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Shifts of finite type on locally finite groups

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Abstract. In this work we prove that every shift of finite type (SFT), sofic shift, and strongly irreducible shift on locally finite groups has strong dynamical properties. These properties include that every sofic shift is an SFT, every SFT is strongly irreducible, every strongly irreducible shift is an SFT, every SFT is entropy minimal, and every SFT has a unique measure of maximal entropy, among others. In addition, we show that if every SFT on a group is strongly irreducible, or if every sofic shift is an SFT, then the group must be locally finite, and this extends to all of the properties we explore. These results are collected in two main theorems which characterize the local finiteness of groups by purely dynamical properties. In pursuit of these results, we present a formal construction of *free extension* shifts on a group G, which takes a shift on a subgroup H of G, and naturally extends it to a shift on all of G.

Key words: symbolic dynamics, shifts of finite type, locally finite groups, group actions
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1. *Introduction*

For a finite set of symbols $\mathcal{A}$ and a group G, the field of symbolic dynamics studies the action of G by translations on the set $\mathcal{A}^G$, called *full G-shift with alphabet $\mathcal{A}$*, and the subsystems within. Equipped with the product topology (with the discrete topology on $\mathcal{A}$), a closed, translation-invariant subset of $\mathcal{A}^G$ is called a *G-shift*, and understanding what properties such subsystems can exhibit is central to symbolic dynamics. In its conception, the primary group of interest was $\mathbb{Z}$, the group of integers under addition. Even in this case, complex behavior arises, though much is known in general about shifts on $\mathbb{Z}$ [16]. A natural extension of this case is the group $\mathbb{Z}^d$ for some natural number d, the study of which has been called multi-dimensional symbolic dynamics. More recently, interest in shifts on $\mathbb{Z}^d$ has grown, though this case already adds much complexity [11, 17, 20], and less is known about $\mathbb{Z}^d$-shifts in general. Interest in the general group case is even more recent, and as may be expected, it is even less tractable than the case of $\mathbb{Z}^d$, though a recent

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result about tilings of amenable groups [8] has made a few results about shifts on amenable groups possible [3, 4, 9].

The class of *G-shifts of finite type* or (G-SFTs) are of particular interest, as they are characterized by a finite amount of information. More precisely, a G-SFT X is a G-shift for which there is a finite collection of *patterns* (an element of $\mathcal{A}^F$ for a finite $F \subset G$) so that X is the collection of all *configurations* in $\mathcal{A}^G$ for which these patterns never appear. The finite nature of G-SFTs makes them amenable to analysis using finitary and combinatorial methods, and in general G-SFTs are well behaved in comparison to general shifts. Furthermore, every shift on a group can be represented as an intersection of SFTs, so in this sense SFTs are plentiful and are good approximations for shifts in general. Formal definitions of G-shifts and G-SFTs can be found in §2.2.

Understanding what properties are possible for SFTs on groups is at the core of symbolic dynamics. One such property is the *entropy* (Definition 2.14) of an SFT on a countable amenable group G, or in particular the set of entropies which are attainable by SFTs on G, which is denoted $\mathcal{E}(G)$. $\mathcal{E}(\mathbb{Z})$ was classified by Lind [15], and more recently, $\mathcal{E}(\mathbb{Z}^d)$ for $d \geq 2$ was classified by Hochman and Meyerovitch [12]. Recent results by Barbieri [1] classify $\mathcal{E}(G)$ as $\mathcal{E}(\mathbb{Z}^d)$ for a certain class of amenable groups. Currently, to the knowledge of the author, there are no known finitely generated groups G for which $\mathcal{E}(G)$ does not coincide with either $\mathcal{E}(\mathbb{Z})$ or $\mathcal{E}(\mathbb{Z}^2)$, and further classifying $\mathcal{E}(G)$ for other groups and classes of groups is an open goal in symbolic dynamics. Another property is *strong irreducibility* (Definition 2.9), which loosely gives that a G-shift is large, and contains a large variety of configurations. In general, a G-SFT need not be strongly irreducible, and a strongly irreducible G-shift need not be a G-SFT. The additional structure which strong irreducibility imposes on a shift has been useful in proving results about shifts [3, 6, 18]. We also explore several other properties of shifts, which are outlined in §2 and discussed informally after the statement of our two main theorems below.

Our motivation for studying locally finite groups comes from the following example. Let $G = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$, the countable direct sum of the two-element group. Elements of G are infinite sequences of 0s and 1s which only contain finitely many 1s, and the group operation is componentwise addition modulo 2. Using elementary methods for computing the entropy on shifts, it is possible to show directly that

$$\mathcal{E}(G) = \left\{ \frac{\log(n)}{2^{m-1}} : n, m \in \mathbb{N} \right\} \subsetneq \mathcal{E}(\mathbb{Z}),$$

providing an example of an infinitely generated group for which $\mathcal{E}(G)$ does not coincide with $\mathcal{E}(\mathbb{Z})$ or $\mathcal{E}(\mathbb{Z}^2)$. In general, classifying the entropies which are attainable by SFTs for a group G is quite difficult; however, the process is made tractable for this group by the fact that

$$H_n = \left(\bigoplus_{k=1}^{n} \mathbb{Z}/2\mathbb{Z} \right) \oplus \left(\bigoplus_{k=n+1}^{\infty} \{0\} \right)$$

is a sequence of finite subgroups of G such that $H_n \leq H_{n+1}$ and $G = \bigcup_{n\in\mathbb{N}} H_n$, which makes $\{H_n\}$ a Følner sequence for G. As it turns out, a countable group with such a sequence $\{H_n\}$ of finite subgroups is necessarily *locally finite*. A group is locally finite

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if every finitely generated subgroup is finite. In fact, any countable locally finite group must have such a sequence of subgroups, and so this property coincides exactly with the property of being locally finite when the group is countable. Locally finite groups naturally extend finite groups in a way that allows for finitary methods to be used when analyzing the groups, despite being possibly infinite. As a result, one may suspect SFTs on locally finite groups are highly structured and have many nice dynamical properties.

The main results of this paper confirm that SFTs on locally finite groups have very strong dynamical properties. Furthermore, we show that locally finite groups are the only groups for which all SFTs exhibit these properties. These results are grouped in two, one in the case where G is an arbitrary group, and the second where G is a countable amenable group. The first is given below, and followed by a brief explanation of each statement in the result, though formal definitions for every term below can be found in §2.

THEOREM I. *Let G be a group. Then the following statements are equivalent.*

(a) *G is locally finite.*
(b) *Every G-SFT is the free extension of some SFT on a finite subgroup of G.*
(c) *Every G-SFT is strongly irreducible.*
(d) *Every strongly irreducible G-shift is a G-SFT.*
(e) *Every sofic G-shift is a G-SFT.*
(f) *For every G-SFT X,* $\mathrm{Aut}(X)$ *is locally finite.*

Statement I(b) is not a typical dynamical property, but involves a specific type of shift defined in §3 called a *free extension* shift. Free extension shifts are by no means a new concept and have been used in the past [**1**, **12**]; however, we present a formal construction and derive many useful properties of free extensions, some of which, to the knowledge of the author, are new. The equivalence between statements I(b) and I(a) is at the core of nearly every argument involved in proving this theorem and the next. Free extension shifts are defined for general groups in §3, and may be useful in studying shifts on groups in general, beyond the study of shifts on locally finite groups. Statement I(e) involves sofic shifts, which are the image of SFTs under continuous, shift-invariant factor maps. Along with SFTs, sofic shifts are a noteworthy class of shifts which are defined by a finite amount of information. Every SFT is necessarily sofic; however, the converse does not hold in general, and Theorem I gives that the converse holds only in the case that the group is locally finite. The definition of factor maps and sofic shifts can be found in §2.2.3. Statement I(c) gives that every SFT on a locally finite group is strongly irreducible. A formal definition is given by Definition 2.9, but informally, strong irreducibility is a property which guarantees that for any two elements in the shift, there exists an element of the shift which is equal to one of the elements on a finite subset, and equal to the other on any sufficiently separated finite subset. In this sense, strongly irreducible shifts are rich with configurations. Statement I(d) is the converse of the previous statement, and is independently equivalent to the group being locally finite. These two statements in combination give that the set of G-SFTs and the set of strongly irreducible G-shifts coincide exactly when G is locally finite, but that neither is contained in the other when G is not locally finite. Statement I(f) involves $\mathrm{Aut}(X)$, the automorphism group of an SFT X.

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This group consists of homeomorphisms from X to itself which preserve the action of G, and is formally defined in §2.2.3.

For the second result, we restrict to the case that G is a countable amenable group, which permits the development of topological entropy, and each of the statements in the result involves this entropy. A brief discussion of each statement follows the statement of the result, and the formal definitions of every term can be found in §2. In statement II(d), we use the non-standard notation $H \ll G$ to denote that H is a *finite* subgroup of G.

THEOREM II. *Let G be a countable amenable group. Then the following statements are equivalent.*

(a) *G is locally finite.*

(b) *If X is a non-empty G-SFT with $h(X) = 0$, then $X = \{x\}$, where x is a fixed point.*

(c) *Every G-SFT is entropy minimal.*

(d) *G is locally non-torsion and*

$$\mathcal{E}(G) = \left\{ \frac{\log(n)}{|H|} : H \ll G, n \in \mathbb{N} \right\} \subset \mathbb{Q}^{+}_{\log} = \left\{ \frac{\log(n)}{m} : n, m \in \mathbb{N} \right\}.$$

(e) *Every G-SFT has a unique measure of maximal entropy.*

We remark that while we restrict the results to countable amenable groups, entropy can also be extended to the more general class of countable sofic groups [**5**]; however, we will not need this more general definition, since any countable locally finite group is necessarily amenable. The definition of entropy can be found in §2.2.4. Statement II(b) is about what sorts of zero topological entropy SFTs can exist, and in the case of locally finite groups there is a single zero-entropy SFT (up to conjugacy). This result indirectly answers a question of Barbieri in the affirmative.

Question 3.19. [**1**] Does there exist an amenable group G and a G-SFT which does not contain a zero-entropy G-SFT?

Since the only zero-entropy SFTs on locally finite groups are single fixed points, it suffices to construct an SFT which contains no fixed points, which is trivial to do using free extensions. There is further discussion about this construction in §5.

Statement II(c) involves entropy minimality, which is the property that a shift has no proper subshift with the same entropy as the entire shift. A formal definition is given by Definition 2.16. Statement II(d) consists of two parts. The first is that G is locally non-torsion, and means that every finitely generated subgroup of G either is finite or contains an element of infinite order. The need for this requirement in this statement is discussed further in §§4.2.3 and 5. The second part of the statement classifies the set of entropies attainable on any locally finite group, and gives the following corollary which may be of independent interest to the remainder of the theorem.

COROLLARY. *Let G be a countable locally finite group. Then*

$$\mathcal{E}(G) = \left\{ \frac{\log(n)}{|H|} : H \ll G, n \in \mathbb{N} \right\}$$

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Finally, the last statement in the theorem, Statement II(e), involves measure-theoretic entropy and measures of maximal entropy, which are invariant measures on the SFT that have a measure-theoretic entropy equal to the topological entropy of the system. Formal definitions for these can be found starting at Definition 2.20.

