## The Geometrography of Euclid's Problems.

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The term Geometrography is new to mathematical science, and it may be defined, in the words of its inventor, as "the art of geometrical constructions."

Certain constructions are, it is well known, simpler than certain others, but in many cases the simplicity of a construction does not consist in the practical execution, but in the brevity of the statement, of what has to be done. Can then any criterion be laid down by which an estimate may be formed of the relative simplicity of several different constructions for attaining the same end?

This is the question which Mr Émile Lemoine put to himself some years ago, and which he very ingeniously answered in a memoir read at the Oran meeting (1888) of the French Association for the Advancement of the Sciences. Mr Lemoine has since returned to the subject, and his maturer views will be found in another memoir read at the Pau meeting (1892) of the same Association. The object of the present paper is to give an account of Mr Lemoine's method of estimation, to suggest a slight modification of it, and to apply it to the problems contained in the first six books of Euclid's *Elements*.

In the first place Mr Lemoine restricts himself, as Euclid does, to constructions executed with the ruler and the compasses, and these he divides into the following elementary operations:

| | |
|---|---|
| To place the edge of the ruler in coincidence with a point ................ | $R_1$ |
| To draw a straight line ................ | $R_2$ |
| To put a point of the compasses on a determinate point ................ | $C_1$ |
| To put a point of the compasses on an indeterminate point of a line ................ | $C_2$ |
| To describe a circle ................ | $C_3$ |

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No account is taken of the length of the lines that are described; if any portion of a straight line be drawn the operation is $R_2$, if a small arc only or the whole circumference be described, the operation is $C_3$.

It ought also to be added that to place the edge of the ruler in coincidence with two points is $2R_1$; to put one point of the compasses on a determinate point and the other point of the compasses on another determinate point is $2C_1$.

Every construction therefore is finally represented by

$$l_1R_1 + l_2R_2 + m_1C_1 + m_2C_2 + m_3C_3$$

where $l_1$, $m_1$, etc., are coefficients denoting the number of times any particular operation is performed.

The number $(l_1 + l_2 + m_1 + m_2 + m_3)$ is called the *coefficient of simplicity*, or more shortly, *the simplicity of the construction;* it denotes the total number of operations. The number $l_1 + m_1 + m_2$ is called the *coefficient of exactitude*, or more shortly, the *exactitude of the construction**; it denotes the number of preparatory operations, on which and not on the tracing operations, the exactitude of the construction depends. The number of straight lines drawn is $l_2$; the number of circles $m_3$.

An objection at once presents itself to the reader, as it did to Mr Lemoine. Is it legitimate to suppose the operations $R_1$, $R_2$, $C_1$, $C_2$, $C_3$ identical in value, in order to make up the coefficient of simplicity or exactitude? They are evidently not identical in execution, and hence Geometrography does not furnish us with an *absolute* measure of simplicity or exactitude in the sense in which measure is usually employed, the comparison of one magnitude with a unit of the same kind. The various operations however are assimilated because they are incapable of decomposition into others more simple, and because, speculatively, any one is neither more simple nor less simple than another.

In one case it may be said that Geometrography does furnish an absolute measure, the case namely when all the coefficients in one construction are smaller than the respective coefficients in the other. This case occurs pretty frequently.

* Mr Lemoine remarks that the simplicity and the exactitude of an operation vary inversely as the numbers he sums; but since no confusion is possible, he prefers names recalling the object aimed at to the more logical terms *coefficient of complication* and *coefficient of inexactitude.*

Such is Mr Lemoine's scheme of comparison, which he applies to more than sixty of the principal problems of elementary geometry, with some very unexpected results.

To justify his procedure in denoting by $2R_1$ and $2C_1$ the operations of placing the edge of the ruler and the two points of the compasses in coincidence with two given points, Mr Lemoine says in a note on the problem

*To take with the compasses a given length AB :*

"It is clear that the operation of putting the first point of the compasses on A is not the same as that of keeping the first point on A and placing the second on B; and yet we denote them both by $C_1$. We believe that there is no inconvenience in that, because we are only making an *ideal* theory of operations. Thus we suppose that all the lines of the figure intersect within the limits of the drawing, that it is indifferent whether these lines intersect at a very acute angle, and so on; so that it appears to us quite sufficient to denote by the symbol $C_1$ the general operation which consists in putting *one* of the points of the compasses on *one* point. The reader however who, after reflection, does not share our opinion has only to denote by $C_1'$ the operation which consists in putting on a given point the movable point of the compasses while the other is kept fixed.

