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C-TOTALLY REAL SUBMANIFOLDS IN (κ, μ)-CONTACT SPACE FORMS

MUKUT MANI TRIPATHI AND JEONG-SIK KIM

We obtain a basic B.-Y. Chen's inequality for a C-totally real submanifold in a (κ, μ)-contact space form involving intrinsic invariants, namely the scalar curvature and the sectional curvatures of the submanifold on left hand side and the main extrinsic invariant, namely the squared mean curvature on the right hand side. Inequalities between the squared mean curvature and Ricci curvature and between the squared mean curvature and k-Ricci curvature are also obtained. These results are applied to get corresponding results for C-totally real submanifolds in a Sasakian space form.

1. INTRODUCTION

According to the well-known Nash immersion theorem, every n-dimensional Riemannian manifold admits an isometric immersion into the Euclidean space $\mathbb{E}^{n(n+1)(3n+11)/2}$. Thus, Nash's theorem enables us to consider any Riemannian manifold as a submanifold of Euclidean space; and this provides a natural motivation for the study of submanifolds of Riemannian manifolds. To find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold is one of the basic interests in the submanifold theory. The Gauss—Bonnet Theorem, isoperimetric inequality, and Chern—Lashof Theorem provide relations between extrinsic and intrinsic invariants for a submanifold in a Euclidean space.

In [**5**], B.-Y. Chen established a sharp inequality for a submanifold in a real space form involving intrinsic invariants, namely the sectional curvatures and the scalar curvature of the submanifold; and the main extrinsic invariant, namely the squared mean curvature. In [**7**], he gave a sharp relationship between the squared mean curvature and the Ricci curvature for the submanifolds in a real space form. He also studied the basic inequalities of submanifolds of complex space forms ([**6**]). A basic B.-Y. Chen's inequality for C-totally real submanifolds in a Sasakian space form $\widetilde{M}(c)$ is given in [**9**].

On the other hand, the roots of contact geometry go back to 1872, when Sophus Lie introduced the notion of contact transformation (Berührungstransformation) as a

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geometric tool to study systems of differential equations (for more details see [**1**, **3**, **10**]). This subject has manifold connections with other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optics, thermodynamics and control theory. The two large classes of examples of contact manifolds are the principal circle bundles of the Boothby—Wang fibration (including the Hopf-fibration of the odd dimensional sphere over complex projective space) and the tangent sphere bundles (for details see [**3**]). Contact manifolds include the class of Sasakian manifolds. In [**4**], the authors studied (κ, μ)-contact metric manifolds for which the characteristic vector field belongs to the (κ, μ)-nullity distribution. Characteristic examples of non-Sasakian, (κ, μ)-contact metric manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one. A (κ, μ)-contact metric manifold with constant φ-sectional curvature is called a (κ, μ)-contact space form, which includes the class of Sasakian space forms.

Thus it is worthwhile to study relationships between intrinsic and extrinsic invariants of submanifolds in a (κ, μ)-contact space form. In this paper, we establish several such relationships for C-totally real submanifolds in a (κ, μ)-contact space form. The paper is organised as follows. Necessary details about (κ, μ)-contact space forms are reviewed in Section 2. In Section 3, we recall some Riemannian invariants. The definition of C-totally real submanifolds in a (κ, μ)-contact space form appears in Section 4 along with some required formulas. In Section 5, we establish a basic B.-Y. Chen's inequality for C-totally real submanifold M in a (κ, μ)-contact space form involving intrinsic invariants, namely the scalar curvature and the sectional curvatures of M; and the main extrinsic invariant, namely the squared mean curvature. For a C-totally real submanifold in a (κ, μ)-contact space form an inequality between the squared mean curvature and Ricci curvature is proved in Section 6, while Section 7 contains an inequality between the squared mean curvature and k-Ricci curvature. In the last section, we apply these results to get corresponding results for C-totally real submanifolds in a Sasakian space form.

2. (κ, μ)-CONTACT SPACE FORMS

A differentiable 1-form η on a $(2m+1)$-dimensional differentiable manifold $\widetilde{M}$ is called a *contact form* if $\eta \wedge (d\eta)^m \neq 0$ everywhere on $\widetilde{M}$, and $\widetilde{M}$ equipped with a contact form is a *contact manifold*. Since the rank of $d\eta$ is $2m$, there exists a unique global vector field ξ, called the *characteristic vector field*, such that

$$(1) \qquad \eta(\xi) = 1, \qquad \mathfrak{L}_\xi \eta = 0,$$

where $\mathfrak{L}_\xi$ denotes the Lie differentiation by ξ. Moreover, it is well-known that there exists a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$(2) \qquad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi),$$

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$$\varphi^2 = -I + \eta \otimes \xi, \quad d\eta(X,Y) = g(X, \varphi Y), \tag{3}$$

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y) \tag{4}$$

for $X, Y \in T\widetilde{M}$. The structure (η, ξ, φ, g) is called a *contact metric structure* and the manifold $\widetilde{M}$ endowed with such a structure is said to be a *contact metric manifold*.