1.1. *Overview.* In §2 we present the relevant background and notation used in the remainder of the paper. In §3 we define free extension shifts generally for groups, and then prove some properties of these shifts. In §4 we prove Theorems I and II, each of which is broken down into several individual lemmas. Finally, in §5, we discuss some general consequences of Theorems I and II and properties of free extensions, and indicate possible directions for future work.

2. *Definitions and notation*

We begin by defining all necessary background terms and notation. This section is broken up into subsections based on what is being defined.

2.1. *Sets and groups.* For any set A, let $B \Subset A$ denote that B is a *finite* subset of A. The set difference of two sets A and B is denoted by $A \setminus B$. The disjoint union of two sets A and B is denoted by $A \sqcup B$. The symmetric difference of two sets A and B is denoted by $A \triangle B$.

Given two sets A and B, the set A^B refers to the collection of all functions $f : B \to A$. If A is endowed with a topology, then A^B is endowed with the product topology.

For a group G, we denote that a subset $H \subset G$ is a subgroup of G by $H \leq G$, and to additionally specify that H is a finite subgroup of G we use the notation $H \ll G$. For $F \subset G$, the subgroup of G generated by F is denoted by $\langle F \rangle$. A group is *periodic* if all of its elements have finite order, and is *torsion* if it is periodic and infinite. This definition of a torsion group is non-standard, as typically the terms 'torsion' and 'periodic' are equivalent; however, we require the distinction between arbitrary periodic groups and infinite periodic groups. If P is a property which a group can posses, then a group G is said to be *locally* P if, for all $F \Subset G$, the subgroup $\langle F \rangle$ has property P. A group G is then *locally finite* if, for all $F \Subset G$, we have $\langle F \rangle \ll G$, that is, every finitely generated subgroup of G is finite. G is *locally non-torsion* if, for all $F \Subset G$, the subgroup $\langle F \rangle$ is non-torsion, or in other words, either finite or not periodic. In addition, we use the terminology *non-locally finite* to mean that a group is not locally finite (and similarly for non-locally non-torsion).

Given a group G and subgroup $H \leq G$, we denote the set of *right* cosets of H in G by $H \backslash G$. This notation is similar to that used for set difference (though the spacing is different); however, it is generally clear from context which is being referred to.

A countable group G is *amenable* if there exists a sequence $\{F_n\}_{n=1}^{\infty}$ such that $F_n \Subset G$, $\{F_n\}$ exhausts G so that $G = \bigcup_n F_n$, and for all $g \in G$,

$$\lim_{n \to \infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$

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Such a sequence is called a *left Følner sequence*, and similarly, *right Følner sequences* exist for amenable groups, which satisfy

$$\lim_{n\to\infty} \frac{|F_n g \triangle F_n|}{|F_n|} = 0.$$

2.2. *G-shifts.* For the remainder of the section, $\mathcal{A}$ is a finite *alphabet* (set), endowed with the discrete topology.

Definition 2.1. Let G be a group. $\mathcal{A}^G$ is endowed with the product topology, which makes it a compact Hausdorff space. When G is countable $\mathcal{A}^G$ is metrizable, and we will take this fact to be evident when G is countable. For any $g \in G$, define $\sigma^g : \mathcal{A}^G \to \mathcal{A}^G$ by

$$(\sigma^g x)(h) = x(hg)$$

for any $h \in G$. Each σ^g is a homeomorphism from $\mathcal{A}^G$ to $\mathcal{A}^G$, and $\sigma^g \circ \sigma^h = \sigma^{gh}$ for all $g, h \in G$. Also, $\sigma^e x = x$ for each $x \in X$, and therefore the collection $\sigma = \{\sigma^g : g \in G\}$ is a continuous action of G on $\mathcal{A}^G$. The pair $(\mathcal{A}^G, \sigma)$ is called the *full G-shift with alphabet* $\mathcal{A}$, or simply the *full G-shift* when the alphabet $\mathcal{A}$ is clear, which is typically the case. The elements of $\mathcal{A}^G$ are referred to as *configurations*.

Though the full G-shift is interesting in its own right, we are primarily interested in subsystems of the full G-shift, which are called *G-shifts*.

Definition 2.2. Let G be a group. A subset $X \subset \mathcal{A}^G$ is said to be *G-invariant*, or merely *shift invariant* when the group G is clear from context, if for every $x \in X$, and $g \in G$, $\sigma^g x \in X$. A closed, G-invariant subset $X \subset \mathcal{A}^G$, along with the action of G on $\mathcal{A}^G$ restricted to X, is called a *G-shift of* $\mathcal{A}^G$, or just a *G-shift* when the full shift is clear from context.

2.2.1. *Patterns.* Although an element of $\mathcal{A}^G$ is known as a configuration, the term *pattern* is used when considering elements of $\mathcal{A}^F$ for some $F \subset G$. In addition, we define a few operations on patterns that are quite useful when working with shifts.

Definition 2.3. For any group G and $F \subset G$, an element $w \in \mathcal{A}^F$ is called an *$\mathcal{A}$-pattern on F*, or just a *pattern* if $\mathcal{A}$ is clear. The *shape* of a pattern $w \in \mathcal{A}^F$ is the set F itself.

For $E, F \subset G$ and patterns $w \in \mathcal{A}^E$ and $v \in \mathcal{A}^F$, we say that w and v are disjoint if E and F are disjoint. Similarly, $w \in \mathcal{A}^F$ is said to be finite if F is finite, and infinite if F is infinite.

For any $E \subset F \subset G$ (including $E = F = G$), the restriction of a pattern $w \in \mathcal{A}^F$ to E, which is denoted by $w|_E$ and contained in $\mathcal{A}^E$, is defined as $w|_E(g) = w(g)$ for every $g \in E$. Conversely, for some $w \in \mathcal{A}^E$, the set of *F-extensions of w* is defined as

$$[w]_F = \{v \in \mathcal{A}^F : v|_E = w\}.$$

In the case that $F \leq G$, then $[w]_F$ is known as a *cylinder set*. In the case that $F = G$, then $[w]$ is used instead of $[w]_G$, unless clarity is necessary.

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Patterns are very useful in describing the structure of G-shifts. For any G-shift X (including the full G-shift), the set

$$\mathfrak{B} = \{[w]_G \cap X : F \Subset G,\, w \in \mathcal{A}^F\}$$

is a basis for the subspace topology on X as a subspace of the full G-shift. Note that $[w]_G \cap X$ may be empty or non-empty, and we define the following sets in order to distinguish when this is or is not the case.

Definition 2.4. For any G-shift X, and any $F \subset G$, let $\mathcal{L}_F(X)$ denote the *F-language* of X, which is defined as

$$\mathcal{L}_F(X) = \{x|_F : x \in X\} \subset \mathcal{A}^F.$$

We then let $\mathcal{L}(X)$ be the *language* of X, which is defined as

$$\mathcal{L}(X) = \bigsqcup_{F \Subset G} \mathcal{L}_F(X).$$

By this definition, note that $w \in \mathcal{L}(X)$ if and only if $[w]_G \cap X \neq \varnothing$. In addition, let $\mathcal{L}^\infty(X)$ denote the set

$$\mathcal{L}^\infty(X) = \bigsqcup_{F \subset G} \mathcal{L}_F(X).$$

The main difference between this and $\mathcal{L}(X)$ is that $\mathcal{L}^\infty(X)$ also contains infinite patterns.

We also let $\mathcal{F}_F(X) = \mathcal{A}^F \setminus \mathcal{L}_F(X)$ and

$$\mathcal{F}(X) = \bigsqcup_{F \Subset G} \mathcal{F}_F(X).$$

These sets are known as the *forbidden F-patterns* of X and the *forbidden patterns* of X, respectively.

In constructions which appear in §3, we utilize an extension of the shift action σ to $\mathcal{L}^\infty(X)$, as well as a joining operation which allows taking two disjoint patterns and combining them into one pattern. These are defined next.

Definition 2.5. Let G be a group, and X be a G-shift. Let $g \in G$. Then for any $F \subset G$, define $\sigma_F^g : \mathcal{L}_F(X) \to \mathcal{L}_{Fg^{-1}}(X)$ by

$$(\sigma_F^g w)(h) = w(hg), \quad \text{for all } h \in Fg^{-1}.$$

Note that in the case $F = G$, this covers the typical shift maps. We then define $\sigma^g : \mathcal{L}^\infty(X) \to \mathcal{L}^\infty(X)$ for any $F \subset G$ and pattern $w \in \mathcal{A}^F$ as

$$\sigma^g w = \sigma_F^g w.$$

Restricting patterns to subshapes and shifting behave well in relation to each other. Let $E \subset F \subset G$, and let $g \in G$. Then for any $w \in \mathcal{L}_{Eg}(X)$, the pattern $\sigma^g w$ has shape $Egg^{-1} = E$, and for any $h \in E$,

$$\sigma^g(w|_{Eg})(h) = (w|_{Eg})(hg) = w(hg) = (\sigma^g w)(h).$$

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Since this holds for any $h \in E$, it follows that

$$\sigma^g(w|_{Eg}) = (\sigma^g w)|_E.$$

This rule is used in many proofs without reference.

Similar interplay exists between the shifts and extension sets. Let $E \subset F \subset G$, and $g \in G$. Then $Eg \subset Fg$, and for any $w \in \mathcal{L}_{Eg}(X)$,

$$\sigma^g[w]_{Fg} = [\sigma^g w]_F.$$

This is also used in many proofs without reference.

Along with this natural notion of shifting patterns, there is a natural way to define joining two disjoint patterns.

Definition 2.6. Let G be a group, and X be a G-shift. For any disjoint $u, v \in \mathcal{L}^\infty(X)$, with shapes F_u and F_v respectively (so that $F_u \cap F_v = \varnothing$), we define the *join* of u and v, denoted by $u \vee v$, as follows. Let $w = u \vee v$ be defined as

$$w(g) = \begin{cases} u(g), & g \in F_u, \\ v(g), & g \in F_v, \end{cases}$$

which is a pattern with shape $F_u \sqcup F_v$. Since F_u and F_v must be disjoint to take a join, it is clear that $\vee$ is commutative.

Additionally, the shift action distributes over $\vee$. For any disjoint $u, v \in \mathcal{L}^\infty(X)$ and $g \in G$, it is always the case that $\sigma^g(u \vee v) = (\sigma^g u) \vee (\sigma^g v)$.

Furthermore, for any infinite collection of mutually disjoint patterns, all of these patterns can be joined together into one (possibly infinite) pattern, and by this commutativity, the order of the infinite join is irrelevant. Also, the shifts commute with infinite joins for similar reasons. Infinite joins and the commutativity of the shifts with infinite joins are an integral part of several proofs in §3.

2.2.2. *Properties of G-shifts.* Each G-shift X defines a set of forbidden patterns; however, it is also possible to define a G-shift from a set of forbidden patterns.

Definition 2.7. Let G be a group, $\mathcal{A}$ be a finite alphabet, and let $\mathbf{F} \subset \mathcal{L}(\mathcal{A}^G)$ be a set of forbidden patterns. Define

$$\mathcal{X}^G[\mathbf{F}] = \{x \in \mathcal{A}^G : \text{ for all } g \in G, \text{ for all } F \Subset G, (\sigma^g x)|_F \notin \mathbf{F}\}.$$

It is an elementary exercise to show that $\mathcal{X}^G[\mathbf{F}]$ is a G-shift (though possibly empty), so $\mathcal{X}^G[\mathbf{F}]$ is called the *G-shift defined by* $\mathbf{F}$. $\mathcal{X}[\mathbf{F}]$ is used whenever G is clear from the context.