"In like manner, since we call $R_1$ the operation which consists in putting the edge of the ruler in contact with a point, it is evident from the manner in which it is performed that the operation which consists in putting the edge of the ruler in coincidence with two given points is not exactly twice the operation $R_1$. One might also denote by $R_1 + R_1'$ the operation which consists in placing the edge of the ruler in contact with two points; but if one practises Geometrography a little I believe he will come to recognise that this distinction is a useless complication.

"We might also have assimilated the operations $C_1$ and $C_2$ and have kept for the two only one symbol $C_1$; but we have not done so, because if *theoretically* $R_1$ and $R_1'$ come to the same thing, $C_1$ and $C_2$ are *theoretically* different. $C_2$ however occurs much more rarely than the other symbols, and in general with a very small coefficient."

I am not sure that I understand in what respect the practical operations $C_1$ and $C_2$ are theoretically different, unless it be that in performing $C_2$ there is one degree more of freedom than in performing $C_1$. But this is so also in the case of the two operations denoted by $2R_1$; for the ruler can be placed in coincidence with one point by a motion either of translation or of rotation or of both, while it can be placed in coincidence with the other point, the coincidence with the first being maintained, only by rotation.

In the case of the two operations denoted by $2C_1$ it is clear also that there is less freedom in placing the second point of the compasses than there is in placing the first, and hence if, for the sake of convenience, two operations which are not precisely identical may be denoted by the reduplication of the same symbol, there does not seem to be any imperative reason why the operations $C_1$ and $C_2$ should not be regarded as equivalent. The fact also that in estimating the simplicity and exactitude of constructions the symbol $C_2$ rarely occurs, and the manifest advantage of having only four units instead of five have induced me to propose the following modification of Mr Lemoine's scheme:

| | |
|---|---|
| To place the edge of the ruler in coincidence with one point ........................................ | $R_1$ |
| To place the edge of the ruler in coincidence with two points........................................ | $2R_1$ |
| To draw a straight line ................................ | $R_2$ |
| To put one point of the compasses on a determinate point ................................................ | $C_1$ |
| To put the points of the compasses on two determinate points ........................................ | $2C_1$ |
| To describe a circle .................................... | $C_2$ |

On another matter (of small importance) I have ventured to differ from Mr Lemoine.

*Given A, B, C, the three vertices of a triangle, to construct the triangle.*

Mr Lemoine estimates this operation as $6R_1 + 3R_2$. I have estimated it as $4R_1 + 3R_2$. To put the ruler in contact with A, B is $2R_1$; to draw AB is $R_2$. Now as the ruler is in contact with B, I estimate the putting of it in contact with B and C as

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only an additional $R_1$; to draw BC is $R_2$. Again as the ruler is in contact with C, I estimate the putting of it in contact with C and A as another $R_1$; to draw CA is $R_2$—in all, $4R_1 + 3R_2$.

The following remarks are extracted from one of Mr Lemoine's letters:

Geometrography may be divided into several branches.

(1) That of the canonical geometry of the straight line and the circle, the only instruments being the ruler and the compasses.

(2) Add the carpenter's square, with two new symbols. This branch may be applied especially to descriptive geometry.

(3) Add graduated rulers, for application to graphical statics.

(4) The geometrography of the ruler alone.

(5) The geometrography of the compasses alone.*

A sub-section may be made of the geometrography of the ruler and one single opening of the compasses.†

In what follows, the notation I. 1, etc., denotes Euclid's *Elements*, Book *First*, Proposition *First*, etc. It will be seen that, except in the fourth book, Euclid does not group his problems together.

I. 1.

*To describe an equilateral triangle on a given finite straight line.*

$$3R_1 + 2R_2 + 3C_1 + 2C_2$$

Simplicity 10; exactitude 6; lines 2; circles 2.

I. 2.

*From a given point to draw a straight line equal to a given straight line.*

$$5R_1 + 3R_2 + 7C_1 + 4C_2$$

The problem may be solved with much less complication, namely,

$$R_1 + R_2 + 3C_1 + C_2$$

I. 3.

*From the greater of two given straight lines to cut off a part equal to the less.*

$$5R_1 + 3R_2 + 9C_1 + 5C_2$$

* See Mascheroni's *Geometria del compasso* (1795).

† See *Proceedings of the Edinburgh Mathematical Society*, V. 2-22 (1887).