The contact metric structure (η, ξ, φ, g) on $\widetilde{M}$ gives rise to a natural almost Hermitian structure on the product manifold $\widetilde{M} \times \mathbb{R}$. If this structure is integrable, then $\widetilde{M}$ is said to be a *Sasakian manifold*. A Sasakian manifold is characterised by the condition

$$(\widetilde{\nabla}_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \quad X, Y \in T\widetilde{M}, \tag{5}$$

where $\widetilde{\nabla}$ is Levi—Civita connection. Also, a contact metric manifold $\widetilde{M}$ is Sasakian if and only if the curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \quad X, Y \in T\widetilde{M}. \tag{6}$$

In a contact metric manifold $\widetilde{M}$, the $(1,1)$-tensor field h defined by $2h = \mathfrak{L}_\xi \varphi$ is symmetric and satisfies

$$h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \widetilde{\nabla}_X \xi = -\varphi X - \varphi h X, \quad \operatorname{trace} h = \operatorname{trace}(\varphi h) = 0. \tag{7}$$

The (κ, μ)*-nullity distribution* of a contact metric manifold $\widetilde{M}$ is a distribution

$$\begin{aligned} N(\kappa,\mu) : p \to N_p(\kappa,\mu) = \Big\{ Z \in T_pM \mid \widetilde{R}(X,Y)Z = \kappa\big(g(Y,Z)X - g(X,Z)Y\big) \\ + \mu\big(g(Y,Z)hX - g(X,Z)hY\big)\Big\}, \end{aligned}$$

where $\kappa, \mu \in \mathbb{R}$ and $\kappa \leqslant 1$. If the characteristic vector field ξ belongs to the (κ,μ)-nullity distribution, the contact metric manifold is called a (κ,μ)*-contact metric manifold*. For a (κ,μ)*-contact metric manifold* one also has $h^2 = (\kappa - 1)\varphi^2$. Thus, the class of (κ,μ)-contact metric manifolds contains the class of Sasakian manifolds, which we obtain for $\kappa = 1$. Characteristic examples of non-Sasakian, (κ,μ)-contact metric manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one. For $\kappa < 1$, the curvature is completely determined for (κ,μ)-contact metric manifolds; in particular, they have constant scalar curvature. Three dimensional (κ,μ)-contact metric manifolds are either Sasakian or locally isometric to one of the following Lie groups: $SO(3)$, $SL(2,R)$, $E(2)$, $E(1,1)$ with a left invariant metric. For more details see [**4**] and [**11**].

The sectional curvature $\widetilde{K}(X, \varphi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a φ*-sectional curvature*. If the (κ,μ)-contact metric manifold $\widetilde{M}$ has constant φ-sectional curvature c then it is called a (κ,μ)*-contact space form* and is denoted by $\widetilde{M}(c)$. The curvature tensor of $\widetilde{M}(c)$ is given by ([**11**])

$$\widetilde{R}(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}$$

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$$+\frac{c-1}{4}\{2g(X,\varphi Y)\varphi Z+g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X\}$$
$$+\frac{(c+3-4\kappa)}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}$$
$$+\frac{1}{2}\{g(hY,Z)hX-g(hX,Z)hY+g(\varphi hX,Z)\varphi hY-g(\varphi hY,Z)\varphi hX\}$$
$$+g(\varphi Y,\varphi Z)hX-g(\varphi X,\varphi Z)hY+g(hX,Z)\varphi^2Y-g(hY,Z)\varphi^2X$$
$$+\mu\{\eta(Y)\eta(Z)hX-\eta(X)\eta(Z)hY+g(hY,Z)\eta(X)\xi-g(hX,Z)\eta(Y)\xi\}. \tag{8}$$

Moreover, if $\kappa<1$, then $\mu=\kappa+1$ and $c=-2\kappa-1$.

3. Riemannian invariants

The Riemannian invariants are the intrinsic characteristics of a Riemannian manifold. In this section we recall a number of Riemannian invariants in a Riemannian manifold ([8]).

Let M be a Riemannian manifold. We denote by $K(\pi)$ the sectional curvature of M for a plane section π in T_pM, $p\in M$. Then, the scalar curvature τ at p is defined by

$$\tau(p)=\sum_{i<j}K_{ij} \tag{9}$$

where $\{e_1,\dots,e_n\}$ is an orthonormal basis for T_pM and K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $p\in M$. By using

$$(\inf K)(p)=\inf\{K(\pi)|\pi \text{ is a plane section } \subset T_pM\},$$

we introduce the Chen invariant

$$\delta_M(p)=\tau(p)-(\inf K)(p),$$

which is certainly an intrinsic character of M.