Another elementary result is that $X = \mathcal{X}[\mathcal{F}(X)]$ for any G-shift X, and therefore every G-shift is generated by some set of finite forbidden patterns.

While $\mathcal{F}(X)$ is always a set of forbidden patterns which defines the G-shift X, there may be much smaller sets of forbidden patterns which also define X. In some cases, there may be a finite set of forbidden patterns which defines a G-shift X, in which case the G-shift is called a *shift of finite type*.

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Definition 2.8. Let G be a group, and X a G-shift. Then X is called a *G-shift of finite type*, or typically a *G-SFT*, if there exists a finite $\mathbf{F} \Subset \mathcal{L}(X)$ such that $X = \mathcal{X}[\mathbf{F}]$.

For any G-SFT X there always exists some $F \Subset G$ such that $X = \mathcal{X}[\mathcal{F}_F(X)]$. Such a shape F is called a *forbidden shape* for X. Additionally, given some forbidden shape F, any $H \Subset G$ with $F \subset H$ is also a forbidden shape, meaning $X = \mathcal{X}[\mathcal{F}_F(X)] = \mathcal{X}[\mathcal{F}_H(X)]$. This property is used in many results without reference.

The finitary nature of G-SFTs makes them amenable to analysis using more combinatorial methods, and they are generally well behaved in many regards. Another strong property a G-shift can possess is *strong irreducibility*, which is a strong mixing type property that is of general interest in the literature.

Definition 2.9. Let G be a group, and X be a G-shift. Then X is *strongly irreducible* if there exists a finite $K \Subset G$ with the following property. For any $u, v \in \mathcal{L}(X)$ with shapes F_u and F_v, if $F_u \cap KF_v = \varnothing$, then there exists $x \in X$ such that $x|_{F_u} = u$ and $x|_{F_v} = v$.

This definition differs from typical definitions of strong irreducibility of shifts on finitely generated groups [9]. In the case that G is finitely generated, this definition is equivalent to more typical definitions using the distance induced by a word metric, and is merely an extension of the more typical definition to (possibly) infinitely generated groups.

2.2.3. *Factors and sofic shifts.* We begin with the definition of factor maps on shift spaces.

Definition 2.10. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets, G be a group, X be a G-shift of $\mathcal{A}^G$ and Y be a G-shift of $\mathcal{B}^G$. Then a map $\phi : X \to Y$ is a *factor map* if

- ϕ is continuous,
- ϕ is surjective, and
- for every $g \in G$, $\sigma^g \circ \phi = \phi \circ \sigma^g$.

In the case that a factor map ϕ is a homeomorphism, then ϕ is called a *conjugacy*, and X and Y are said to be *conjugate*. The collection of conjugacies from a G-shift X to itself forms a group under composition denoted $\mathrm{Aut}(X)$.

This definition of a factor map applies more generally between actions of a group on two topological spaces; however, in the context of G-shifts, factor maps have a very specific structure. We begin by defining a specific kind of factor map which can be constructed between two G-shifts.

Definition 2.11. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets, let G be a group, and let X be a G-shift of $\mathcal{A}^G$. For some $F \Subset G$, let $\beta : \mathcal{L}_F(X) \to \mathcal{B}$ be any function, called a *block map*. Then β induces a map $\phi_\beta^G : X \to \mathcal{B}^G$ called a *block code* by

$$(\phi_\beta^G(x))(g) = \beta((\sigma^g x)|_F),$$

and $Y = \phi_\beta^G(X)$ is a G-shift of $\mathcal{B}^G$. Rather than $\mathcal{B}^G$, however, we consider the codomain of ϕ_β^G to be Y, which makes ϕ_β^G surjective and therefore a factor map from X to Y.

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Block codes are generally easy to work with, due to the finitary nature of the block map that generates them. Surprisingly, any factor map between G-shifts (on possibly different alphabets) is a block code generated by some block map, and this fact is given by the following theorem.

THEOREM 2.12. (Curtis, Hedlund and Lyndon) *Let G be a group, $\mathcal{A}$ and $\mathcal{B}$ be finite alphabets, X be a G-shift of $\mathcal{A}^G$ and Y a G-shift of $\mathcal{B}^G$, and let $\phi : X \to Y$. Then ϕ is a factor map if and only if there exist $F \Subset G$ and block map $\beta : \mathcal{L}_F(X) \to \mathcal{B}$ such that $\phi = \phi_\beta^G$.*

A proof of the theorem at this level of generality can be found in [21, Corollary 6]. Informally, the theorem gives that factor maps for G-shifts are defined by a finite amount of information. A broader class of G-shifts which are defined by a finite amount of information, which contains all SFTs but generally includes more shifts, is the class of sofic G-shifts.

Definition 2.13. A G-shift Y is called a *sofic G-shift* if there exists a G-SFT X such that Y is a factor of X.

Weiss noted when first introducing sofic $\mathbb{Z}$-shifts that ‘the finite type subshifts are flawed by not being closed under the simplest operation, namely that of taking [factors]’ [23]. The collection of all sofic shifts is clearly closed under taking factors, and this is one of the many reasons why the class of sofic shifts is of interest in symbolic dynamics.

2.2.4. *Entropy.* Another important aspect of shifts which is studied in dynamics is entropy (both topological and measure-theoretic), though this theory is generally restricted to countable amenable groups, as computing averages for shifts on non-amenable groups is not possible in general. Notions of entropy do exist for the broader class of countable sofic groups [5]; however, certain undesirable properties arise from such definitions, such as the potential for the entropy of a factor of a system being higher than the entropy of the system itself [24]. Formal treatment of topological and measure-theoretic entropy for G-shifts (and more generally continuous actions of groups on metric spaces), as well as results about these notions of entropy, can be found in [13].

Definition 2.14. Let G be a countable amenable group. Then the *(topological) entropy* of a non-empty G-shift X is defined as

$$h(X) = \inf_n \frac{\log(|\mathcal{L}_{F_n}(X)|)}{|F_n|} = \lim_{n\to\infty} \frac{\log(|\mathcal{L}_{F_n}(X)|)}{|F_n|},$$

where $\{F_n\}$ is some Følner sequence for G. This limit always exists and is equal to the infimum above [13, §9.9]. The entropy of X is also independent of the choice of Følner sequence.

Furthermore, some results pertain to the set of real numbers which are attained as the (topological) entropies for SFTs on a particular group.

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Definition 2.15. Let G be a countable amenable group. Then let

$$\mathcal{E}(G) = \{h(X) : X \text{ a non-empty } G\text{-SFT}\}.$$

Note that $\mathcal{E}(G)$ is a countable subset of $[0, \infty)$, since there are only countably many G-SFTs for any countable group G. Determining exactly what the set $\mathcal{E}(G)$ is for a given group G is in general quite difficult. A classic result of Lind [15] precisely classifies $\mathcal{E}(\mathbb{Z})$ as non-negative rational multiples of logarithms of Perron numbers. More recently, Hochman and Meyerovitch determined that $\mathcal{E}(\mathbb{Z}^d)$ is the set of non-negative upper semi-computable real numbers [12]. For finitely generated amenable groups G with decidable word problem which admit a translation-like action by $\mathbb{Z}^2$, recent work by Barbieri [1] has classified $\mathcal{E}(G)$ as the set of non-negative upper semi-computable real numbers.

With entropy, we may also define the following notion of minimality.

Definition 2.16. Let G be a countable amenable group, and X a G-shift. Then X is *entropy minimal* if for each subshift $Y \subsetneq X$, we have $h(Y) < h(X)$.

A weaker but related notion of minimality is SFT-entropy minimality.

Definition 2.17. Let G be a countable amenable group, and X a G-shift. Then X is *SFT-entropy minimal* if for each SFT $Y \subsetneq X$, we have $h(Y) < h(X)$.

Although in general SFT-entropy minimality is weaker than entropy minimality, they are in fact equivalent if the shift in question is an SFT. Proving this is a fairly standard argument involving approximating subshifts by SFTs, so we omit its proof. This fact is quite useful for proving that an SFT is entropy minimal, as it significantly reduces the amount of shifts to consider when proving entropy minimality.

Along with topological entropy, measure-theoretic entropy can be defined if the shift X is additionally endowed with a Borel probability measure (which is always Radon, since $\mathcal{A}^G$ is metrizable when G is countable) that behaves nicely with the shift action of G.

Definition 2.18. Let G be a countable amenable group, and let X be a G-shift. A measure μ on X is *G-invariant* if for any $g \in G$ and measurable $E \subset X$, it is the case that $\mu(\sigma^{g^{-1}} E) = \mu(E)$.

Let $\mathcal{M}(X)$ denote the set of all G-invariant Borel probability measures μ on X.

For a G-shift X and $w \in \mathcal{L}(X)$, $\mu[w]$ is used as a shorthand for $\mu([w] \cap X)$. To define the μ-entropy of X, first an associated partition entropy must be defined.

Definition 2.19. Let G be a countable amenable group, X be a G-shift, and $\mu \in \mathcal{M}(X)$. Then, for any $F \Subset G$, the *(F, μ)-entropy* of X is defined as

$$H_\mu(X, F) = -\sum_{w \in \mathcal{L}_F(X)} \mu[w] \log(\mu[w]),$$

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where $0 \cdot \log(0)$ is taken to be 0 by convention. The maximum of $H_\mu(X, F)$ over $\mathcal{M}(X)$ is $\log(|\mathcal{L}_F(X)|)$, and is attained only by any $\mu \in \mathcal{M}(X)$ for which $\mu[w] = 1/|\mathcal{L}_F(X)|$ for all $w \in \mathcal{L}_F(X)$ [22, Corollary 4.2.1].

With this, the measure-theoretic entropy can be defined.

Definition 2.20. Let G be a countable amenable group, X be a G-shift, and $\mu \in \mathcal{M}(X)$. Then for any Følner sequence $\{F_n\}_{n=1}^\infty$ for G, the *μ-entropy* of X is defined as

$$h_\mu(X) = \inf_n \frac{H_\mu(X, F_n)}{|F_n|} = \lim_{n\to\infty} \frac{H_\mu(X, F_n)}{|F_n|}.$$

As with topological entropy, this limit always exists, is equal to this infimum, and is independent of the choice of Følner sequence [13, §9.3]. Furthermore, the Variational Principle [13, Theorem 9.43] gives that

$$h(X) = \sup_{\mu\in\mathcal{M}(X)} h_\mu(X).$$

A measure $\mu \in \mathcal{M}(X)$ satisfying $h(X) = h_\mu(X)$ is called a *measure of maximal entropy*, and for G-shifts there always exists at least one measure of maximal entropy, since shift actions are expansive and the entropy map $\mu \to h_\mu(X)$ is upper semi-continuous in this case [24].

