ABSOLUTE REGULARITY OF THE NÖRLUND MEAN

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Let $\{p_n\}$ be any sequence of real or complex numbers subject to the sole restriction

$$P_n = p_0 + p_1 + \cdots + p_n \neq 0 \qquad (n = 0, 1, 2, \cdots).$$

And let

$$t_n = \frac{p_n s_0 + p_{n-1} s_1 + \cdots + p_0 s_n}{P_n}.$$

If $t_n \to s$ as $n \to \infty$, we say that the sequence $\{s_n\}$ is summable Nörlund or summable (N, p) to s.

If $t_n \to s$ whenever $s_n \to s$, we say that (N, p) is regular.

It is known [3] that the necessary and sufficient conditions for the regularity of (N, p) are that, for any fixed k,

$$p_{n-k} = o(P_n) \tag{1}$$

as $n \to \infty$ and that

$$|p_0| + |p_1| + \cdots + |p_n| = 0(P_n). \tag{2}$$

We shall say that the sequence $\{s_n\}$ is absolutely summable Nörlund or summable $|N, p|$ to s if

$$\sum_{n=0}^{\infty} |t_n - t_{n-1}| < \infty, \tag{3}$$

where $t_{-1} = 0$, and

$$t_n \to s \tag{4}$$

as $n \to \infty$.

If (3) and (4) hold whenever $s_n \to s$ and

$$\sum_{n=0}^{\infty} |a_n| < \infty, \tag{5}$$

where $a_0 = s_0$, $a_n = s_n - s_{n-1}$ $(n \geqq 1)$, then we say that (N, p) is absolutely regular.

The aim of this paper is to discuss the relation between regularity and

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absolute regularity of the Nörlund Summability, and for this purpose we require the following theorem.

THEOREM 1. *In order that (N, p) should be absolutely regular it is necessary and sufficient that*

$$\sum_{n=m}^{\infty} \left| \frac{P_{n-m}}{P_n} - \frac{P_{n-m-1}}{P_{n-1}} \right| \leqq H, \tag{6}$$

where H is independent of m and $P_{-1} = 0$ and that (1) *should hold.*

For the proof of this theorem we require the following theorem on general summation matrices (c_{nk}), proved by F. M. Mears [1].

THEOREM. *The necessary and sufficient conditions that $\sum_{n=1}^{\infty} |u_n|$, where $u_n = U_n - U_{n-1}$, $U_n = \sum_{k=1}^{\infty} c_{nk} s_k$, and $s_k = a_0 + a_1 + \cdots + a_k$, should converge whenever $\sum_{n=1}^{\infty} |a_n|$ converges are*

(A) *$\sum_{k=1}^{\infty} c_{nk}$ converges, for all n;*

(B) *$\sum_{n=1}^{\infty} |\sum_{k=m}^{\infty} (c_{nk} - c_{n-1,k})| \leqq H$, for all m where H is a positive constant.*

PROOF OF THEOREM 1 [1]. The conditions are necessary. We suppose (N, p) is absolutely regular and wish to prove (1) and (6) must then hold. Since

$$t_n = \frac{p_n s_0 + p_{n-1} s_1 + \cdots + p_0 s_n}{P_n}$$

$$= \frac{p_n}{P_n} s_0 + \frac{p_{n-1}}{P_n} s_1 + \cdots + \frac{p_0}{P_n} s_n,$$

we have

[1] An alternative proof of this theorem is possible by appeal to a theorem of H. Hahn, Monatshefte für Math. und Phys. 32 (1922), 3—88. This theorem is quoted in Math. Rev. 9 (1948), 579, by R. P. Agnew in a review of a paper on absolute regularity by Z. Schurr. The theorem of Hahn is as follows.