Let L be a k-plane section of T_pM and U a unit vector in L. We choose an orthonormal basis $\{e_1,\dots,e_k\}$ of L such that $e_1=U$. The Ricci curvature Ric_L of L at U is given by

$$\underset{L}{\mathrm{Ric}}(U)=K_{12}+K_{13}+\cdots+K_{1k}. \tag{10}$$

$\mathrm{Ric}_L(U)$ is called a *k-Ricci curvature*.

The scalar curvature τ of the k-plane section L is given by

$$\tau(L)=\sum_{1\leqslant i<j\leqslant k}K_{ij}. \tag{11}$$

The scalar curvature $\tau(p)$ of M at p is nothing but the scalar curvature of the tangent space of M at p. And if L is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature K of L.

For each integer k, $2 \leqslant k \leqslant n$, the Riemannian invariant θ_k on an n-dimensional Riemannian manifold M is defined by

$$(12) \qquad \theta_k(p) = \left(\frac{1}{k-1}\right) \inf_{L,X} \operatorname{Ric}_L(X), \quad p \in M,$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

4. C-TOTALLY REAL SUBMANIFOLDS

Let M be an n-dimensional submanifold in a manifold $\widetilde{M}$ equipped with a Riemannian metric g. The Gauss and Weingarten formulae are given respectively by $\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ and $\widetilde{\nabla}_X N = -A_N X + \nabla^\perp_X N$ for all $X, Y \in TM$ and $N \in T^\perp M$, where $\widetilde{\nabla}$, ∇ and $\nabla^\perp$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^\perp M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $g\big(\sigma(X,Y), N\big) = g(A_N X, Y)$. The equation of Gauss is given by

$$(13) \quad \widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - g\big(\sigma(X,W),\sigma(Y,Z)\big) + g\big(\sigma(X,Z),\sigma(Y,W)\big)$$

for all $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M respectively. The *relative null space* of M at a point $p \in M$ is defined by

$$\mathcal{N}_p = \big\{X \in T_pM \,|\, \sigma(X,Y) = 0 \text{ for all } Y \in T_pM\big\}.$$

For any orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space T_pM, the mean curvature vector $H(p)$ is given by

$$(14) \qquad H(p) = \frac{1}{n}\sum_{i=1}^{n} \sigma(e_i, e_i).$$

The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X,Y) = g(X,Y)H$ for all $X, Y \in TM$, then M is *totally umbilical.*

A submanifold M in a contact manifold is called a *C-totally real submanifold* ([**13**]) if every tangent vector of M belongs to the contact distribution. Thus, a submanifold M in a contact metric manifold is a *C-totally real submanifold* if ξ is normal to M. A submanifold M in an almost contact metric manifold is called *anti-invariant* ([**14**]) if $\varphi(TM) \subset T^\perp M$. If a submanifold M in a contact metric manifold is normal to the structure vector field ξ, then it is anti-invariant. The simplest possible proof of this fact is given in [**12**]. Thus C-totally real submanifolds in a contact metric manifold are anti-invariant, as they are normal to ξ.

For a C-totally real submanifold in a contact metric manifold we have

$$g(A_\xi X, Y) = g(-\widetilde{\nabla}_X \xi, Y) = g(\varphi X + \varphi h X, Y),$$

which implies that

$$A_\xi = (\varphi h)^T, \tag{15}$$

where $(\varphi h)^T X$ is the tangential part of $\varphi h X$ for all $X \in TM$.

Let M be an n-dimensional C-totally real submanifold in a (κ, μ)-contact space form $\widetilde{M}(c)$, such that $\xi \in T^\perp M$. Then, in view of (8) and the Gauss equation (13), we get

$$\begin{aligned} R(X,Y,Z,W) = {}& \frac{c+3}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ & + \frac{1}{2}\Big\{g(h^TY,Z)g(h^TX,W) - g(h^TX,Z)g(h^TY,W) \\ & \qquad + g\big((\varphi h)^TX,Z\big)g\big((\varphi h)^TY,W\big) - g\big((\varphi h)^TY,Z\big)g\big((\varphi h)^TX,W\big)\Big\} \\ & - g(X,Z)g(h^TY,W) + g(Y,Z)g(h^TX,W) \\ & - g(h^TX,Z)g(Y,W) + g(h^TY,Z)g(X,W) \\ & + g\big(\sigma(X,W),\sigma(Y,Z)\big) - g\big(\sigma(X,Z),\sigma(Y,W)\big). \end{aligned} \tag{16}$$

