The possible generalisation of Pascal's Theorem to three dimensions.

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§1. Extensions of Pascal's theorem are already known if we look at the matter from certain particular points of view – an extension of the theorem from the more general point of view is still a desideratum.

Thus an Oxford examination paper contains the question "*ABCDEF* is a skew hexagon in space, and the lines of intersection of pairs of planes (*ABC*, *DEF*) (*BCD*, *EFA*) (*CDE*, *FAB*) lie in one plane. Prove that the six sides of the hexagon lie on a quadric surface," while Salmon's Geometry of Three Dimensions, Vol I, p. 142 (5th edition, 1912) gives the theorem

"The edges of a tetrahedron intersect a quadric in 12 points, through which can be drawn 4 planes, each containing 3 points lying on edges passing thro' the same angle of the tetrahedron: the lines of intersection of each such plane with the opposite face of the tetrahedron are generators of the same system of a certain hyperboloid."

These are generalisations of particular aspects of Pascal's theorem. The type of generalisation hinted at by Professor Turnbull at the June meeting is to be made from the following form of the theorem "given six points $(x_r\, y_r\, z_r)$ $r = 1 \ldots 6$, and points *ABC* determined as (12, 45) (23, 56) (34, 61) then '*ABC* are collinear' implies 'the six points $(x_r\, y_r\, z_r)$ lie on a conic,' and *vice versa.*"

Algebraically, the determinant $|\,\xi_1\, \eta_2\, \zeta_3\,|$, where *ABC* are respectively the points $(\xi_1\, \eta_1\, \zeta_1)$ $(\xi_2\, \eta_2\, \zeta_2)$ $(\xi_3\, \eta_3\, \zeta_3)$, is equal to

$$\pm \begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & y_1 z_1 & z_1 x_1 & x_1 y_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_6^2 & y_6^2 & z_6^2 & y_6 z_6 & z_6 x_6 & x_6 y_6 \end{vmatrix} \qquad (\alpha_1)$$

This result is established by Reiss, *Math. Annalen* **2** (1870), and also by Mertens *Crelle* **84** (1878).

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In three dimensions, *nine* points determine a locus of the second degree and the condition that *ten* points should lie on such a locus is expressed by the vanishing of

$$\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & w_1^2 & x_1 y_1 & y_1 z_1 & z_1 x_1 & x_1 w_1 & y_1 w_1 & z_1 w_1 \\ \cdots & & & & & & & & & \cdots \\ x_{10}^2 & y_{10}^2 & \cdots & & & & & & & z_{10} w_{10} \end{vmatrix} \qquad (\alpha_2)$$

The algebraical form of the generalisation of Pascal's theorem, if such exist, is that some determinant or other expression vanishes when α_2 does, and that the vanishing of the expression implies the coplanarity of some four points $ABCD$, or a particular relationship of some similar kind. Professor Turnbull has worked out an expression for (α_2) in the form

$$\Sigma\,(abcd)\,(aefg)\,(bejk)\,(cfkh)\,(dghj) = 0 \qquad (\beta_2)$$

where $(1\,2\,3\,4)$ is used to denote the determinant $|\,x_1 y_2 z_3 w_4\,|$. He suggests that the form (β_2), by analogy with known facts in two dimensions (compare Reiss, 1870) may imply the coplanarity of some four points $ABCD$.

§ 2. *Considerations of order and relation among the points.*

In two dimensions, with the order 1 2 3 4 5 6, the three points ABC which are to be collinear are given as the intersections of

1 2	with	4 5	A
2 3	,,	5 6	B
3 4	,,	6 1	C.

With the same cyclical order but a different first member, *e.g.* 2 3 4 5 6 1, the points $A'\,B'\,C'$ are merely ABC in some order,

e.g.	2 3	with	5 6	$A' = B$
	3 4	,,	6 1	$B' = C$
	4 5	,,	1 2	$C' = A$.

It is obvious that the determinant $|\,\xi_1 \eta_2 \zeta_3\,|$ is unaltered, save possibly in sign. That its absolute value remains unaltered for *any* change of order among the points 1 2...6 must be established by means of the algebra which proves $|\,\zeta_1 \eta_2 \zeta_3\,| = \pm \alpha_1$.

When we extend to 3 dimensions, though we cannot expect cyclical order to play an important part, yet there is much to be

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said for finding some expression which is obviously unaltered when we write 2 3...10 1 instead of 1 2 3...10. [desideratum (1).]

Moreover, in the two-dimensional scheme

1 2	4 5
2 3	5 6
3 4	6 1

1 occurs with 2 and 6 and not with the others.
2 ,, ,, 3 and 1 ,, ,, ,, ,, ,, , and so on.

The relationship of any one point to the remaining five is of the same type whatever point we take. [desideratum (2).]

I have been quite unable to determine a grouping of the 10 points in the three dimensional case which will define, by a simple linear construction, four points $ABCD$, and at the same time have either of the properties called above desiderata (1) and (2) respectively.

On the other hand it is easy to find groupings of the 10 points, used once or three times each, which define *five lines* and have one or both of the properties in question.

§ 3. *Possible groupings to determine four points.*

The determinant (α_2) is of degree 20 in all, and of degree 2 in each of the sets $(x_r\, y_r\, z_r\, w_r)$.