"Necessary and sufficient conditions that $t_n = \sum_{n=0}^{\infty} c_{nk} s_k$ should converge (as $n \to \infty$) whenever s_n converges absolutely are:

(i) $$c_{nk} \to d_k,$$

(ii) $$\sum_{k=1}^{\infty} c_{nk} \to d,$$

(iii) $$\left| \sum_{k=m}^{\infty} c_{nk} \right| < k \quad \text{for all } m \text{ and } n;$$

and then

$$t_n \to ds + \sum_{k=1}^{\infty} d_k (s_k - s),$$

where

$$s = \lim_{n\to\infty} s_n.$$

This theorem was pointed out to me by the referee.

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$$c_{n,k} = \frac{p_{n-k}}{P_n} \qquad (k = 0, \cdots, n),$$

$$c_{n,k} = 0 \qquad (k > n).$$

Since (N, p) is absolutely regular Mears's theorem tells us that

$$\sum_{n=1}^{\infty} \left| \sum_{k=m}^{\infty} (c_{n,k} - c_{n-1,k}) \right|$$

is bounded. Putting in the c_{nk} appropriate to (N, p) we find that the above sum is equal to

$$\sum_{n=m}^{\infty} \left| \frac{P_{n-m}}{P_n} - \frac{P_{n-m-1}}{P_{n-1}} \right|.$$

Hence (6) holds.

Take $s_k = 1$, $s_n = 0$ $(n \neq k)$. Then $\sum_{n=0}^{\infty} a_n$ converges absolutely to 0 and hence, when k is fixed,

$$t_n = \frac{p_{n-k}}{P_n} \to 0$$

as $n \to \infty$. (1) is also necessary.

The conditions are sufficient. There are two things to be proved:

(i) that, if $s_n \to s$ absolutely, then $t_n \to s$,

(ii) that t_n converges absolutely, i.e. $\sum_{n=0}^{\infty} |t_n - t_{n-1}| < \infty$.

Since $\sum_{k=0}^{\infty} c_{nk}$ is a terminating series for each n, it is convergent, and Mears's condition (A) is fulfilled. And since, by the algebra above,

$$\sum_{n=m}^{\infty} \left| \frac{P_{n-m}}{P_n} - \frac{P_{n-m-1}}{P_{n-1}} \right| = \sum_{n=1}^{\infty} \left| \sum_{k=m}^{\infty} (c_{nk} - c_{n-1,k}) \right|,$$

and the left side is bounded by (6), Mears's condition (B) is fulfilled. So by Mears's theorem, $\sum_{n=0}^{\infty} |t_n - t_{n-1}| < \infty$ and (ii) is established.

To prove that (1) and (6) imply (i), suppose first that $\sum_{n=0}^{\infty} a_n$ converges absolutely to 0. Then we can choose k so that

$$|a_k| + |a_{k+1}| + \cdots + |a_n| < \frac{\varepsilon}{3H} \tag{7}$$

and

$$|s_{k-1}| < \frac{\varepsilon}{3H}. \tag{8}$$

Now, by partial summation,

$$\begin{aligned} t_n &= \frac{p_n s_0 + p_{n-1} s_1 + \cdots + p_0 s_n}{P_n} \\ &= \frac{P_n a_0 + P_{n-1} a_1 + \cdots + P_0 a_n}{P_n} \\ &= A_n + B_n, \end{aligned}$$

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where

$$A_n = \frac{P_n a_0 + P_{n-1} a_1 + \cdots + P_{n-k+1} a_{k-1}}{P_n},$$

$$B_n = \frac{P_{n-k} a_k + P_{n-k-1} a_{k+1} + \cdots + P_0 a_n}{P_n}.$$

Since, for all $m \leqq n$,

$$\text{(9)} \qquad \left|\frac{P_{n-m}}{P_n}\right| = \left|\sum_{r=m}^{n}\left(\frac{P_{r-m}}{P_r} - \frac{P_{r-m-1}}{P_{r-1}}\right)\right| \leqq \sum_{r=m}^{\infty}\left|\frac{P_{r-m}}{P_r} - \frac{P_{r-m-1}}{P_{r-1}}\right| \leqq H,$$