From the above equation, it follows that the scalar curvature and the mean curvature of M satisfy

$$\begin{aligned} 2\tau = {}& n^2\|H\|^2 - \|\sigma\|^2 + \frac{1}{4}n(n-1)(c+3) + 2(n-1)\operatorname{trace}(h^T) \\ & + \frac{1}{2}\Big\{\big(\operatorname{trace}(h^T)\big)^2 - \|h^T\|^2 - \Big(\operatorname{trace}\big((\varphi h)^T\big)\Big)^2 + \|(\varphi h)^T\|^2\Big\}, \end{aligned} \tag{17}$$

where $c = -2\kappa - 1$ if $\kappa < 1$ and $h^T X$ is the tangential part of hX for all $X \in TM$. This formula will play crucial role in establishing several inequalities for C-totally real submanifolds in a (κ, μ)-contact space form.

We also recall the following algebraic Lemma, which will be used later.

Lemma 4.1. ([5]) *If $a_1, \dots, a_n, a_{n+1}$ are $n+1$ $(n > 1)$ real numbers such that*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + a_{n+1}\Big),$$

then $2a_1a_2 \geqslant a_{n+1}$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

5. B.-Y. Chen's inequality

In [5], B.-Y. Chen established a sharp inequality for submanifolds M in a real space form involving intrinsic invariants, namely the sectional curvature and the scalar curvature of M; and the main extrinsic invariant, namely the squared mean curvature as follows.

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THEOREM 5.1. ([5]) *Let M be an n-dimensional ($n \geqslant 3$) submanifold in a real space form $R^m(c)$. Then*

$$\tau - \inf K(\pi) \leqslant \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)c. \tag{18}$$

Equality holds if and only if, with respect to suitable orthonormal frame fields $e_1, \dots, e_n$, $e_{n+1}, \dots, e_m$, the shape operators $A_r = A_{e_r}$, $r = n+1, \dots, m$ take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)I_{n-2} \end{pmatrix}, \tag{19}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad r \in \{n+2, \dots, m\}. \tag{20}$$

He also established similar inequality in [**6**, Theorem 2] for a submanifold in a complex space form. Now, we prove the following contact version of [**6**, Theorem 2] for C-totally real submanifolds in a (κ, μ)-contact space form.

THEOREM 5.2. *Let M be an n-dimensional ($n \geqslant 3$) C-totally real submanifold in a $(2m+1)$-dimensional (κ, μ)-contact space form $\widetilde{M}(c)$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have*

$$\begin{aligned} \tau - K(\pi) \leqslant{} & \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}(n+1)(n-2)(c+3) \\ & - \frac{1}{2}\Big\{2\,\mathrm{trace}(h|_\pi) + \det(h|_\pi) - \det\big((\varphi h)|_\pi\big)\Big\} + (n-1)\mathrm{trace}(h^T) \\ & + \frac{1}{4}\Big\{\big(\mathrm{trace}(h^T)\big)^2 - \|h^T\|^2 - \big(\mathrm{trace}\big((\varphi h)^T\big)\big)^2 + \|(\varphi h)^T\|^2\Big\}. \end{aligned} \tag{21}$$

The equality in (21) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_{2m}, \xi\}$ of $T_p^\perp M$ such that $\pi = \mathrm{Span}\{e_1, e_2\}$ and the shape operators $A_r = A_{e_r}$, $r = n+1, \dots, 2m+1$ take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)I_{n-2} \end{pmatrix}, \tag{22}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad r \in \{n+2, \dots, 2m+1\}. \tag{23}$$

PROOF: Let

$$\begin{aligned}(24)\quad \rho = 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - \frac{1}{4}n(n-1)(c+3) - 2(n-1)\,\mathrm{trace}(h^T)\\ - \frac{1}{2}\Big\{(\mathrm{trace}\, h^T)^2 - \|h^T\|^2 - \big(\mathrm{trace}(\varphi h)^T\big)^2 + \|(\varphi h)^T\|^2\Big\}.\end{aligned}$$

From (17) and (24), we get

$$n^2\|H\|^2 = (n-1)\big(\|\sigma\|^2 + \rho\big). \tag{25}$$

Let $\pi \subset T_pM$ be a plane section. We choose an orthonormal basis $\{e_1, \dots, e_n\}$ for T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_{2m}, e_{2m+1} = \xi\}$ for the normal space $T_p^\perp M$ at p such that $\pi = \mathrm{Span}\{e_1, e_2\}$ and the mean curvature vector $H(p)$ is parallel to e_{n+1}, then the equation (25) can be written as

$$\Big(\sum_{i=1}^{n}\sigma_{ii}^{n+1}\Big)^2 = (n-1)\Big(\sum_{i=1}^{n}(\sigma_{ii}^{n+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^{r})^2 + \rho\Big). \tag{26}$$