3. *Free extension shifts*

Though the primary purpose of this paper is to prove that locally finite groups are precisely the groups which exhibit strong dynamical properties for all SFTs, proving many of these properties directly is somewhat tedious. Instead, we develop a general theory of *free extension* shifts, which simplifies (and even trivializes) many of the results for locally finite groups. Essentially all of the primary results in this paper use properties of free extensions, which are constructed in this section.

The notion of a free extension shift is not new, however. Hochman and Meyerovitch [12] used them (though not explicitly by name) in their landmark paper characterizing the possible entropies of $\mathbb{Z}^d$ SFTs. The term *free extension* and some associated notation used were coined by Barbieri [1], with free extensions appearing as a special case of a far more general method of constructing ‘extensions’ of shifts. The definition given here is far less general than Barbieri’s construction, but it is perhaps more amenable to specifically analyzing free extensions.

3.1. *Definition of free extensions.*

Though there are a few equivalent ways of defining free extensions, we use the following as the primary definition, and prove its equivalence to other definitions.

Definition 3.1. Let G be a group, $H \leq G$, and Y be an H-shift with alphabet $\mathcal{A}$. Then the *free G-extension* of Y, which is denoted by $Y^{\uparrow G}$, is defined as

$$Y^{\uparrow G} = \{x \in \mathcal{A}^G : \text{ for all } g \in G, (\sigma^g x)|_H \in Y\}.$$

Given a free extension $Y^{\uparrow G}$, we call Y the *base* shift.

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While it is clear from this definition that $Y^{\uparrow G}$ is always G-invariant, it is less clear that $Y^{\uparrow G}$ is necessarily closed, and therefore a G-shift. Rather than proving this directly, we use the following lemma to deduce that $Y^{\uparrow G}$ is a G-shift.

LEMMA 3.2. *Let G be a group, $H \leq G$. Then for any $\mathbf{F} \subset \mathcal{L}(\mathcal{A}^H)$, we have $(\mathcal{X}^H[\mathbf{F}])^{\uparrow G} = \mathcal{X}^G[\mathbf{F}]$.*

Proof. First, we show that $(\mathcal{X}^H[\mathbf{F}])^{\uparrow G} \subset \mathcal{X}^G[\mathbf{F}]$. To do so, let $x \in (\mathcal{X}^H[\mathbf{F}])^{\uparrow G}$, $g \in G$, and $F \Subset G$, and since $\mathbf{F} \subset \mathcal{L}(\mathcal{A}^H)$, we may consider only when $F \Subset H$. By definition, we have that $y = (\sigma^g x)|_H \in \mathcal{X}^H[\mathbf{F}]$, which gives that $y|_F = (\sigma^e y)|_F \notin \mathbf{F}$. With $F \Subset H$, we have $y|_H = ((\sigma^g x)|_H)|_F = (\sigma^g x)|_F \notin \mathbf{F}$. Since x, g, and F were arbitrary, we obtain the desired inclusion.

To show that $\mathcal{X}^G[\mathbf{F}] \subset (\mathcal{X}^H[\mathbf{F}])^{\uparrow G}$, let $x \in \mathcal{X}^G[\mathbf{F}]$ and $g \in G$, and we must show that $(\sigma^g x)|_H \in \mathcal{X}^H[\mathbf{F}]$. Let $h \in H$ and $F \Subset H$. Then

$$(\sigma^h(\sigma^g x)|_H)|_F = ((\sigma^{hg} x)|_H)|_F = (\sigma^{hg} x)|_F \notin \mathbf{F},$$

by the fact that $x \in \mathcal{X}^G[\mathbf{F}]$. As such, $(\sigma^g x)|_H \in \mathcal{X}^H[\mathbf{F}]$, so we have shown the desired result. □

With this lemma, we can easily see that $Y^{\uparrow G}$ is a G-shift.

COROLLARY 3.3. *Let G be a group, $H \leq Y$, and Y an H-shift. Then $Y^{\uparrow G} = \mathcal{X}^G[\mathcal{F}(Y)]$, and in particular $Y^{\uparrow G}$ is a G-shift.*

Proof. Taking $\mathbf{F} = \mathcal{F}(Y)$ in Lemma 3.2 gives the desired result, along with noting that $\mathcal{X}^G[\mathcal{F}(Y)]$ is a G-shift. □

In addition to proving that free extensions are shifts, the previous corollary gives an alternative characterization of free extensions, namely the free extension of an H-shift Y is the G-shift defined by the set of finite forbidden patterns for Y. We now provide another characterization of free extensions which is useful in constructing elements of the free extension of a shift. We begin by defining the following function.

Definition 3.4. With $H \backslash G$ denoting the set of right cosets of H in G, let $\mathscr{C}(H \backslash G)$ denote the set of all choice sets for $H \backslash G$, whose existence is given by the axiom of choice. In particular, an element $C \in \mathscr{C}(H \backslash G)$ is a subset of G such that $\{Hc\}_{c \in C}$ is an enumeration of the right cosets of H in G.

For any group G, $H \leq G$, $C \in \mathscr{C}(H \backslash G)$, and alphabet $\mathcal{A}$, define a map $\kappa_C^{\mathcal{A}} : (\mathcal{A}^H)^C \to \mathcal{A}^G$ by

$$\kappa_C^{\mathcal{A}}(\{w_c\}_{c \in C}) = \bigvee_{c \in C} \sigma^{c^{-1}} w_c.$$

Such a map is called a *construction* function. When $\mathcal{A}$ is clear from context, κ_C is used instead.

Note that with $\{w_c\}_{c \in C} \in (\mathcal{A}^H)^C$, each w_c has shape H, which makes $\sigma^{c^{-1}} w_c$ have shape Hc. Since $\{Hc\}_{c \in C}$ is an enumeration of all right cosets of H in G, this makes all

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$\sigma^{c^{-1}} w_c$ disjoint, and so we may join them together to form a configuration on G, giving that κ_C is well defined.

We now show that every construction function is bijective.

LEMMA 3.5. *Let G be a group, $H \leq G$, and $C \in \mathscr{C}(H\backslash G)$. Then κ_C is a bijection, and $\kappa_C^{-1}(x) = \{(\sigma^c x)|_H\}_{c \in C}$.*

Proof. First, we show that κ_C is injective. Let $\{w_c\}_{c \in C}, \{v_c\}_{c \in C} \in (\mathcal{A}^H)^C$ be such that $\kappa_C(\{w_c\}) = \kappa_C(\{v_c\})$. For any $d \in C$, it must be that $\kappa_C(\{w_c\})|_{Hd} = \kappa_C(\{v_c\})|_{Hd}$. Since

$$\kappa_C(\{w_c\})|_{Hd} = \bigg(\bigvee_{c \in C} \sigma^{c^{-1}} w_c\bigg)\bigg|_{Hd},$$

and each $\sigma^{c^{-1}} w_c$ has shape Hc, this restriction must result in $\sigma^{d^{-1}} w_d$. Similarly, $\kappa_C(\{v_c\})|_{Hd} = \sigma^{d^{-1}} v_d$, and so $\sigma^{d^{-1}} w_d = \sigma^{d^{-1}} v_d$. Applying σ^d to both sides gives that $w_d = v_d$. Since $d \in C$ was arbitrary, we have that $\{w_c\} = \{v_c\}$, and therefore κ_C is injective.

Next, let us show that κ_C is surjective. Let $x \in \mathcal{A}^G$. Then

$$\begin{aligned} x &= \bigvee_{c \in C} x|_{Hc} = \bigvee_{c \in C} \sigma^{c^{-1}}(\sigma^c(x|_{Hc})) \\ &= \bigvee_{c \in C} \sigma^{c^{-1}}((\sigma^c x)|_H) = \kappa_C(\{(\sigma^c x)|_H\}_{c \in C}), \end{aligned}$$

and so κ_C is surjective.

This makes κ_C a bijection, and is therefore invertible, and the previous display gives the exact rule for κ_C^{-1}. □

In fact, with $(\mathcal{A}^H)^C$ endowed with the product topology, κ_C is a homeomorphism from $(\mathcal{A}^H)^C$ to $\mathcal{A}^G$; however, we do not use this. For the remainder of the paper, it will be taken as a given that κ_C is a bijection. With κ_C, we may now give our last characterization of free extensions.

LEMMA 3.6. *Let G be a group, $H \leq G$, and Y be an H-shift. Then for any $C \in \mathscr{C}(H\backslash G)$, we have $Y^{\uparrow G} = \kappa_C(Y^C)$.*

Proof. First, let $x \in Y^{\uparrow G}$. Then $\kappa_C^{-1}(x) = \{(\sigma^c x)|_H\}_{c \in C}$, and by definition of $Y^{\uparrow G}$, we have $(\sigma^c x)|_H \in Y$ for each $c \in C$, and therefore $\{(\sigma^c x)|_H\}_{c \in C} \in Y^C$, so $x \in \kappa_C(Y^C)$. As such, $Y^{\uparrow G} \subset \kappa_C(Y^C)$.

Now let $x \in \kappa_C(Y^C)$. Then, by definition, we have that $\kappa_C^{-1}(x) = \{(\sigma^c x)|_H\}_{c \in C} \in Y^C$, and so $(\sigma^c x)|_H \in Y$ for each $c \in C$. Let $g \in G$. Then there exists a unique $c \in C$ and $h \in H$ such that $g = hc$. Since $(\sigma^c x)|_H \in Y$, by shift invariance we also have $\sigma^h(\sigma^c x)|_H \in Y$, and therefore we have

$$(\sigma^g x)|_H = (\sigma^{hc} x)|_H = \sigma^h(\sigma^c x)|_H \in Y.$$

As $g \in G$ was arbitrary, this gives that $x \in Y^{\uparrow G}$, and therefore $\kappa_C(Y^C) \subset Y^{\uparrow G}$. □

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Each of these characterizations provides a useful perspective on the structure of free extensions, and is useful in proving different properties of free extensions. The definition used here indicates that a free extension shift is locally a shift on a subgroup, which is broadly useful in ensuring restrictions of configurations in the free extension are an element of the shift on the subgroup. The second characterization by forbidden patterns clearly makes free extensions a type of shift, and provides an implicit connection between free extensions and SFTs. The final characterization with construction functions gives free extensions a natural strong mixing condition, in the sense that for any $C \in \mathscr{C}(H\backslash G)$ and collection $\{x_c\}_{c\in C} \subset Y^{\uparrow G}$, there exists an element of $Y^{\uparrow G}$ which is equal to x_c on the coset Hc. This strong mixing condition is at the core of the utility of studying free extensions in general.

3.2. *Properties of free extensions and their base shifts.* Having established that free extensions are well defined, we now prove that many useful properties can be transferred from a base shift to its extension, and vice versa.

First, we observe that the topological entropy of a free extension is the same as its base shift.

PROPOSITION 3.7. [1, Proposition 5.2] *Let G be a countable amenable group, $H \leq G$, and X an H-shift. Then*

$$h(X^{\uparrow G}) = h(X).$$

Second, if there are three groups $K \leq H \leq G$, then taking a K-shift and extending it to G produces the same thing as first extending to H and then to G.

