REMARKS ON AN ARITHMETIC DERIVATIVE

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1. Introduction. Let $D(n)$ denote a function of an integral variable $n \geq 0$ such that[1]

(1) $D(1) = D(0) = 0$

(2) $D(p) = 1$ for every prime p

(3) $D(n_1 n_2) = n_1 D(n_2) + n_2 D(n_1)$ for every pair of non-negative integers n_1, n_2.

The property (3) is analagous to the product rule for derivatives, and its extension to k terms

(4) $D(n) = n \Sigma_{i=1}^{k} n_i^{-1} D(n_i)$ for $n = n_1 n_2 \ldots n_k$

is immediate. The above properties are consistent and determine $D(n)$ uniquely for all non-negative integers n. In fact, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, we have, on using (4),

(5) $$D(n) = n \Sigma_{i=1}^{r} \alpha_i p_i^{-1}$$

so that, once the prime factor decomposition of n is known, the first derivative $D(n)$ is given explicitly. However, the "higher" derivatives, defined successively by

$$D^0(n) = n, \; D^1(n) = D(n), \; D^2(n) = D[D(n)], \; \ldots, \; D^k(n) = D[D^{k-1}(n)]$$

[1] I have not been able to trace explicit references to previous work on $D(n)$. However, it appeared in a question on the Putnam Prize competition (1950); see American Mathematical Monthly 57 (1950), p. 469. I am indebted to Dr. J. H. H. Chalk for suggesting a note on this topic and for assistance during its preparation.

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present an unsolved problem. For fixed n, the function $D^k(n)$ of k exhibits irregular behaviour as k increases. For example, using (3) with $n = p^p n_1$, where p is a prime, we obtain

$$(6) \qquad D(n) = p^p[n_1 + D(n_1)] \geq n$$

equality holding if and only if $n_1 = 1$. Hence, for integers n possessing a proper divisor of the form p^p, $\lim D^k(n) = \infty$, and if $n = p^p$, $D^k(n) = n$ for all k. On the other hand, $D^k(p) = 0$ for all $k > 1$ and all primes p. Numerical considerations suggest the following.

CONJECTURE. For each $n > 1$, there exists a constant $k_o = k_o(n) \geq 1$ such that, for all $k \geq k_o$,
either

1) $D^k(n) = 0$

or

2) $D^k(n) \neq 0$,

and there exists a prime p such that $D^k(n) \equiv 0 \pmod{p}$.

2. Some remarks about $D(n)$. Although the function $D(n)$ behaves erratically, it is easy to obtain exact upper and lower bounds, depending on n, for its values. We suppose that $n = q_1 q_2 \ldots q_\nu$ has prime factors q_i which are not necessarily distinct.

(a) $D(n) \leq \dfrac{n \log n}{2 \log 2}$ for all n, equality occurring if and only if n is a power of 2. In fact, n satisfies $2^k \leq n < 2^{k+1}$ for some k. Clearly, $\nu \leq k$ and

$$D(n) = n \Sigma_{i=1}^{\nu} \frac{1}{q_i} \leq n \Sigma_{i=1}^{\nu} \tfrac{1}{2} \leq \frac{nk}{2} \leq \frac{n \log n}{2 \log 2}.$$

If $n = 2^k$, $D(n) = k2^{k-1} = \dfrac{2^k \log 2^k}{2 \log 2}$. If $n \neq 2^k$, then some $q_i \neq 2$ and strict inequality holds in the above.

(b) $D(n) \geq \nu n^{1 - \frac{1}{\nu}}$, equality holding if, and only if, all the factors q_i are equal. For, by (5) and the inequality of the arithmetic and geometric means,

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$$D(n) = n\Sigma_{i=1}^{\nu} \frac{1}{q_i} \geq n\nu \frac{1}{(q_1 q_2 \cdots q_\nu)^{\frac{1}{\nu}}} = \nu n^{1-\frac{1}{\nu}}.$$

Hence, if n is not a prime or unity, $D(n) \geq 2\sqrt{n}$, with equality if and only if $n = p^2$ where p is a prime.

In addition, we can relate the value of $D(n)$ to n in the following ways.