Applying Lemma 4.1, from (26) we obtain

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1} \geqslant \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^{r})^2 + \rho. \tag{27}$$

From equation (16), we also have

$$\begin{aligned}(28)\quad K(\pi) = \frac{c+3}{4} + \frac{1}{2}\Big\{2\,\mathrm{trace}(h|_\pi) + \det(h|_\pi) - \det\big((\varphi h)|_\pi\big)\Big\}\\ + \sigma_{11}^{n+1}\sigma_{22}^{n+1} - (\sigma_{12}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\big(\sigma_{11}^{r}\sigma_{22}^{r} - (\sigma_{12}^{r})^2\big),\end{aligned}$$

which in view of (27) gives

$$\begin{aligned}K(\pi) \geqslant \frac{c+3}{4} + \frac{1}{2}\Big\{2\,\mathrm{trace}(h|_\pi) + \det(h|_\pi) - \det\big((\varphi h)|_\pi\big)\Big\} + \frac{1}{2}\rho\\ + \sum_{r=n+1}^{2m+1}\sum_{j>2}\{(\sigma_{1j}^{r})^2 + (\sigma_{2j}^{r})^2\} + \frac{1}{2}\sum_{i\neq j>2}(\sigma_{ij}^{n+1})^2\\ + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}(\sigma_{ij}^{r})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}(\sigma_{11}^{r} + \sigma_{22}^{r})^2,\end{aligned} \tag{29}$$

or

$$K(\pi) \geqslant \frac{c+3}{4} + \frac{1}{2}\Big\{2\,\mathrm{trace}(h|_\pi) + \det(h|_\pi) - \det\big((\varphi h)|_\pi\big)\Big\} + \frac{1}{2}\rho. \tag{30}$$

In view of (24) and (30), we obtain (21).

If the equality in (21) holds, then the inequalities given by (27) and (29) become equalities. In this case, we have

$$\begin{aligned} &\sigma_{1j}^{n+1}=0,\ \sigma_{2j}^{n+1}=0,\ \sigma_{ij}^{n+1}=0, \quad i\neq j>2;\\ &\sigma_{1j}^{r}=\sigma_{2j}^{r}=\sigma_{ij}^{r}=0,\ r=n+2,\dots,2m+1;\quad i,j=3,\dots,n;\\ &\sigma_{11}^{n+2}+\sigma_{22}^{n+2}=\cdots=\sigma_{11}^{2m+1}+\sigma_{22}^{2m+1}=0. \end{aligned} \tag{31}$$

Furthermore, we may choose e_1 and e_2 so that $\sigma_{12}^{n+1}=0$. Moreover, by applying Lemma 4.1, we also have

$$\sigma_{11}^{n+1}+\sigma_{22}^{n+1}=\sigma_{33}^{n+1}=\cdots=\sigma_{nn}^{n+1}. \tag{32}$$

Thus, after choosing a suitable orthonormal basis, the shape operator of M becomes of the form given by (22) and (23). The converse is straightforward. ∎

REMARK 5.3. The above theorem is different from [**2**, Theorem 4.2], which seems not to be true. In fact, the assumption of statement of [**2**, Theorem 4.2(i)] is at least not true, because the submanifold M in a contact metric manifold normal to the structure vector field ξ is always anti-invariant (that is, $P=0$), as $X,Y\in TM$ implies that $[X,Y]\in TM$ and therefore $g(X,\varphi Y)=d\eta(X,Y)=0$ ([**12**]). Thus, in this case we can not find an invariant plane section in the submanifold.

6. Squared mean curvature and Ricci curvature

In [**7**], B.-Y. Chen established a sharp relationship between the squared mean curvature and the Ricci curvature for the submanifolds in a real space form. In this section, we prove similar result for C-totally real submanifolds in a (κ,μ)-contact space form as follows.