LEMMA 3.8. *Let G be a group, $K \leq H \leq G$, and Y a K-shift. Then $Y^{\uparrow G} = (Y^{\uparrow H})^{\uparrow G}$.*

Proof. By Corollary 3.3, we have $Y^{\uparrow G} = \mathcal{X}^G[\mathcal{F}(Y)]$ and $Y^{\uparrow H} = \mathcal{X}^H[\mathcal{F}(Y)]$. Then, by Lemma 3.2, we have

$$(Y^{\uparrow H})^{\uparrow G} = (\mathcal{X}^H[\mathcal{F}(Y)])^{\uparrow G} = \mathcal{X}^G[\mathcal{F}(Y)] = Y^{\uparrow G}. \qquad \square$$

Next, we prove a stability result for free extensions, namely that the intersection of free extensions is the free extension of an intersection of shifts.

LEMMA 3.9. *Let G be a group, $H \leq G$, and $\{Y_i\}_{i\in I}$ be a collection of H-shifts. Then $\bigcap_{i\in I} Y_i^{\uparrow G} = (\bigcap_{i\in I} Y_i)^{\uparrow G}$.*

Proof. Let $C \in \mathscr{C}(H\backslash G)$. Then by Lemma 3.6 and the fact that κ_C is a bijection,

$$\bigcap_{i\in I} Y_i^{\uparrow G} = \bigcap_{i\in I} \kappa_C(Y_i^C) = \kappa_C\bigg(\bigcap_{i\in I} Y_i^C\bigg) = \kappa_C\bigg(\bigg(\bigcap_{i\in I} Y_i\bigg)^C\bigg) = \bigg(\bigcap_{i\in I} Y_i\bigg)^{\uparrow G}. \qquad \square$$

Lemma 3.2 readily gives that the free extension of an SFT remains an SFT. Perhaps surprisingly, the converse also holds; if the free extension of a shift is an SFT, then the base shift must have been an SFT to begin with.

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LEMMA 3.10. *Let G be a group, $H \leq G$, and Y be an H-shift. Then Y is an H-SFT if and only if $Y^{\uparrow G}$ is a G-SFT.*

Proof. First, suppose Y is an SFT. Let $F \Subset H$ be a forbidden shape for Y, and $\mathbf{F} \Subset \mathcal{A}^F$ be a set of forbidden F-patterns so that $Y = \mathcal{X}^H[\mathbf{F}]$. By Lemma 3.2, $Y^{\uparrow G} = (\mathcal{X}^H[\mathbf{F}])^{\uparrow G} = \mathcal{X}^G[\mathbf{F}]$, which is clearly an SFT.

Now suppose that $Y^{\uparrow G}$ is an SFT. Let $C \in \mathscr{C}(H\backslash G)$, and let $d \in C$ such that $H = Hd$. Let $F \Subset G$ be a forbidden shape for $Y^{\uparrow G}$ so that $Y^{\uparrow G} = \mathcal{X}^G[\mathcal{F}_F(Y^{\uparrow G})]$. Now define $C_0 \subset C$ to be the set of all $c \in C$ for which $F \cap Hc \neq \varnothing$, and for $c \in C_0$ let $F_c = F \cap Hc$. This partitions F into the finitely many disjoint subsets F_c, which are each contained within a separate coset of H. Then define

$$E = \bigcup_{c \in C_0} F_c c^{-1} \subset H$$

and

$$\hat{F} = \bigcup_{c \in C_0} Ec.$$

Note that for each $c \in C_0$ we have $F_c c^{-1} \subset E$, and therefore $F_c = (F_c c^{-1})c \subset Ec \subset \hat{F}$, so $F \subset \hat{F}$. As such, $Y^{\uparrow G} = \mathcal{X}[\mathcal{F}_{\hat{F}}(Y^{\uparrow G})]$. For any $w \in \mathcal{A}^E$, let

$$P(w) = \bigcup_{c \in C_0} [\sigma^{c^{-1}} w]_{\hat{F}} \subset \mathcal{A}^{\hat{F}},$$

and define

$$\mathbf{F} = \{w \in \mathcal{A}^E : P(w) \subset \mathcal{F}_{\hat{F}}(Y^{\uparrow G})\}.$$

Let us now show that $Y = \mathcal{X}^H[\mathbf{F}]$, which clearly shows Y is a H-SFT, since $\mathbf{F}$ is finite.

First, let $x \in Y^{\uparrow G} = \mathcal{X}^G[\mathcal{F}_{\hat{F}}(Y^{\uparrow G})]$. Let $g \in G$, and pick any $c \in C_0$. Then, from the definition of $P(w)$,

$$(\sigma^{c^{-1}g} x)|_{\hat{F}} \in [(\sigma^{c^{-1}g} x)|_{Ec}]_{\hat{F}} = [\sigma^{c^{-1}} (\sigma^g x)|_E]_{\hat{F}} \subset P((\sigma^g x)|_E).$$

Since $x \in \mathcal{X}^G[\mathcal{F}_{\hat{F}}(Y^{\uparrow G})]$, it follows that $(\sigma^{c^{-1}g} x)|_{\hat{F}} \notin \mathcal{F}_{\hat{F}}(Y^{\uparrow G})$, and therefore, $P((\sigma^g x)|_E) \not\subset \mathcal{F}_{\hat{F}}(Y^{\uparrow G})$. This gives that $(\sigma^g x)|_E \notin \mathbf{F}$. Since g was arbitrary, this implies $x \in \mathcal{X}^G[\mathbf{F}]$, and therefore $Y^{\uparrow G} \subset \mathcal{X}^G[\mathbf{F}]$.

Now let $x \in \mathcal{X}^G[\mathbf{F}]$. By definition, for each $g \in G$, it must be that $(\sigma^g x)|_E \notin \mathbf{F}$, and therefore for every $c \in C_0$, we have $P((\sigma^{cg} x)|_E) \not\subset \mathcal{F}_{\hat{F}}(Y^{\uparrow G})$. As such, for each $c \in C_0$, there exists $w_c \in P((\sigma^{cg} x)|_E) \setminus \mathcal{F}_{\hat{F}}(Y^{\uparrow G})$. Note that this means that for each $c \in C_0$, there exists $d_c \in C_0$ such that $(\sigma^{d_c} w_c)|_E = (\sigma^{cg} x)|_E$. Let $x_c \in Y^{\uparrow G}$ be such that $x_c|_{\hat{F}} = w_c$. This must be possible since $w_c \notin \mathcal{F}_{\hat{F}}(Y^{\uparrow G})$, and $Y^{\uparrow G}$ is shift invariant. Let $\{a_d^c\}_{d \in C} = \kappa_C^{-1}(x_c)$, and note that $a_d^c \in Y$ for each $c \in C_0$ and $d \in C$ by Lemma 3.6. Furthermore, for each $c \in C_0$, we have $a_{d_c}^c|_E = (\sigma^{cg} x)|_E$, which gives that

$$(\sigma^{c^{-1}} a_{d_c}^c)|_{Ec} = \sigma^{c^{-1}} (a_{d_c}^c|_E) = \sigma^{c^{-1}} ((\sigma^{cg} x)|_E) = (\sigma^g x)|_{Ec}.$$

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Define $\{y_d\}_{d\in C} \in Y^C$ as follows. For each $c \in C_0$ let $y_c = a^c_{d_c}$, and for $d \in C \setminus C_0$ let $y_d \in Y$ (it does not matter how these are chosen). Then $y = \kappa_C(\{y_d\}) \in Y^{\uparrow G}$, so for each $c \in C_0$ it is the case that $y|_{Ec} = (\sigma^{c^{-1}} a_{c,d_c})|_{Ec} = (\sigma^g x)|_{Ec}$, which gives that $y|_{\hat{F}} = (\sigma^g x)|_{\hat{F}}$. Since $y \in Y^{\uparrow G}$, this implies $(\sigma^g x)|_{\hat{F}} \notin \mathcal{F}_{\hat{F}}(Y^{\uparrow G})$. As this is true for all $g \in G$, this implies $x \in \mathcal{X}[\mathcal{F}_{\hat{F}}(Y^{\uparrow G})] = Y^{\uparrow G}$, and therefore $\mathcal{X}^G[\mathbf{F}] \subset Y^{\uparrow G}$.

The results of the two previous paragraphs give that $Y^{\uparrow G} = \mathcal{X}^G[\mathbf{F}]$. By Lemma 3.2, since $\mathbf{F} \subset \mathcal{A}^E$, and $E \subset H$, we have $Y^{\uparrow G} = \mathcal{X}^G[\mathbf{F}] = (\mathcal{X}^H[\mathbf{F}])^{\uparrow G}$, and so $\kappa_C(Y^C) = \kappa_C((\mathcal{X}^H[\mathbf{F}])^C)$. Since κ_C is a bijection, it must be that $Y^C = (\mathcal{X}^H[\mathbf{F}])^C$, and therefore $\mathcal{X}^H[\mathbf{F}] = Y$. □

Next, we show a similar result to the previous one, replacing the property of being a G-SFT with being strongly irreducible.

LEMMA 3.11. *Let G be a group, $H \leq G$, and Y be an H-shift. Then Y is strongly irreducible if and only if $Y^{\uparrow G}$ is strongly irreducible.*

Proof. First, suppose that Y is strongly irreducible. Then there exists $K \Subset H$ such that for any $u, v \in \mathcal{L}(Y)$ with shapes F_u and F_v such that $F_u \cap KF_v = \varnothing$, there exists $x \in Y^{\uparrow G}$ such that $x|_{F_u} = u$ and $x|_{F_v} = v$. We will now show that $X = Y^{\uparrow G}$ is strongly irreducible with the same K. Let $u, v \in \mathcal{L}(X)$ with shapes F_u and F_v such that $F_u \cap KF_v = \varnothing$. Since $u, v \in \mathcal{L}(X)$, let $x_u, x_v \in X$ such that $x_u|_{F_u} = u$ and $x_v|_{F_v} = v$. Let $C \in \mathscr{C}(H\backslash G)$. Since $x_u, x_v \in X = Y^{\uparrow G}$, we may take $\{y_c\}_{c\in C} = \kappa_C^{-1}(x_u)$ and $\{z_c\}_{c\in C} = \kappa_C^{-1}(x_v)$ with $y_c, z_c \in Y$ for all $c \in C$ by Lemma 3.6. Let $c \in C$, and define $U_c = F_u c^{-1} \cap H$ and $V_c = F_v c^{-1} \cap H$. With U_c and V_c being subsets of H, we have that $y_c|_{U_c} \in \mathcal{L}(Y)$ and $z_c|_{V_c} \in \mathcal{L}(Y)$. Furthermore, we have

$$U_c \cap KV_c \subset F_u c^{-1} \cap KF_v c^{-1} = (F_u \cap KF_v)c^{-1} = \varnothing.$$

As such, the strong irreducibility of Y gives that there exists $w_c \in Y$ such that $w_c|_{U_c} = y_c|_{U_c}$ and $w_c|_{V_c} = z_c|_{V_c}$. Let $x = \kappa_C(\{w_c\}_{c\in C}) \in X$. We now show that $x|_{F_u} = u$ and $x|_{F_v} = v$. Indeed, as

$$F_u = \bigsqcup_{c\in C} F_u \cap Hc = \bigsqcup_{c\in C} (F_u c^{-1} \cap H)c = \bigsqcup_{c\in C} U_c c$$

(and similarly for F_v with V_c in place of U_c), we may obtain that $x|_{F_u} = u$ by checking $x|_{U_c c} = u|_{U_c c}$ for every $c \in C$ (and similarly for $x|_{F_v} = v$). For $c \in C$, we have $U_c c \subset Hc$, and so

$$\begin{aligned} x|_{U_c c} = (x|_{Hc})|_{U_c c} &= (\sigma^{c^{-1}} x_c)|_{U_c c} = \sigma^{c^{-1}}(x_c|_{U_c}) = \sigma^{c^{-1}}(y_c|_{U_c}) \\ &= \sigma^{c^{-1}}((\sigma^c x_u)|_{U_c}) = \sigma^{c^{-1}}\sigma^c(x_u|_{U_c c}) = u|_{U_c c}, \end{aligned}$$

where the final equality follows from the fact that $x_u|_{F_u} = u$. By the same argument, replacing U_c with V_c, y_c with z_c, x_u with x_v, and u with v, we obtain $x|_{V_c c} = v|_{V_c c}$, and therefore $x|_{F_u} = u$ and $x|_{F_v} = v$. As such, $Y^{\uparrow G}$ is strongly irreducible.