The problem may be solved with much less complication, namely,

$$3C_1 + C_2$$

## I. 9.

*To bisect a given rectilineal angle.*

$$2R_1 + R_2 + 4C_1 + 3C_2$$

In this estimate the operations for drawing the sides of the equilateral triangle which occurs in Euclid's construction are omitted. The construction may be effected by

$$2R_1 + R_2 + 3C_1 + 3C_2$$

## I. 10.

*To bisect a given finite straight line*

$$2R_1 + R_2 + 3C_1 + 2C_2.$$

Some of Euclid's operations are not counted, as they are needed only for the demonstration. The construction may be effected by

$$2R_1 + R_2 + 2C_1 + 2C_2$$

## I. 11.

*To draw a straight line perpendicular to a given straight line from a given point in the same.*

$$2R_1 + R_2 + 4C_1 + 3C_2$$

The construction may be effected by

$$2R_1 + R_2 + 3C_1 + 3C_2$$

Or thus:

FIGURE 1.

Let AB be the straight line, C the point in it.

Take any point D outside AB; with D as centre and DC as radius describe a circle cutting AB again at E.

Join ED, and produce it to meet the circle at F; join FC.

$$3R_1 + 2R_2 + C_1 + C_2$$

## I. 12.

*To draw a straight line perpendicular to a given straight line from a given point outside it.*

$$2R_1 + R_2 + 5C_1 + 3C_2$$

From the way in which Euclid describes his construction, the formula for it would be

$$4R_1 + 2R_2 + 5C_1 + 3C_2$$

But if the construction be fully carried out it will be seen that the drawing of the final straight line is unnecessary. Hence the formula is as first stated.

The construction may be effected by

$$2R_1 + R_2 + 3C_1 + 3C_2$$

Or thus:

FIGURE 2.

Let AB be the straight line, C the point outside it.

Take any point D in AB; with D as centre and DC as radius describe a circle cutting AB at E.
With E as centre and EC as radius describe a circle cutting the previous one again at F; join FC.

$$2R_1 + R_2 + 4C_1 + 2C_2$$

## I. 22.

*To make a triangle the sides of which shall be equal to three given straight lines.*

Euclid does not use any of the given straight lines as a side of the triangle.

$$3R_1 + 3R_2 + 9C_1 + 4C_2$$

## I. 23.

*At a given point in a given straight line to make an angle equal to a given angle.*

$$2R_1 + R_2 + 9C_1 + 3C_2$$

The construction may be effected by

$$2R_1 + R_2 + 5C_1 + 3C_2$$

## I. 31.

*Through a given point to draw a straight line parallel to a given straight line.*

$$3R_1 + 2R_2 + 9C_1 + 3C_2$$

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The construction is frequently effected by

$$2R_1 + R_2 + 5C_1 + 3C_2$$

The following method is due to Mr Gaston Tarry.

FIGURE 3.

Let A be the given point, BC the given straight line.

Draw any circle passing through A and cutting BC at D and E. With E as centre and radius AD describe a circle to cut the previous one at F. Join AF.

$$2R_1 + R_2 + 4C_1 + 2C_2$$

I. 42.

*To describe a parallelogram that shall be equal to a given triangle and have one of its angles equal to a given angle.*

Euclid constructs his parallelogram on the half of one of the sides of the triangle.

$$10R_1 + 6R_2 + 30C_1 + 11C_2$$

The construction may be effected by

$$8R_1 + 4R_2 + 15C_1 + 9C_2$$

I. 44.

*To a given straight line to apply a parallelogram which shall be equal to a given triangle and have one of its angles equal to a given angle.*

$$28R_1 + 17R_2 + 81C_1 + 28C_2$$

I. 45.

*To describe a parallelogram equal to a given rectilineal figure and having an angle equal to a given angle.*

Euclid takes a quadrilateral for the given rectilineal figure.

$$40R_1 + 24R_2 + 111C_1 + 39C_2$$

I. 46.

*To describe a square on a given straight line.*

$$10R_1 + 6R_2 + 24C_1 + 10C_2$$

The construction may be effected by

$$6R_1 + 3R_2 + 7C_1 + 5C_2$$

II. 11.

*To divide a given straight line in medial section.*

$$6R_1 + 3R_2 + 12C_1 + 7C_2$$

I have left out several of Euclid's operations, as they are necessary only for the demonstration.

If the given straight line AB be denoted by 2, the greater segment of it will be denoted by $\sqrt{5} - 1$. Hence to obtain the required section of AB, a geometrical construction for $\sqrt{5}$ must be found. This geometrical construction can be found from a right-angled triangle whose sides containing the right angle are 2 and 1 (Euclid's method). It may also be found from a right-angled triangle whose hypotenuse is 3 and one of its sides 2.

The following method (which in substance has been long known) depends upon the second construction for $\sqrt{5}$, and was communicated to me by Mr Lemoine, to whom it had been sent by Mr Bernès. Mr Bernès remarked that he would probably not have discovered it without the aid of Geometrography, or that if he had, he would have attached no special importance to it. And yet it is the simplest of all the solutions yet discovered.