THEOREM 6.1. *Let M be an n-dimensional C-totally real submanifold in a $(2m+1)$-dimensional $(\kappa,\mu F)$-contact space form $\widetilde{M}(c)$. Then*

(i) *For each unit vector $U\in T_pM$, we have*

$$\begin{aligned} \mathrm{Ric}(U)\leqslant{}& \frac{1}{4}\{n^2\|H\|^2+(n-1)(c+3)\}+\mathrm{trace}(h^T)+(n-2)g(h^TU,U)\\ &+\frac{1}{2}\Big\{\mathrm{trace}(h^T)g(h^TU,U)-\|h^TU\|^2\\ &\qquad -\mathrm{trace}\big((\varphi h)^T\big)g\big((\varphi h)^TU,U\big)+\|(\varphi h)^TU\|^2\Big\} \end{aligned} \tag{33}$$

(ii) *For $H(p)=0$, a unit tangent vector $U\in T_pM$ satisfies theequalitycaseof (33) if and only if U belongs to the relative null space $\mathcal{N}_p$.*

(iii) *The equality in (33) holds identically for all unit tangent vectors at pif and only if either p is a totally geodesic point or $n=2$ and p is a totally umbilical point.*

PROOF: Let $U \in T_pM$ be a unit tangent vector. We choose an orthonormal basis $e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}$ such that $e_1, \dots, e_n$ are tangential to M at p with $e_1 = U$. Then, the squared second fundamental form and the squared mean curvature satisfy the following relation

$$(34)\quad \|\sigma\|^2 = \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{2m+1}(\sigma_{11}^r - \sigma_{22}^r - \dots - \sigma_{nn}^r)^2 + 2\sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(\sigma_{1j}^r)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\leqslant i<j\leqslant n}\left(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2\right).$$

From (17) and (34) we get

$$\begin{aligned}\frac{1}{4}n^2\|H\|^2 = \tau - \frac{1}{8}n(n-1)(c+3) - (n-1)\operatorname{trace}(h^T) \\ -\frac{1}{4}\left\{\left(\operatorname{trace}(h^T)\right)^2 - \left\|h^T\right\|^2 - \left(\operatorname{trace}\left((\varphi h)^T\right)\right)^2 + \left\|(\varphi h)^T\right\|^2\right\} \\ +\frac{1}{4}\sum_{r=n+1}^{2m+1}(\sigma_{11}^r - \sigma_{22}^r - \dots - \sigma_{nn}^r)^2 + \sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(\sigma_{1j}^r)^2 \\ -\sum_{r=n+1}^{2m+1}\sum_{2\leqslant i<j\leqslant n}\left(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2\right).\end{aligned}\tag{35}$$

From (16), we also have

$$\begin{aligned}K_{ij} = \sum_{r=n+1}^{2m+1}\left(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2\right) + \frac{c+3}{4} + g(h^Te_i, e_i) + g(h^Te_j, e_j) \\ +\frac{1}{2}\Big\{g(h^Te_i, e_i)g(h^Te_j, e_j) - g(h^Te_i, e_j)^2 \\ -g\left((\varphi h)^Te_i, e_i\right)g\left((\varphi h)^Te_j, e_j\right) + g\left((\varphi h)^Te_i, e_j\right)^2\Big\},\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{2\leqslant i<j\leqslant n}K_{ij} = \sum_{r=n+1}^{m}\sum_{2\leqslant i<j\leqslant n}\left(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2\right) + \frac{1}{8}(n-1)(n-2)(c+3) \\ +\frac{1}{4}\Big\{\left(\operatorname{trace}(h^T)\right)^2 - 2\operatorname{trace}(h^T)g(h^Te_1, e_1) - \left\|h^T\right\|^2 + 2\left\|h^Te_1\right\|^2 \\ -\left(\operatorname{trace}\left((\varphi h)^T\right)\right)^2 + 2\operatorname{trace}\left((\varphi h)^T\right)g\left((\varphi h)^Te_1, e_1\right) + \left\|(\varphi h)^T\right\|^2 \\ -2\left\|(\varphi h)^Te_1\right\|^2\Big\} + (n-2)\left(\operatorname{trace}(h^T) - g(h^Te_1, e_1)\right)\end{aligned}\tag{36}$$

From (35) and (36), we get

$$\operatorname{Ric}(U) = \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)(c+3)\right\} + \operatorname{trace}(h^T) + (n-2)g(h^TU, U)$$

$$
\begin{aligned}
&+\frac{1}{2}\Big\{\operatorname{trace}(h^T)g(h^TU,U)-\|h^TU\|^2\\
&\qquad\qquad -\operatorname{trace}\big((\varphi h)^T\big)g\big((\varphi h)^TU,U\big)+\|(\varphi h)^TU\|^2\Big\}\\
&-\frac{1}{4}\sum_{r=n+1}^{2m+1}(\sigma_{11}^r-\sigma_{22}^r-\cdots-\sigma_{nn}^r)^2-\sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}\sigma_{1j}^r)^2 \qquad (37)
\end{aligned}
$$

which implies (33).