Now suppose that $X = Y^{\uparrow G}$ is strongly irreducible. Let $L \Subset G$ be such that if $u, v \in \mathcal{L}(X)$ with shapes E_u and E_v satisfy $E_u \cap LE_v = \varnothing$, then there exists $x \in X$ such

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that $x|_{E_u} = u$ and $x|_{E_v} = v$. Let $K = L \cap H$, and we will now show that Y is strongly irreducible with this K. Note that K must be non-empty because L must contain an element of H (otherwise, if E_u and E_v are finite subsets of H which intersect, then $E_u \cap LE_v = \varnothing$, but if u and v disagree on some element of $E_u \cap E_v$, there clearly cannot be an element in X that contains both u and v). Let $u, v \in \mathcal{L}(Y)$ with shapes F_u and F_v such that $F_u \cap KF_v = \varnothing$. Since $u, v \in \mathcal{L}(Y)$, we clearly have that $u, v \in \mathcal{L}(X)$. We now show that $F_u \cap LF_v = \varnothing$. Indeed, since $K = L \cap H$, $F_u \cap KF_v = \varnothing$, and $F_u, F_v \subset H$,

$$\begin{aligned} F_u \cap LF_v &= (F_u \cap (L \cap H)F_v) \cup (F_u \cap (L \setminus H)F_v) \\ &\subset (F_u \cap KF_v) \cup (H \cap (L \setminus H)H) \\ &= \bigcup_{l \in L \setminus H} H \cap lH. \end{aligned}$$

Since H is a subgroup of G, for any $l \in L \setminus H$ we have $H \cap lH = \varnothing$, since lH is a proper left coset of H. As such, $F_u \cap LF_v = \varnothing$, and so by the strong irreducibility of X, there exists $x \in X$ such that $x|_{F_u} = u$ and $x|_{F_v} = v$. Let $y = x|_H \in Y$, and so $y|_{F_u} = u$ and $y|_{F_v} = v$. Therefore, Y is strongly irreducible. □

Lastly, factor maps which are defined by block maps on a base shifts can be extended to a factor map of the free extension of the base shift in a natural manner. This does not apply to arbitrary factor maps from a free extension shift, however, and only applies to factor maps whose block maps are defined on a subset of the group for the base shift. In the result, for a function $\phi : X \to X$, we denote by $(\phi)^C$ the product function on X^C.

LEMMA 3.12. *Let G be a group, $F \Subset H \leq G$ and $C \in \mathscr{C}(H\backslash G)$. For finite alphabets $\mathcal{A}$ and $\mathcal{B}$, let Y be an H-shift of $\mathcal{A}^H$, and let $\beta : \mathcal{L}_F(Y) \to \mathcal{B}$ be a block map. Then for any $x \in Y^{\uparrow G}$,*

$$\phi_\beta^G(x) = (\kappa_C^{\mathcal{B}} \circ (\phi_\beta^H)^C \circ (\kappa_C^{\mathcal{A}})^{-1})(x).$$

Proof. Let $x \in Y^{\uparrow G}$. Then for any $g \in G$, let $h \in H$ and $d \in C$ such that $g = hd$. Expanding the definition of $\kappa_C^{\mathcal{B}}$, we have

$$\begin{aligned} ((\kappa_C^{\mathcal{B}} \circ (\phi_\beta^H)^{H\backslash G} \circ (\kappa_C^{\mathcal{A}})^{-1})(x))(g) &= \bigg(\bigvee_{c \in C} \sigma^{c^{-1}}(\phi_\beta^H((\kappa_C^{\mathcal{A}})^{-1}(x)_c))\bigg)(g) \\ &= (\sigma^{d^{-1}}(\phi_\beta^H((\kappa_C^{\mathcal{A}})^{-1}(x)_d)))(hd), \end{aligned}$$

where $g \in Hd$ implies that we only need to observe the pattern in the join on Hd. Applying the shift $\sigma^{d^{-1}}$, and expanding the definition of ϕ_β^H at h, we obtain

$$(\sigma^{d^{-1}}(\phi_\beta^H((\kappa_C^{\mathcal{A}})^{-1}(x)_d)))(hd) = \phi_\beta^H((\sigma^d x)|_H)(h) = \beta((\sigma^h((\sigma^d x)|_H))|_F).$$

Simplifying, we have

$$\beta((\sigma^h((\sigma^d x)|_H))|_F) = \beta(((\sigma^{hd}x)|_{Hh^{-1}})|_F) = \beta((\sigma^g x)|_F) = (\phi_\beta^G(x))(g)$$

by the definition of ϕ_β^G at g. With this, we have shown the desired result. □

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A direct consequence of the previous result is that certain factors of a free extension are equal to the free extension of a factor of the base shift. This is not the case for all factors, but the property is quite useful nevertheless.

COROLLARY 3.13. *Let G be a group, $H \le G$, $\mathcal{A}$ and $\mathcal{B}$ be finite alphabets, and Y be an H-shift of $\mathcal{A}^H$. Let $F \Subset H$ and $\beta : \mathcal{L}_F(Y) \to \mathcal{B}$ be a block map. Then*

$$\phi_\beta^G(Y^{\uparrow G}) = \phi_\beta^H(Y)^{\uparrow G}.$$

Proof. Let $C \in \mathscr{C}(H\backslash G)$. By Lemma 3.12, for any $x \in Y^{\uparrow G}$, we have $\phi_\beta^G(x) = (\kappa_C^{\mathcal{B}} \circ (\phi_\beta^H)^C \circ (\kappa_C^{\mathcal{A}})^{-1})(x)$, and therefore, using Lemma 3.6,

$$\begin{aligned}\phi_\beta^G(x) &= (\kappa_C^{\mathcal{B}} \circ (\phi_\beta^H)^C \circ (\kappa_C^{\mathcal{A}})^{-1})(x)\\ &= (\kappa_C^{\mathcal{B}} \circ (\phi_\beta^H)^C)((\kappa_C^{\mathcal{A}})^{-1}(x))\\ &\in \kappa_C^{\mathcal{B}}((\phi_\beta^H)^C(Y^C))\\ &= \kappa_C^{\mathcal{B}}(\phi_\beta^H(Y)^C)\\ &= \phi_\beta^H(Y)^{\uparrow G}.\end{aligned}$$

Similarly, for any $y \in \phi_\beta^H(Y)^{\uparrow G}$, we have $(\kappa_C^{\mathcal{B}})^{-1}(y) \in \phi_\beta^H(Y)^C = (\phi_\beta^H)^C(Y^C) = (\phi_\beta^H)^C((\kappa_C^{\mathcal{A}})^{-1}(Y^{\uparrow G}))$. Applying $\kappa_C^{\mathcal{B}}$ to both sides, we obtain by Lemma 3.12 that

$$y \in (\kappa_C^{\mathcal{B}} \circ (\phi_\beta^H)^C \circ (\kappa_C^{\mathcal{A}})^{-1})(Y^{\uparrow G}) = \phi_\beta^G(Y^{\uparrow G}).$$ □

3.3. *Applications of free extensions to shifts on groups.* Using free extensions, it is possible to analyze shifts on arbitrary groups, though only to an extent. First, we can look at SFTs on arbitrary groups. We use the following result extensively in the study of SFTs on locally finite groups in particular; however, it applies in full generality to all groups.

LEMMA 3.14. *Let G be a group, and X a G-SFT. Then there exist $F \Subset G$ and $\langle F\rangle$-SFT Y such that $X = Y^{\uparrow G}$. In other words, every SFT on a group G is the free extension of an SFT on a finitely generated subgroup of G.*

Proof. Since X is an SFT, let $F \Subset G$ be a forbidden shape for X, so $X = \mathcal{X}^G[\mathcal{F}_F(X)]$. Let $H = \langle F\rangle \le G$, which makes H finitely generated. Additionally, since $F \Subset H$, the H-shift $Y = \mathcal{X}^H[\mathcal{F}_F(X)]$ is an H-SFT. By Lemma 3.2,

$$Y^{\uparrow G} = \mathcal{X}^H[\mathcal{F}_F(X)]^{\uparrow G} = \mathcal{X}^G[\mathcal{F}_F(X)] = X,$$

which proves the desired result. □

In the case that G itself is finitely generated, it may be that $F \Subset G$ is such that $\langle F\rangle = G$, and so $X = Y = Y^{\uparrow G}$, which is a trivial result. In the case that G is infinitely generated, however, this can never be the case and leads to interesting results such as the following corollary.

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COROLLARY 3.15. *Let G be an infinitely generated amenable group. Then*

$$\mathcal{E}(G) = \bigcup_{F \Subset G} \mathcal{E}(\langle F \rangle).$$

Proof. This follows immediately from the previous lemma and Proposition 3.7. □

In addition to SFTs, which are defined by a finite forbidden shape, sofic shifts are defined by an SFT and a finite block map, and using a technique similar to the lemma above, we can show that any sofic shift on a group is the free extension of a sofic shift on a finitely generated subgroup.

LEMMA 3.16. *Let G be a group and X be a sofic G-shift. Then there exist $F \Subset G$ and sofic $\langle F \rangle$-shift Y such that $X = Y^{\uparrow G}$.*

Proof. Since X is sofic, let Z be a G-SFT, and $\beta : \mathcal{L}_E(Z) \to \mathcal{A}$ be a block map such that $X = \phi_\beta^G(Z)$ with $E \Subset G$. Since Z is a G-SFT, by Lemma 3.14 that there exist $F \Subset G$ and $\langle F \rangle$-SFT W such that $W^{\uparrow G} = Z$. Let $H = \langle F \cup E \rangle$, and we have that $U = W^{\uparrow H}$ is an H-SFT by Lemma 3.10, and that $U^{\uparrow G} = (W^{\uparrow H})^{\uparrow G} = W^{\uparrow G} = Z$ by Lemma 3.8. As such, Lemma 3.12 gives us that

$$X = \phi_\beta^G(Z) = \phi_\beta^G(U^{\uparrow G}) = (\phi_\beta^H(U))^{\uparrow G},$$

since $E \Subset H$. Since U is an H-SFT, we have that $Y = \phi_\beta^H(U)$ is a sofic H-shift, and clearly H is finitely generated with $X = Y^{\uparrow G}$. □

Furthermore, the finite nature of the strong irreducibility condition (namely, the finiteness of K) allows us to prove the same result as for SFTs and sofic shifts.