(c) Let $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, where $p_1, \ldots, p_r$ are distinct primes. Then $D(n) \equiv 0 \pmod{n}$ if, and only if, $\alpha_1 \equiv 0 \pmod{p_1}, \ldots, \alpha_r \equiv 0 \pmod{p_r}$. In particular, $D(n) = n$ if and only if, $n = p^p$. The sufficiency of the conditions is obvious. Their necessity is seen by noting that, if $n = p^\alpha n'$, where $(p, n') = 1$, then $D(n) = n'\alpha p^{\alpha-1} + p^\alpha D(n') \equiv 0 \pmod{n}$ implies $n'\alpha p^{\alpha-1} = 0 \pmod{p^\alpha}$ and, hence, $\alpha \equiv 0 \pmod{p}$, since $(n', p) = 1$.

(d) If $D(n) \geq n$, then $D(kn) = kD(n) + nD(k) > kn$ for all $k > 1$.

3. <u>The average order of D(n)</u>. Let

$$S(n) = \Sigma_{r=1}^{n} D(r), \quad T(n) = \Sigma_{r=1}^{n} K(r)$$

where $K(n) = n^{-1}D(n)$. Since $K(n)$ is totally additive, i.e. $K(n_1 n_2) = K(n_1) + K(n_2)$ for all integer pairs n_1, n_2, it is easier to estimate $T(n)$ first, and then use partial summation to deduce the average order of $D(n)$. Let

$$j(n,p) = \Sigma_{t=1}^{\infty} \left[\frac{n}{p^t}\right], \quad \alpha(n) = \left[\frac{\log n}{\log 2}\right];$$

then $j(n,p)$ denotes [1; p. 342] the exponent of the highest power of p dividing $n!$ and $\alpha(n)$ denotes the exponent of the highest power of $2 \leq n$. Observe that

$$T(n) = K(n!) = \Sigma_{p \leq n} \frac{1}{p} j(n,p)$$
$$= \Sigma_{p \leq n} \frac{1}{p} \left(\Sigma_{t=1}^{\infty} \left[\frac{n}{p^t}\right]\right)$$

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$$= \Sigma_{p \leq n} \frac{1}{p} (\Sigma_{t=1}^{\alpha(n)} [\frac{n}{p^t}])$$

$$= \Sigma_{p \leq n} \frac{1}{p} \{\Sigma_{t=1}^{\alpha(n)} \frac{n}{p^t} + O(\log n)\}$$

$$= \Sigma_{p \leq n} \{\Sigma_{t=2}^{\infty} \frac{n}{p^t} - \Sigma_{\alpha(n)+2}^{\infty} \frac{n}{p^t}\} + O\{(\log n) \Sigma_{p \leq n} \frac{1}{p}\}$$

$$= n\Sigma_{p \leq n} \frac{1}{p(p-1)} - \Sigma_{p \leq n} \frac{1}{p^{\alpha(n)+1}(p-1)} + O\{(\log n)\Sigma_{p \leq n} \frac{1}{p}\}$$

$$= n\Sigma_{p=2}^{\infty} \frac{1}{p(p-1)} - \Sigma_{p > n} \frac{n}{p(p-1)} - \Sigma_{p \leq n} \frac{n}{p^{\alpha(n)+1}(p-1)}$$

$$+ O\{(\log n) \Sigma_{p \leq n} \frac{1}{p}\}$$

$$= T_o n + O\{(\log n)(\log \log n)\}$$

where $T_o = \Sigma_{p=2}^{\infty} \frac{1}{p(p-1)} = 0.749\ldots$

since

$$\Sigma_{p > n} \frac{n}{p(p-1)} < n\Sigma_{k > n} \frac{1}{k(k-1)} \leq 1,$$

$$p^{\alpha+1} > p^{\frac{\log n}{\log 2}} \geq 2^{\frac{\log n}{\log 2}} \geq n,$$

$$\Sigma_{p \leq n} \{\frac{1}{p-1} - \frac{1}{p}\} \leq 1,$$

$$\Sigma_{p \leq n} \frac{1}{p} = O(\log \log n). \qquad [1; p. 351]$$

For $S(n)$, we have

$$S(n) = \Sigma_{r=1}^{n} rK(r) = T(n) + \Sigma_{r=1}^{n-1} \{T(n) - T(r)\}$$

$$= nT(n) - \Sigma_{r=1}^{n-1} T(r)$$

$$= n\{T_o n + O(n^\delta)\} - T_o \Sigma_{r=1}^{n-1} r + O(n^{1+\delta})$$

$$= T_o n^2 - T_o \frac{n(n-1)}{2} + O(n^{1+\delta})$$

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$$= \frac{1}{2} T_o n^2 + O(n^{1+\delta})$$

where[2] $\frac{1}{2} T_o = 0.374\ldots$, for each fixed $\delta > 0$.