Assuming $U=e_1$, from (37), the equality case of (33) is valid if and only if

$$
\begin{aligned}
&\sigma_{11}^r=\sigma_{22}^r+\cdots+\sigma_{nn}^r,\\
&\sigma_{12}^r=\cdots=\sigma_{1n}^r=0,\quad r\in\{n+1,\dots,2m+1\}. \qquad (38)
\end{aligned}
$$

If $H(p)=0$, (38) implies that $U=e_1$ lies in the relative null space $\mathcal{N}_p$. Conversely, if $U=e_1$ lies in the relative null space $\mathcal{N}_p$, then (38) holds, since $H(p)=0$ is assumed. Thus (ii) is proved.

Now we prove (iii). The equality case of (33) for all unit tangent vectors to M at p happens if and only if

$$
\begin{aligned}
&2\sigma_{ii}^r=\sigma_{11}^r+\sigma_{22}^r+\cdots+\sigma_{nn}^r,\quad i\in\{1,\dots,n\},\ r\in\{n+1,\dots,2m+1\},\\
&\sigma_{ij}^r=0,\quad i\neq j,\ r\in\{n+1,\dots,2m+1\}. \qquad (39)
\end{aligned}
$$

Thus, we have two cases, namely either $n=2$ or $n\neq 2$. In the first case p is a totally umbilical point, while in the second case p is a totally geodesic point. The proof of converse part is straightforward. ∎

REMARK 6.2. Theorem 6.1 is different from [**2**, Theorem 3.1]. Moreover, the assumption of the statement in [**2**, Corollary 3.2] is also not true, because the submanifold Min a contact metric manifold normal to the structure vector ξ can not be invariant.

7. Squared mean curvature and k-Ricci curvature

In this section, we prove a relationship between the k-Ricci curvature and the squared mean curvature for C-totally real submanifolds in (κ,μ)-contact space form $\widetilde{M}(c)$. First, we prove the following theorem.

Theorem 7.1. *Let M be an n-dimensional C-totally real submanifold in a $(2m+1)$-dimensional (κ,μ)-contact space form $\widetilde{M}(c)$. Then we have*

$$
\begin{aligned}
\|H\|^2\geqslant{}&\frac{2\tau}{n(n-1)}-\frac{c+3}{4}-\frac{2}{n}\operatorname{trace}(h^T)\\
&-\frac{1}{2n(n-1)}\Big\{\big(\operatorname{trace}(h^T)\big)^2-\|h^T\|^2-\big(\operatorname{trace}((\varphi h)^T)\big)^2+\|(\varphi h)^T\|^2\Big\}. \qquad (40)
\end{aligned}
$$

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PROOF: Let $X \in T_pM$ be a unit tangent vector X at p. We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}\}$ such that $e_1, \dots, e_n$ are tangential to M at p with $e_1 = X$. We recall the equation (17) as

$$(41)\quad n^2\|H\|^2 = 2\tau + \|\sigma\|^2 - \frac{1}{4}n(n-1)(c+3) - 2(n-1)\operatorname{trace}(h^T) \\ - \frac{1}{2}\Big\{\big(\operatorname{trace}(h^T)\big)^2 - \|h^T\|^2 - \big(\operatorname{trace}((\varphi h)^T)\big)^2 + \|(\varphi h)^T\|^2\Big\},$$

where $c = -2\kappa - 1$ if $\kappa < 1$. Let the orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}\}$ be such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \dots, e_n$ diagonalise the shape operator A_{n+1}. Then the shape operators take the forms

$$(42)\qquad A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

$$(43)\qquad A_r = (\sigma_{ij}^r),\ i,j = 1, \dots, n;\ r = n+2, \dots, 2m+1, \quad \operatorname{trace} A_r = \sum_{i=1}^{n} \sigma_{ii}^r = 0.$$

From (41), we get

$$(44)\quad n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^r)^2 - \frac{1}{4}n(n-1)(c+3) - 2(n-1)\operatorname{trace}(h^T) \\ - \frac{1}{2}\Big\{\big(\operatorname{trace}(h^T)\big)^2 - \|h^T\|^2 - \big(\operatorname{trace}((\varphi h)^T)\big)^2 + \|(\varphi h)^T\|^2\Big\}.$$

Since

$$0 \leqslant \sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j,$$

therefore, we get

$$(45)\qquad n^2\|H\|^2 = \Big(\sum_{i=1}^{n} a_i\Big)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leqslant n\sum_{i=1}^{n} a_i^2,$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geqslant n\|H\|^2.$$