LEMMA 3.17. *Let G be a group and X be a strongly irreducible G-shift. Then there exist $F \Subset G$ and strongly irreducible $\langle F \rangle$-shift Y such that $X = Y^{\uparrow G}$.*

Proof. Since X is strongly irreducible, let $K \Subset G$ be such that, for any $u, v \in \mathcal{L}(X)$ with shapes F_u and F_v respectively such that $F_u \cap K F_v = \varnothing$, there exists $x \in X$ such that $x|_{F_u} = u$ and $x|_{F_v} = v$. We will now show that K is an option for the finite set F in the statement of the lemma. For any $F \Subset H$ let $\mathbf{F}_F = \mathcal{F}_F(X)$, and define $Y_F = \mathcal{X}^H[\mathbf{F}_F]$. Then since F is finite, Y_F is an H-SFT for each F. Furthermore, by Lemma 3.2, we have that $Y_F^{\uparrow G} = (\mathcal{X}^H[\mathbf{F}_F])^{\uparrow G} = \mathcal{X}^G[\mathbf{F}_F]$, and so clearly $X \subset Y_F^{\uparrow G}$. As such, $X \subset \bigcap_{F \Subset H} Y_F^{\uparrow G}$, and by Lemma 3.9 we have that

$$\bigcap_{F \Subset H} Y_F^{\uparrow G} = \bigg(\bigcap_{F \Subset H} Y_F \bigg)^{\uparrow G}.$$

Let $Y = \bigcap_{F \Subset H} Y_F$, which is an H-shift, so we have $X \subset Y^{\uparrow G}$. We now show that $Y^{\uparrow G} \subset X$.

Let $C \in \mathscr{C}(H \backslash G)$, let $z \in Y^{\uparrow G}$, and let $g \in G$ and $F \Subset H$. By the construction of $Y^{\uparrow G}$, we have $z \in Y_F^{\uparrow G} = \mathcal{X}^G[\mathbf{F}_F]$, and so $z|_F \notin \mathbf{F}_F = \mathcal{F}_F(X)$. Therefore, there exists $x_F \in X$ such that $z|_F = x_F|_F$. In particular, this shows that the set $E_F = [z|_F] \cap X$ is non-empty

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and closed. Additionally, since for each $g \in G$ we have $\sigma^g z \in Y^{\uparrow G}$, we also have that $[(\sigma^g z)|_F] \cap X = \sigma^g([z|_{Fg}] \cap X)$ is non-empty and closed. Since σ^g is a homeomorphism on $\mathcal{A}^G$ and X, we have that $E_{Fg} = [z|_{Fg}] \cap X$ is a non-empty closed subset of X. As such,

$$\mathcal{G} = \{E_{Fg} : F \Subset H, g \in G\}$$

is a collection of non-empty closed subsets of X. We now show that $\mathcal{G}$ has the finite intersection property.

Let $E_{F_1 g_1}, E_{F_2 g_2}, \ldots, E_{F_n g_n} \in \mathcal{G}$. Note that since $F_i \Subset H$ and $g_i \in G$, we have that $F_i g_i \Subset Hc_i$ for some unique $c_i \in C$. If we have that $F_j g_j \Subset Hc_i$ for some $j \neq i$, then $F_i g_i \cup F_j g_j \Subset Hc_i$, giving $(F_i g_i \cup F_j g_j)c_i^{-1} \Subset H$, and so we have

$$E_{F_i g_i} \cap E_{F_j g_j} = [z|_{F_i g_i}] \cap [z|_{F_j g_j}] \cap X = [z|_{(F_i g_i \cup F_j g_j)c_i^{-1} c_i}] \cap X = E_{F_i g_i \cup F_j g_j},$$

which is an element of $\mathcal{G}$, and so we may assume without loss of generality that $c_i \neq c_j$ for $i \neq j$. For finite induction, we have that $E_{F_1 g_1}$ is non-empty, so suppose that we have shown $\bigcap_{i=1}^k E_{F_i g_i}$ is non-empty for some $k < n$. As such, there exists an element $x \in X$ such that $u = x|_{\bigcup_{i=1}^k F_i g_i} = z|_{\bigcup_{i=1}^k F_i g_i}$, meaning that $u \in \mathcal{L}(X)$. Let $v = z|_{F_{k+1} g_{k+1}}$, and note that $v \in \mathcal{L}(X)$, as $E_{F_{k+1} g_{k+1}}$ is non-empty. Now, since $F_i g_i \subset Hc_i$ for each i, and $K \subset H$ by definition of H, we have that

$$\left(\bigcup_{i=1}^k F_i g_i\right) \cap K(F_{k+1} g_{k+1}) \subset \left(\bigcup_{i=1}^k Hc_i\right) \cap H(Hc_{k+1}) = \left(\bigcup_{i=1}^k Hc_i\right) \cap Hc_{k+1}.$$

Since $c_i \neq c_j$ for $i \neq j$, we have in the rightmost set an intersection of a right coset with a union of distinct right cosets, which is necessarily empty, and so we have

$$\left(\bigcup_{i=1}^k F_i g_i\right) \cap K(F_{k+1} g_{k+1}) = \varnothing.$$

By the strong irreducibility of X, there exists $x \in X$ such that $x|_{\bigcup_{i=1}^k F_i g_i} = u$ and $x|_{F_{k+1} g_{k+1}} = v$. This gives that $x \in \bigcap_{i=1}^{k+1} E_{F_i g_i}$, so this set is non-empty. By inducing until n, we obtain that $\bigcap_{i=1}^n E_{F_i g_i}$ is non-empty. As such, $\mathcal{G}$ has the finite intersection property.

Since X is a closed subset of $\mathcal{A}^G$, which is compact, we have that X is compact. As such, since $\mathcal{G}$ is a collection of closed subsets of X with the finite intersection property, $\bigcap \mathcal{G}$ is also non-empty; in particular, there exists $x \in X$ such that

$$x \in \bigcap_{g \in G} \bigcap_{F \Subset H} E_{Fg}.$$

With $\{e\} \Subset H$, this gives that for each $g \in G$ we have $x \in E_{\{g\}} = [z|_{\{g\}}] \cap X$, which gives that $x(g) = z(g)$ and therefore $x = z \in X$. Since $z \in Y^{\uparrow G}$ was arbitrary, we have shown that $Y^{\uparrow G} \subset X$, and therefore $X = Y^{\uparrow G}$.

Finally, by Lemma 3.11, since $X = Y^{\uparrow G}$ is strongly irreducible, we have that Y is strongly irreducible. □

As is the case with SFTs, the two previous results may be trivial in the case that G is finitely generated; however, this is not the case when G is infinitely generated. Further study

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into the properties of free extensions and which properties translate from a free extension to its base shift and vice versa, may prove to show that the study of SFTs on arbitrary groups may be reducible to studying SFTs on finitely generated groups. While we do not require any further properties for the results of this paper, it may be fruitful to explore other such properties in the context of free extensions.

4. *Locally finite groups*

With the theory of free extensions sufficiently developed, we may move on to proving properties of SFTs on locally finite groups. This section contains all parts of the proofs of Theorems I and II.

We first begin by introducing the following construction, which applies to any group G which is not locally finite, and which will be referenced throughout the remainder of the section.

Definition 4.1. Let G be a non-locally finite group, and $\mathcal{A} = \{0, 1\}$. Since G is non-locally finite, there exists an infinite, finitely generated group $H \leq G$. Let $S \Subset H$ be such that $e \in S$ and $\langle S \rangle = H$. Then, taking $\mathbf{F} = \mathcal{A}^S \setminus \{0^S, 1^S\}$, where 0^S and 1^S are the constant 0 and 1 patterns, let $\mathcal{2}_H = \mathcal{X}^H[\mathbf{F}]$.

$\mathcal{2}_H$ is clearly an SFT from this construction, and in particular contains exactly two points, the constant 0 and 1 patterns on H, which will be denoted by 0^H and 1^H, respectively. By Lemma 3.10, $\mathcal{2}_H^{\uparrow G}$ is also an SFT.

4.1. *Proof of Theorem I.*

We now have everything needed to prove Theorem I. Each of the results in this section which contributes to the theorem will be marked with the implication that it provides. For instance, Proposition 4.2 is marked as (I(a) $\Longrightarrow$ I(b)) to indicate that it provides the implication that if G is locally finite, then every G-SFT is the free extension of an SFT on a finite subgroup of G. Many of these results follow readily from the properties of free extensions developed in the previous section. Theorem I is restated below for convenience.

THEOREM I. *Let G be a group. Then the following statements are equivalent.*
(a) *G is locally finite.*
(b) *Every G-SFT is the free extension of some SFT on a finite subgroup of G.*
(c) *Every G-SFT is strongly irreducible.*
(d) *Every strongly irreducible G-shift is a G-SFT.*
(e) *Every sofic G-shift is a G-SFT.*
(f) *For every G-SFT X,* Aut(X) *is locally finite.*

We begin by proving the following chain of implications:

$$\text{I(a)} \Longrightarrow \text{I(b)} \Longrightarrow \text{I(c)} \Longrightarrow \text{I(a)}$$

The first of these implications follows directly from Lemma 3.14.

PROPOSITION 4.2. (I(a) $\Longrightarrow$ I(b)) *Let G be a locally finite group, and X a G-SFT. Then there exist $H \ll G$ and an H-SFT Y such that $X = Y^{\uparrow G}$.*

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Proof. By Lemma 3.14 there exists $F \Subset G$ and $\langle F\rangle$-SFT Y such that $X = Y^{\uparrow G}$. But G is locally finite, so $H = \langle F\rangle$ is finite, which gives the desired result. □

Next we show that if G is a group for which every G-SFT is the free extension of a shift on a finite subgroup of H, then every G-SFT is strongly irreducible. In fact, we can show the following result, which is stronger; if X is a G-SFT for which there exist $H \ll G$ and H-SFT Y such that $X = Y^{\uparrow G}$, then X is strongly irreducible.

LEMMA 4.3. (I(b) $\Longrightarrow$ I(c)) *Let G be a group and X be a G-SFT such that there exist $H \ll G$ and H-SFT Y such that $X = Y^{\uparrow G}$. Then X is strongly irreducible.*

Proof. Since H is finite, Y is vacuously strongly irreducible with $K = H$. By Lemma 3.11, $X = Y^{\uparrow G}$ is strongly irreducible. □

Lastly, we prove the final implication by contrapositive, where we use the SFT $2_H^{\uparrow G}$ as an example of an SFT on non-locally finite groups which is not strongly irreducible.

LEMMA 4.4. (I(c) $\Longrightarrow$ I(a)) *Let G be a non-locally finite group. Then there exists a G-SFT which is not strongly irreducible.*

Proof. Let $H \leq G$ be an infinite, finitely generated subgroup of G, which must exist because G is not locally finite.

To show that $2_H^{\uparrow G}$ is not strongly irreducible, it is necessary to show that for all $K \Subset G$, there exist patterns $u, v \in \mathcal{L}(2_H^{\uparrow G})$ with shapes F_u and F_v respectively such that $F_u \cap KF_v = \varnothing$, but there is no $x \in X$ with $x|_{F_u} = u$ and $x|_{F_v} = v$.