we have, by (7),

$$|B_n| \leqq \left|\frac{P_{n-k}}{P_n}\right| |a_k| + \left|\frac{P_{n-k-1}}{P_n}\right| |a_{k+1}| + \cdots + \left|\frac{P_0}{P_n}\right| |a_n| \leqq H(|a_k| + |a_{k+1}| + \cdots + |a_n|) < \frac{\varepsilon}{3}.$$

and, by partial summation,

$$|A_n| = \left|\frac{p_n s_0 + p_{n-1} s_1 + \cdots + p_{n-k+2} s_{k-2} + P_{n-k+1} s_{k-1}}{P_n}\right| \leqq \left|\frac{p_n s_0 + p_{n-1} s_1 + \cdots + p_{n-k+2} s_{k-2}}{P_n}\right| + \left|\frac{P_{n-k+1}}{P_n}\right| |s_{k-1}|.$$

By (8) and (9),

$$\left|\frac{P_{n-k+1}}{P_n}\right| |s_{k-1}| < \frac{\varepsilon}{3}.$$

Since k is a constant, it follows from (1) that

$$\left|\frac{p_n s_0 + p_{n-1} s_1 + \cdots + p_{n-k+2} s_{k-2}}{P_n}\right| < \frac{\varepsilon}{3}$$

for all sufficiently large n. Therefore

$$|A_n| < \frac{2\varepsilon}{3},$$

and hence, for sufficiently large n,

$$|t_n| < \varepsilon.$$

Thus $t_n \to 0$ as $n \to \infty$.

If $\sum_{n=0}^{\infty} a_n$ converges absolutely to s, which is not zero, then $\sum_{n=0}^{\infty} a'_n$, where $a'_0 = a_0 - s$, $a'_n = a_n$ $(n \neq 0)$, converges absolutely to 0 so that

$$t'_n = \frac{p_n s'_0 + p_{n-1} s'_1 + \cdots + p_0 s'_n}{P_n} \to 0.$$

As $n \to \infty$. But, on substituting $s_n - s$ for s'_n,

$$t'_n = t_n - s.$$

Hence $t_n \to s$ as $n \to \infty$.

If we take $p_{2n} = 1$, $p_{2n+1} = 0$, we see that (1) and (2) are satisfied but (6) is not when m is an odd integer. Hence a Nörlund method can be regular without being absolutely regular. The question naturally arises as to whether it is true that an absolutely regular Nörlund method is necessarily regular. I have not been able to solve this problem. I have, however, obtained the following two theorems.

THEOREM 2. *If (N, p) is absolutely regular and P_n is bounded, then (N, p) is regular.*

PROOF. It follows from the absolute regularity of (N, p) that

$$\sum_{n=m}^{\infty} \left| \frac{P_{n-m}}{P_n} - \frac{P_{n-m-1}}{P_{n-1}} \right| \tag{11}$$

is bounded and this implies that p_0/P_m is bounded, so that for all m

$$|P_m| \geqq c > 0, \tag{12}$$

where c is a constant.

Now the sum (11), which is equal to

$$\sum_{n=m}^{\infty} \frac{1}{|P_n P_{n-1}|} |P_{n-1} P_{n-m} - P_{n-m-1} P_n|,$$

is bounded. Since we are restricting ourselves to the case in which P_n is bounded, $|P_n P_{n-1}|$ is also bounded so that the boundedness of (11) implies the boundedness of

$$\begin{aligned}
&\sum_{n=m}^{\infty} |P_{n-1} P_{n-m} - P_{n-m-1} P_n| \\
&\quad = \sum_{n=m}^{\infty} |(P_n - p_n) P_{n-m} - (P_{n-m} - p_{n-m}) P_n| \\
&\quad = \sum_{n=m}^{\infty} |P_n p_{n-m} - p_n P_{n-m}| \\
&\quad = \sum_{n=0}^{\infty} |P_{n+m} p_n - p_{n+m} P_n|.
\end{aligned}$$