In view of (44), we obtain

$$(46)\quad n^2\|H\|^2 \geqslant 2\tau + n\|H\|^2 - \frac{1}{4}n(n-1)(c+3) - 2(n-1)\operatorname{trace}(h^T) \\ - \frac{1}{2}\Big\{\big(\operatorname{trace}(h^T)\big)^2 - \|h^T\|^2 - \big(\operatorname{trace}((\varphi h)^T)\big)^2 + \|(\varphi h)^T\|^2\Big\},$$

which gives (40). $\square$

Now, we are able to prove the following

Theorem 7.2. *Let M be an n-dimensional C-totally real submanifold in a $(2m+1)$-dimensional (κ,μ)-contact space form $\widetilde{M}(c)$. Then, for each integer k, $2 \leqslant k \leqslant n$, and every point $p \in M$, we have*

$$\begin{aligned}(47)\quad \|H\|^2 \geqslant \theta_k(p) - \frac{c+3}{4} - \frac{2}{n}\operatorname{trace}(h^T) \\ - \frac{1}{2n(n-1)}\Big\{\big(\operatorname{trace}(h^T)\big)^2 - \|h^T\|^2 - \big(\operatorname{trace}((\varphi h)^T)\big)^2 + \|(\varphi h)^T\|^2\Big\}.\end{aligned}$$

Proof: Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of T_pM. We denote by $L_{i_1\dots i_k}$ the k-plane section spanned by $e_{i_1},\dots,e_{i_k}$. From (10) and (11), it follows that

$$\tau(L_{i_1\dots i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\dots,i_k\}} \operatorname{Ric}_{L_{i_1\dots i_k}}(e_i), \tag{48}$$

and

$$\tau(p) = \frac{1}{C_{k-2}^{n-2}} \sum_{1\leqslant i_1<\dots<i_k\leqslant n} \tau(L_{i_1\dots i_k}). \tag{49}$$

Combining (12), (48) and (49), we obtain

$$\tau(p) \geqslant \frac{n(n-1)}{2}\theta_k(p), \tag{50}$$

which in view of (40) implies (47). ∎

8. Some applications

In this section, we apply the results of previous sections to get corresponding results for C-totally real submanifolds in Sasakian space forms. If $\kappa = 1$, the (κ,μ)-contact space form reduces to Sasakian space form $\widetilde{M}(c)$; thus $h = 0$ and (8) becomes

$$\begin{aligned}\widetilde{R}(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ + \frac{c-1}{4}\{2g(X,\varphi Y)\varphi Z + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X \\ + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}.\end{aligned} \tag{51}$$

Moreover, for a C-totally real submanifolds in Sasakian space forms, from (15), we also get

$$A_\xi = 0. \tag{52}$$

Thus, in view of Theorem 5.2, we can state the following.

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Theorem 8.1. [9, Theorem 1]) *Let M be an n-dimensional $(n > 2)$ C-totally real submanifold in a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then*

$$\delta_M \leqslant \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{8}(n+1)(n-2)(c+3). \tag{53}$$

Moreover, the equality holds at a point $p \in M$ if and only if there exist a tangent basis $\{e_1, \dots, e_n\} \subset T_pM$ and a normal basis $\{e_{n+1}, \dots, e_{2m}, \xi\} \subset T_p^{\perp}M$ such that the shape operators take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)I_{n-2} \end{pmatrix}, \tag{54}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad r \in \{n+2, \dots, 2m\}, \tag{55}$$

and $A_\xi = 0$.

Now, we state the following Sasakian version for C-totally real submanifolds, which follows from the Theorem 6.1.

Theorem 8.2. *Let M be an n-dimensional C-totally real submanifold in a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then*

1. *For each unit vector $U \in T_pM$, we have*

$$4\,\mathrm{Ric}(U) \leqslant n^2\|H\|^2 + (n-1)(c+3). \tag{56}$$

2. *If $H(p) = 0$, then a unit tangent vector $U \in T_pM$ satisfies*

$$4\,\mathrm{Ric}(U) = (n-1)(c+3) \tag{57}$$

if and only if U belongs to the relative null space $\mathcal{N}_p$.

3. *For each $p \in M$*

$$4S \leqslant \big(n^2\|H\|^2 + (n-1)(c+3)\big)g, \tag{58}$$

where S is the Ricci tensor of the submanifold. The equality in (58) holds if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

In the last, in view of Theorems 7.1 and 7.2, we have the following relationship in case of an n-dimensional C -totally real submanifold in a Sasakian space form $\widetilde{M}(c)$ as follows.

THEOREM 8.3. *Let M be an n-dimensional C-totally real submanifold in a Sasakian space form $\widetilde{M}(c)$. Then we have*

$$\frac{2\tau}{n(n-1)} - \|H\|^2 \leqslant \theta_k(p) - \|H\|^2 \leqslant \frac{c+3}{4}. \tag{59}$$

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Department of Mathematics and Astronomy
Lucknow University
Lucknow 226 007
India
e-mail: mm_tripathi@hotmail.com

Department of Mathematics education
Sunchon National University
Sunchon 540-742
Korea
e-mail: jskim01@hanmir.com

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