FIGURE 4.

Produce BA, the given straight line.
With centre A and radius AB describe a circle cutting BA produced at C. With centre C and the same radius describe a circle cutting the previous circle in D, D′.
Join DD′, cutting AC in E.
With centre E and radius AB cut DD′ in F. With centre F and radius EB describe a circle cutting BA in G, and BA produced in G′. These are the required points of internal and external section.

$$4R_1 + 2R_2 + 7C_1 + 4C_2$$

For other solutions see the *Proceedings of the Edinburgh Mathematical Society*, IV. 60 (1886), and Mr Lemoine's memoir of 1892, already cited.

II. 14.

*To describe a square equal to a given rectilineal figure.*

Euclid describes a rectangle equal to the given rectilineal figure, which can be done by the extremely complicated construction of I. 45; and then finds the side of a square equal to the rectangle. This latter process he performs by

$$6R_1 + 3R_2 + 7C_1 + 4C_2$$

### III. 1.

*To find the centre of a given circle.*

$$4R_1 + 3R_2 + 6C_1 + 4C_2.$$

The following solution, due to J. H. Swale of Liverpool (1830), is probably the simplest yet discovered.

FIGURE 5.

Take any point P on the given circumference, and with P as centre describe a circle ABC cutting the given circle at A and B. In this circle place the chord BC equal to BP; and join AC cutting the given circumference in D. Then BD or CD is the radius of the given circle.

$$2R_1 + R_2 + 5C_1 + 4C_2$$

### III. 17.

*To draw a tangent to a circle from an external point.*

Euclid begins by finding the centre of the circle. I shall suppose the centre to be given.

$$8R_1 + 4R_2 + 6C_1 + 4C_2$$

To draw the two tangents, there would be required

$$10R_1 + 6R_2 + 6C_1 + 4C_2$$

A common solution is to join the external point to the centre of the circle, and on this line as diameter to describe a circle. This, giving the two tangents, is effected by

$$7R_1 + 4R_2 + 4C_1 + 3C_2$$

If the ruler alone is used, the two tangents can be obtained by

$$14R_1 + 10R_2$$

### III. 25.

*A segment of a circle being given, to describe the circle of which it is the segment.*

$$6R_1 + 3R_2 + 14C_1 + 6C_2$$

The construction can be effected by

$$5R_1 + 3R_2 + 6C_1 + 5C_2$$

III. 30.

*To bisect a given arc of a circle.*

$$4R_1 + 2R_2 + 3C_1 + 2C_2$$

The construction may be effected by

$$2R_1 + R_2 + 2C_1 + 2C_2$$

III. 33.

*On a given straight line to describe a segment of a circle containing an angle equal to a given angle.*

$$6R_1 + 3R_2 + 18C_1 + 9C_2$$

The construction may be effected by

$$4R_1 + 2R_2 + 11C_1 + 6C_2$$

III. 34.

*From a given circle to cut off a segment containing an angle equal to a given angle.*

I shall suppose the centre of the given circle to be known.

$$6R_1 + 3R_2 + 13C_1 + 6C_2$$

The construction may be effected by

$$4R_1 + 2R_2 + 8C_1 + 4C_2$$

IV. 1.

*In a given circle to place a chord of given length.*

$$3R_1 + 2R_2 + 3C_1 + C_2$$

The construction may be effected by

$$2R_1 + R_2 + 3C_1 + C_2$$

IV. 2.

*In a given circle to inscribe a triangle equiangular to a given triangle.*

I shall suppose the centre of the circle to be known.

$$10R_1 + 4R_2 + 22C_1 + 9C_2$$

The construction may be effected by

$$9R_1 + 5R_2 + 10C_1 + 6C_2$$

IV. 3.

*About a given circle to circumscribe a triangle equiangular to a given triangle.*

$$14R_1 + 7R_2 + 30C_1 + 15C_2$$

The construction may be effected by

$$10R_1 + 7R_2 + 12C_1 + 8C_2$$

IV. 4.

*To inscribe a circle in a given triangle.*

$$6R_1 + 3R_2 + 14C_1 + 10C_2$$

The construction may be effected by

$$4R_1 + 2R_2 + 11C_1 + 6C_2$$

IV. 5.

*To circumscribe a circle about a given triangle.*

$$4R_1 + 2R_2 + 8C_1 + 5C_2$$

The construction may be effected by

$$4R_1 + 2R_2 + 5C_1 + 4C_2$$

IV. 6.

*To inscribe a square in a given circle.*

$$8R_1 + 6R_2 + 3C_1 + 2C_2$$

IV. 7.

*To circumscribe a square about a given circle.*

$$11R_1 + 6R_2 + 19C_1 + 14C_2$$

IV. 8.

*To inscribe a circle in a given square.*

$$10R_1 + 6R_2 + 26C_1 + 11C_2$$

### IV. 9.

*To circumscribe a circle about a given square.*

$$4R_1 + 2R_2 + 2C_1 + C_2$$

### IV. 10.

*To describe an isosceles triangle having each of the base angles double of the vertical angle.*