Thus we have

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$$\sum_{n=0}^{\infty} |P_{n+m}p_n - p_{n+m}P_n| \leqq k.$$

Hence, for any fixed N and all m,

$$\sum_{n=0}^{N} |P_{n+m}p_n - p_{n+m}P_n| \leqq k,$$

and hence

$$\sum_{n=0}^{N} |P_{n+m}p_n| \leqq k + \sum_{n=0}^{N} |p_{n+m}P_n|. \tag{13}$$

Take N as fixed and make $m \to \infty$, then

$$\sum_{n=0}^{N} |p_{n+m}P_n| \to 0,$$

because, by (1), $p_n = o(P_n) = o(1)$.

Since $|P_{n+m}| \geqq c > 0$, by (12), for all n, m, it follows from (13) that

$$\sum_{n=0}^{N} |p_n| \leqq \frac{k}{c}.$$

Hence

$$\sum_{n=0}^{\infty} |p_n|$$

converges.

Hence, by (12),

$$|p_0|+|p_1|+ \cdots +|p_n| = 0(1) = 0(P_n),$$

and (N, p) is regular.

THEOREM 3. *If* (N, p) *is absolutely regular and* P_n *is not bounded, then* $|P_n| \to \infty$.

PROOF. If $|P_n|$ does not tend to infinity, we can find a positive number G such that $|P_n| < G$ for arbitrarily large values of n. Also by Theorem 1,

$$\sum_{n=m}^{\infty} \left| \frac{P_{n-m}}{P_n} - \frac{P_{n-m-1}}{P_{n-1}} \right| \leqq H \tag{6}$$

for all m. Since P_n is unbounded, there is k such that $|P_k| > HG$. Then there is $N > k$ such that $|P_N| < G$. Let $m = H-k$, a positive integer. Then

$$\left| \frac{P_{N-m}}{P_N} \right| > H.$$

Hence

$$\sum_{n=m}^{\infty}\left|\frac{P_{n-m}}{P_n}-\frac{P_{n-m-1}}{P_{n-1}}\right| \geqq \sum_{n=m}^{N}\left|\frac{P_{n-m}}{P_n}-\frac{P_{n-m-1}}{P_{n-1}}\right|$$
$$\geqq \left|\sum_{n=m}^{N}\left(\frac{P_{n-m}}{P_n}-\frac{P_{n-m-1}}{P_{n-1}}\right)\right|$$
$$\geqq \left|\frac{P_{N-m}}{P_N}\right| > H,$$

which is in contradiction to (6). Therefore $|P_n| \to \infty$.

Using the above two theorems, we see that only the case in which $|P_n| \to \infty$ is left to investigate.

It is worth remarking that it is possible for a Nörlund method to be absolutely conservative without being conservative. This will be shown by an example.

We say that a Nörlund method is conservative if $s_n \to s$ implies $t_n \to t$.

If follows from Theorem 1 of Hardy's book [2] that necessary and sufficient conditions for (N, p) to be conservative are that (2) should hold and that for some δ_m

$$P_{n-m} = [\delta_m + o(1)]P_n \tag{14}$$

as $n \to \infty$.

A Nörland method is said to be absolutely conservative if (3) holds and $t_n \to t$ whenever $s_n \to s$ and (5) holds.

By Mears's Theorem, a necessary and sufficient condition that (N, p) should be absolutely conservative is that (6) should hold.

We note that (6) does not imply (2). For if we take $P_n = e^{ni\theta}$ where θ is any constant not a multiple of 2π, we see that (6) is satisfied but (10) is not. Thus the remark is proved.

Finally I should like to express my thanks to the referee for some useful suggestions and to Dr. B. Kuttner for help in writing this paper.

References

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[2] G. H. Hardy, Divergent Series.
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