A determinant which expresses the coplanarity of four points will be one of 4 rows and 4 columns. If this determinant, Δ say, is to be equal to $\pm\alpha_2$, its degree in all must be 20 and the possible degrees of the columns (assuming row 1 to contain the coordinates of A, row 2 those of B, etc.) are

5 5 5 5 ....(1)
5 5 9 1
1 9 9 1
4 6 6 4 etc.

Of these, the only one which offers any hope of satisfying desideratum (2) is the first, and, on closer examination, it is found that even this does not satisfy it. The only simple linear construction which uses 5 points to determine a point A is that which makes A the point of intersection of the plane 1 2 3 with the line 4 5. In the arrangement (1), *i.e.* 5 pts. to determine each of $ABCD$, the 10 points from which we start must be used twice each and must then fall into four groups of three and four groups of two. This means that *two of the points at least cannot be members of a group of two.*

Suppose 1 is a member of a group of two while 2 is not. The relation of 1 to the remaining nine is quite different from the relation of 2 to the remainder. Unless the algebra removes this asymmetry, a possible but not very probable event, the determinant expressing the coplanarity of $ABCD$ will not be symmetrical in each of the sets $(x_r\, y_r\, z_r\, w_r)$ $r = 1 \ldots 10$.

To sum up the argument of this section, it does not seem probable that we can find four points $ABCD$, determined by a linear construction from the original ten points, whose coplanarity will be expressed by the vanishing of an expression which is symmetrical in the coordinates of the ten points and of degree 20 in these coordinates.

§ 4. It is possible of course that Δ instead of being $\pm\alpha_2$, may be $(\alpha_2)^2$ or $(\alpha_2)^3$ or α_2 times some other factor. The algebra of these possibilities I have not examined in any great detail.

It may be remarked, at this point, that the algebra of reducing Δ, once any mode of defining $ABCD$ has been adopted, into the form

$$\Sigma\,(abcd)\,(abfg)\,(\ldots)\,(\ldots)\,(\ldots),$$

where $(1\,2\,3\,4)$ denotes $|\, x_1\, y_2\, z_3\, w_4 \,|$, is by no means as troublesome as it might first appear.

e.g. if A is the point of intersection of plane 1 2 3 with line 6 7

B of 2 3 4 with 7 8

C of 8 9 10 with 5 4

D of 9 10 1 with 6 5

we can, by considering A as the point of intersection of the three planes 1 2 3, 5 6 7, 6 7 8 readily express Δ in the above form, and deduce the somewhat trivial theorem that, if the points 2...10 are fixed, and $ABCD$ are coplanar, the locus of 1 is a conicoid through the points 2, 3, 6, 9, 10.

§ 5. *The condition that a line may be drawn to meet each of five lines.*

As we have already stated in § 2, the symmetry of the grouping in Pascal's two-dimensional theorem is easily retained in the grouping of 10 points used 3 times each, or once only, to define 5 lines.

Using line coordinates $(p^{(\alpha)}\, q^{(\alpha)}\, r^{(\alpha)}\, s^{(\alpha)}\, t^{(\alpha)}\, u^{(\alpha)})$ $\alpha = 1 \ldots 5$, the conditions that each meets $(PQRSTU)$ are

$$P\, s^{(\alpha)} + Q\, t^{(\alpha)} + R\, u^{(\alpha)} + S\, p^{(\alpha)} + T\, q^{(\alpha)} + U\, r^{(\alpha)} = 0 \quad \alpha = 1, 2 \ldots 5.$$

From these $\dfrac{P}{f_1} = \dfrac{Q}{f_2} = \ldots = \dfrac{U}{f_6}$ where the f's are of degree 1 in each of the sets $(pqrstu)$.

But since $PS + QT + RU = 0$, $f_1 f_4 + f_2 f_5 + f_3 f_6 = 0$.

Consider now the scheme *

line (1) is the intersection of the planes	1 2 3	and	6 7 8
,, (2) ,, ,, ,, ,,	2 3 4	,,	7 8 9
,, (3) ,, ,, ,, ,,	3 4 5	,,	8 9 10
,, (4) ,, ,, ,, ,,	4 5 6	,,	9 10 1
,, (5) ,, ,, ,, ,,	5 6 7	,,	10 1 2

This scheme has the symmetry of the scheme defining the Pascal line of six points in a plane: it has the properties "desiderata 1 and 2" noted in §2. The condition that these five lines may all meet one line is symmetrical in the points 1, 2 ... 10. The degree is, however, *sixty* in all the sets $(x_r y_r z_r w_r)$, whereas (α_2) is of degree *twenty*.

Again, the condition that a line may be drawn to meet each of the lines 16, 27, 38, 49, 5 10, is of the second degree in each of the sets $(x_r y_r z_r w_r)$ and of degree *twenty* in all.

§ 6. *Summary*.

To sum up the two lines of attack we have indicated in this note, we see that the failures are due to the following general characters,

(*a*) when we keep the degree right and try *four coplanar points*, no scheme gives any semblance of symmetry in defining the points,

(*b*) when we concentrate on the symmetry of the defining scheme we are led either to one which resembles Pascal's closely, but leads to an equation of degree 60 instead of 20, or to one which leads to an equation of degree 20, but the geometry of whose derivation in no way suggests Pascal's theorem.

* That 5 lines may be the basis of a possible generalisation of Pascal's theorem has occurred independently to Dr A. Young and Professor Turnbull.