4. <u>The congruence $D(n) \equiv 0 \pmod 4$</u>. A key problem is to find a characterization of those numbers for which $\lim_{k \to \infty} D^k(n) = \infty$. This limit is known for numbers n of the form p, p^p, kp^p where p is any prime. Further investigation is hampered by the absence of explicit formulae for the higher derivatives. If there were some way of dealing with $D(m + n)$ for any integers m and n, then $D^2(n)$ could be determined from $D(n) = \Sigma_{i=1}^{k} F_i$, where $n = \prod_{i=1}^{k} f_i$, $F_i = n/f_i$, f_i prime. However, it is known only that, if $D(m + n) = D(m) + D(n)$, then $D(km + kn) = D(km) + D(kn)$ for every integer k; in particular, $D(h) + D(2h) = D(3h)$.

Another approach to the problem is to find a characterization of those numbers, excluding p, p^p, kp^p for which $p^p | D^k(n)$ for some positive integer k and some prime p. According to our conjecture, this would be sufficient to characterize those numbers for which $D^k(n) \to \infty$ as $k \to \infty$, provided $D^k(n) \neq 0$ for all k. We deal with the special case $p = 2$, $k = 1$.

Let $n = 2^\alpha p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ where $p_i \equiv 1 \pmod 4$, $q_j \equiv -1 \pmod 4$ are primes, not necessarily distinct. We have the following results:

(i) if $\alpha = 0$, then $D(n) \equiv (-1)^s (r - s) \pmod{2^2}$

(ii) if $\alpha = 1$, then $D(n) \equiv (-1)^s [1 + 2(r - s)] \equiv (-1)^{r-1} \pmod{2^2}$

(iii) if $\alpha > 1$, then $D(n) \equiv 0 \pmod{2^2}$.

In order to prove (i), let $P = p_1 p_2 \cdots p_r \equiv (+1) \pmod 4$

$$Q = q_1 q_2 \cdots q_s \equiv (-1)^s \pmod 4$$

$$P_i = \frac{P}{p_i} \equiv 1 \pmod 4$$

$$Q_i = \frac{Q}{q_i} \equiv (-1)^{s-1} \pmod 4.$$

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[2] The approximation $0.374\ldots n^2$ for $S(n)$ is good, even for small values of n. For example, $S(10) = 38 \doteqdot (0.374\ldots)(100)$.

Then

$$D(n) = D(PQ) = \Sigma_{i=1}^{r} P_i Q + \Sigma_{i=1}^{s} PQ_i \equiv r(-1)^s + s(-1)^{s-1}$$

$$\equiv (-1)^s (r - s) \pmod 4.$$

In case (ii),

$$D(2PQ) = PQD(2) + 2D(PQ)$$

$$\equiv (-1)^s + 2(-1)^s (r - s)$$

$$\equiv (-1)^s [1 + 2(r - s)] \pmod 4.$$

Result (iii) follows from the fact that $4 | n$. We conclude that $D(n) \equiv 0 \pmod 4$ if and only if

(a) $\alpha = 0$, $r \equiv s \pmod 4$

(b) $\alpha > 1$.

The numbers in (a) have a density of $\frac{1}{8}$ in the integers; those in (b) have a density of $\frac{1}{4}$. Hence, those integers n satisfying $\lim_{k \to \infty} D^k(n) = \infty$ (which include the numbers of (a) and (b)) have a density exceeding $\frac{3}{8}$. What this density is remains an open question.

REFERENCE

1. G. H. Hardy and E. M. Wright, Introduction to the Theory of Numbers, 4th edition, (Oxford, 1960).

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