent except for the sojourn times at nodes 1 and 3 of a customer that follows route $r_1$. We show that for such a customer the sojourn time at node 3 is stochastically increasing in the sojourn time at node 1. This implies that the random variables are associated and positively correlated (cf. [1], pp. 142–146). Let $S_1$ and $S_3$ be the sojourn times at nodes 1 and 3 of a customer that follows route $r_1$. From [1], it suffices to show that $S_3$ is stochastically increasing in $S_1$, i.e.,

$$\Pr\{S_3 > t \mid S_1\} \tag{1}$$

is increasing in $S_1$. To do this it suffices to show that the queue length at node 3 upon the customer's arrival there is stochastically increasing in $S_1$. We use the following coupling argument which is similar to that given in [4]. Let $0 \leqq u_1 < u_2$. Construct two systems. System 1 assumes $S_1 = u_1$ and system 2 assumes $S_1 = u_2$. Let time 0 be the time at which the tagged customer arrives at node 2. From [3], the joint probability of having $i$ at queue 1, $j$ at queue 2,

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and $k$ at queue 3 at time 0 is

$$\frac{\exp(-\lambda S_1)(\lambda S_1)^i}{i!}\left(1-\frac{\lambda p}{\mu_2}\right)\left(\frac{\lambda p}{\mu_2}\right)^{j-1}\left(1-\frac{\lambda}{\mu_3}\right)\left(\frac{\lambda}{\mu_3}\right)^k. \tag{2}$$

In fact, (2) is the conditional probability of the joint queue length embedded at a customer's departure from node 1 which goes to node 2 given its sojourn time in node 1 is $S_1$. The tagged customer is the last customer in line at node 2. Because of this distribution, we can assume at time 0 the queue lengths at nodes 2 and 3 are the same for both systems, but the queue length at node 1 in system 1 is less than or equal to that in system 2. Now assume that the two systems have exactly the same arrival process to node 1. Assume that there are three independent Poisson processes each with parameter $\mu_i$, $i = 1, 2, 3$. These define the potential departure times at the nodes. If there are customers present at node $i$ at an epoch of the corresponding Poisson process, then there is a departure from node $i$. Along with the first Poisson process is a Bernoulli process with parameter $p$. The $n$th Bernoulli random variable indicates whether the $n$th potential departure from node 1 would have gone to node 2 or 3. Clearly, the tagged customer arrives at node 3 at the same time in both systems. Also, whenever there is an arrival to node 3 in system 1, there is also an arrival to node 3 in system 2, though system 2 may have additional arrivals. Call the extra customers in node 1 at time 0 the special customers and all other customers regular customers. Give priority to regular customers at node 3. Thus if there is a potential departure time at node 3, a regular customer will be selected over a special customer. Also, the departure process of regular customers in systems 1 and 2 are identical. Hence, the number of regular customers at node 3 when the tagged customer arrives is the same in both systems, but node 3 may have additional special customers in system 2. Consequently, (1) holds.

Our proof does not extend to more general Jackson networks. It relies on the tagged customer reaching node 3 at the same time in both systems. This is not the case in more general Jackson networks.

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