$$9R_1 + 6R_2 + 17C_1 + 9C_2$$

### IV. 11.

*To inscribe a regular pentagon in a given circle.*

$$28R_1 + 16R_2 + 47C_1 + 24C_2$$

The following construction, given in the first book of Ptolemy's *Almagest*, is much simpler than Euclid's.

FIGURE 6.

Draw AB any diameter of the given circle. From the centre C draw CD perpendicular to AB and meeting the circumference at D. Bisect AC at E; and from EB cut off EF equal to ED. Then DF is a side of the inscribed regular pentagon.

$$11R_1 + 8R_2 + 11C_1 + 9C_2$$

### IV. 12.

*To circumscribe a regular pentagon about a given circle.*

$$43R_1 + 22R_2 + 67C_1 + 39C_2$$

### IV. 13.

*To inscribe a circle in a regular pentagon.*

$$6R_1 + 3R_2 + 12C_1 + 8C_2$$

### IV. 14.

*To circumscribe a circle about a regular pentagon.*

$$4R_1 + 2R_2 + 7C_1 + 5C_2$$

### IV. 15.

*To inscribe a regular hexagon in a given circle.*

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Euclid states as a corollary to this problem that the side of the regular hexagon is equal to the radius of the circle. Hence his construction, if he were not concerned with demonstration, would be as simple as possible.

## IV. 16.

*To inscribe in a circle a regular figure of fifteen sides.*

It does not seem worth while to evaluate the simplicity of Euclid's solution of this problem. The solution depends on the inscription of a regular pentagon in the circle, and Euclid's construction for this is more than twice as complicated as it need be.

## V.

The propositions in Euclid's fifth book are all theorems.

## VI. 9.

*From a given straight line to cut off any aliquot ($n^{th}$) part.*

On the supposition that all the points of division are to be marked on the auxiliary line, the compasses being lifted from the paper each time, the result is

$$6R_1 + 4R_2 + (n+9)C_1 + (n+3)C_2$$

## VI. 10.

*To divide a given straight line similarly to a given divided straight line.*

Euclid's given divided straight line consists of three consecutive segments.

$$9R_1 + 6R_2 + 27C_1 + 9C_2$$

## VI. 11.

*To find a third proportional to two given straight lines.*

Euclid's two given straight lines are drawn from the same point, and the third proportional is found on one of them.

$$8R_1 + 5R_2 + 12C_1 + 4C_2$$

## VI. 12.

*To find a fourth proportional to three given straight lines.*

$$5R_1 + 5R_2 + 18C_1 + 6C_2$$

VI. 13.

*To find a mean proportional between two given straight lines.*

Euclid's two given straight lines are placed contiguous to each other and in the same straight line.

$$4R_1 + 2R_2 + 9C_1 + 6C_2$$

If the lengths of the two given straight lines had to be measured off on another straight line, the result would be

$$4R_1 + 3R_2 + 15C_1 + 8C_2$$

The following is the simplest solution yet obtained.

FIGURE 7.

Let M, N be the two given straight lines, M being greater than N.

Draw any straight line AB, and with A as centre and M as radius describe a circle cutting AB in B. With B as centre and N as radius describe a circle cutting BA, between A and B, at C; with C as centre and N as radius describe a circle cutting the second circle in D and E. Join DE and let it cut the first circle at F. BF is the mean proportional.

$$2R_1 + 2R_2 + 7C_1 + 3C_2$$

If BF be drawn, the result is

$$4R_1 + 3R_2 + 7C_1 + 3C_2$$

This solution is practically identical with that communicated in 1684 by Thomas Strode to Dr John Wallis of Oxford. See Wallis's *Treatise of Algebra, Additions and Emendations*, p. 164 (1685), or his *Opera Mathematica*, I. 301 (1695).

It does not seem worth while to consider the remaining problems of the sixth book. Euclid's construction of VI. 18,

*On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure,*

is as simple as possible; his construction of VI. 25,

*To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure,*

depends on I. 45, and is therefore unnecessarily complicated. The problems VI. 28, 29 have now no practical but only a historical interest, and VI. 30 is merely II. 11 over again.