Let $K \Subset G$. Since K is finite, it must be that $H \setminus K$ is non-empty, so let $h \in H \setminus K$. Let $u = 0^{\{h\}}$ and $v = 1^{\{e\}}$, which are trivially in $\mathcal{L}(2_H^{\uparrow G})$. Then $F_u = \{h\}$ and $F_v = \{e\}$, and clearly, since $h \notin K$, we have $F_u \cap KF_v = \{h\} \cap K = \varnothing$. But, for any $x \in X$, it must be that $x|_H \in \{0^H, 1^H\}$, and therefore $x|_{\{h\}} = x|_{\{e\}}$, so it cannot be that $x|_{F_u} = u$ and $x|_{F_v} = v$ simultaneously.

Therefore, $2_H^{\uparrow G}$ is not strongly irreducible. □

Next, we shall prove that I(a) $\Longrightarrow$ I(d), and prove the converse direction in the subsection immediately following, as we will need an example introduced then.

LEMMA 4.5. (I(a) $\Longrightarrow$ I(d)) *Let G be a locally finite group, and X a strongly irreducible G-shift. Then X is a G-SFT.*

Proof. By Lemma 3.17, there exist $F \Subset G$ and strongly irreducible $\langle F\rangle$-shift Y such that $X = Y^{\uparrow G}$. Since G is locally finite and F is finite, $H = \langle F\rangle$ is finite, and therefore Y is an H-SFT. By Lemma 3.10, $Y^{\uparrow G} = X$ is a G-SFT. □

4.1.1. *Sofic shifts on locally finite groups.* Next, we prove the following implication involving the statement that every sofic G-shift is a G-SFT.

$$\text{I(a)} \Longleftrightarrow \text{I(e)}$$

First, we show directly that all sofic G-shifts on locally finite groups are G-SFTs.

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LEMMA 4.6. (I(a) $\Longrightarrow$ I(e)) *Let G be a locally finite group, X be an SFT, and ϕ be a factor map. Then $\phi(X)$ is an SFT.*

Proof. By Proposition 4.2, there exist $H \ll G$ and H-SFT Y such that $X = Y^{\uparrow G}$. Let $F \Subset G$ and let $\beta : \mathcal{L}_F(X) \to \mathcal{B}$ be a block map such that $\phi = \phi_\beta^G$. Let $K = \langle H \cup F \rangle$, which is finite because G is locally finite, and let $Z = Y^{\uparrow K}$. By Lemma 3.8, we have $X = Y^{\uparrow G} = (Y^{\uparrow K})^{\uparrow G} = Z^{\uparrow G}$. Since $F \subset K$, Lemma 3.13 gives that

$$\phi(X) = \phi_\beta^G(Z^{\uparrow G}) = \phi_\beta^K(Z)^{\uparrow G}.$$

Since K is finite the K-shift $\phi_\beta^K(Z) \subset \mathcal{A}^K$ is an SFT, and by Lemma 3.10 we obtain that $\phi_\beta^K(Z)^{\uparrow G} = \phi(X)$ is an SFT. □

For the converse result, we will prove the contrapositive by constructing, for any non-locally finite group, a sofic shift which is not an SFT. We begin with the construction.

Definition 4.7. [14, Example 1.11] Let H be an infinite, finitely generated group, and let $S \Subset H$ such that $S = S^{-1}$, $e \notin S$, and $H = \langle S \rangle$. The *even H-shift* $S_{\text{even}} \subset \{0, 1\}^H$ is the set of all configurations x such that any maximal finite connected component of $x^{-1}(1) \subset H$ in the Cayley graph $\Gamma(H, S)$ has even size. To put it another way, each finite connected component of 1s has even size.

Proposition 1.10 of [14] gives that S_{even} is a sofic H-shift, but not an H-SFT. Using this, we can prove the converse result.

LEMMA 4.8. (I(e) $\Longrightarrow$ I(a)) *Let G be a non-locally finite group. Then there exists a sofic G-shift which is not a G-SFT.*

Proof. Let $H \le G$ be infinite and finitely generated. Then S_{even} as defined above is a sofic H-shift, but not an H-SFT. Let X be an H-SFT and $\phi : X \to S_{\text{even}}$ be a factor map, which must exist by the soficity of S_{even}. Then there exist $F \Subset H$ and a block map $\beta : \mathcal{L}_F(X) \to \{0, 1\}$ such that $\phi = \phi_\beta^H$. Then by Lemma 3.10 it follows $X^{\uparrow G}$ is a G-SFT, and by Lemma 3.13, we obtain

$$S_{\text{even}}^{\uparrow G} = \phi_\beta^H(X)^{\uparrow G} = \phi_\beta^G(X^{\uparrow G}),$$

and therefore $S_{\text{even}}^{\uparrow G}$ is sofic. By the contrapositive of Lemma 3.10, however, $S_{\text{even}}^{\uparrow G}$ is not an SFT. □

In addition to S_{even} being a sofic H-shift, we also have that it is strongly irreducible. With $K = (S \cup \{e\})^2$, and two patterns $u, v \in S_{\text{even}}$ with shapes F_u and F_v such that $F_u \cap KF_v = \varnothing$, we may extend u to a pattern on $(S \cup \{e\})F_u$ by using the symbol 0 or 1 in a manner that ensures this extension has an even number of 1s in any connected component of 1s. The same can be done for v. By the definition of K, we have that these extensions are disjoint, so let $x \in \{0, 1\}^H$ which matches these extensions of u and v, and is 0 elsewhere. Since the extensions of u and v have connected components of 1s of even size, x only has connected components of 1s of even size, even if a connected component

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in the extension of u is connected with a connected component of the extension of v, since both individually have even size. As such, we have the following result.

LEMMA 4.9. (I(d) $\Longrightarrow$ I(a)) *Let G be a non-locally finite group. Then there exists a strongly irreducible G-shift which is not a G-SFT.*

Proof. Let $H \leq G$ be infinite and finitely generated. Then S_{even} as defined above is a strongly irreducible H-shift, but not an H-SFT. By Lemma 3.10, $S_{\text{even}}^{\uparrow G}$ is not a G-SFT, and by Lemma 3.11, $S_{\text{even}}^{\uparrow G}$ is strongly irreducible. □

4.1.2. *Automorphism groups for locally finite SFTs.* Finally, we prove the last implications for Theorem I in the following manner.

$$\text{I(a)} \Longleftrightarrow \text{I(f)}$$

First, we show that the automorphism group for any SFT on a locally finite groups is locally finite.

LEMMA 4.10. (I(a) $\Longrightarrow$ I(f)) *Let G be locally finite and X a G-SFT. Then* $\mathrm{Aut}(X)$ *is locally finite.*

Proof. Let $F \Subset G$ be a forbidden shape for X so that $X = \mathcal{X}^G[\mathcal{F}_F(X)]$. Let $E = \{\phi_1, \phi_2, \ldots, \phi_n\} \subset \mathrm{Aut}(X)$ be a finite set of autoconjugacies, and let $K = \langle E \rangle$. Without loss of generality, E may be assumed to be symmetric. Then, for each ϕ_i, there exist $F_i \Subset G$ and block maps $\beta_i : \mathcal{L}_{F_i}(X) \to \mathcal{A}$ such that $\phi_i = \phi_{\beta_i}^G$. Now let

$$H = \left\langle F \cup \bigcup_{i=1}^{n} F_i \right\rangle.$$

H must be finite, since G is locally finite. Then, since $F \subset H$, it is the case that $X = \mathcal{X}^G[\mathcal{F}_H(X)]$, and by Lemma 3.2, we have $X = \mathcal{X}^G[\mathcal{F}_H(X)] = \mathcal{X}^H[\mathcal{F}_H(X)]^{\uparrow G}$. For simplicity, let $Y = \mathcal{X}^H[\mathcal{F}_H(X)]$. Additionally, by Corollary 3.13, for each i, we obtain

$$Y^{\uparrow G} = \phi_i(Y^{\uparrow G}) = \phi_{\beta_i}^H(Y)^{\uparrow G},$$

and therefore $Y = \phi_{\beta_i}^H(Y)$, which gives $\phi_{\beta_i}^H \in \mathrm{Aut}(Y)$. Let $C \in \mathscr{C}(H\backslash G)$. Then, for each i and j and using Lemma 3.12, we have for every $x \in Y^{\uparrow G}$ that

$$\begin{aligned}(\phi_i \circ \phi_j)(x) &= (\kappa_C \circ (\phi_{\beta_i}^H)^C \circ \kappa_C^{-1} \circ \kappa_C \circ (\phi_{\beta_j}^H)^C \circ \kappa_C^{-1})(x) \\ &= (\kappa_C \circ (\phi_{\beta_i}^H \circ \phi_{\beta_j}^H)^C \circ \kappa_C^{-1})(x),\end{aligned}$$

with $\phi_{\beta_i}^H \circ \phi_{\beta_j}^H \in \mathrm{Aut}(Y)$. As such, the behavior of each $\phi \in K$ on $Y^{\uparrow G}$ is entirely determined by an element in $\mathrm{Aut}(Y)$. Since H is finite, Y is finite, and therefore $\mathrm{Aut}(Y) \subset Y^Y$ is also finite, which gives that K must be finite. Since E was arbitrary, this gives that $\mathrm{Aut}(X)$ is locally finite. □

Lastly, we show that if the automorphism group of an SFT is locally finite, then the group on which the SFT is defined must be locally finite.

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LEMMA 4.11. (I(f) $\implies$ I(a)) *Let G be a group. If, for every G-SFT X,* Aut(X) *is locally finite, then G is locally finite.*

Proof. Since the full G-shift Σ is a G-SFT, then by assumption Aut(Σ) is locally finite. Clearly, the map $\psi : G \to \mathrm{Aut}(\Sigma)$ defined by $\psi(g) = \sigma^g$ is an injective homomorphism, since for any $h \neq g$, we have $\sigma^h \neq \sigma^g$ on Σ, as it is possible to describe a configuration which gets sent to different configurations under σ^h and σ^g. As such, $\psi(G) \leq \mathrm{Aut}(\Sigma)$. Since Aut$(\Sigma)$ is locally finite, $\psi(G)$ is locally finite. But $\psi(G)$ and G are isomorphic, and therefore G is locally finite. □

4.2. *Proof of Theorem II.* Next, we will prove Theorem II. As with the previous section, results pertaining to certain implications in Theorem II are marked. The main additional assumption we will need is that G is a countable amenable group, rather than any group. Most of these results also depend heavily on the properties of free extensions developed in the previous section. We restate Theorem II below for convenience.

THEOREM II. *Let G be a countable amenable group. Then the following statements are equivalent.*

(a) *G is locally finite.*
(b) *If X is a non-empty G-SFT with $h(X) = 0$, then $X = \{x\}$, where x is a fixed point.*
(c) *Every G-SFT is entropy minimal.*
(d) *G is locally non-torsion and*

$$\mathcal{E}(G) = \left\{ \frac{\log(n)}{|H|} : H \ll G, n \in \mathbb{N} \right\} \subset \mathbb{Q}^+_{\log}.$$

(e) *Every G-SFT has a unique measure of maximal entropy.*

Each of the equivalences in the theorem will be shown individually to be equivalent to statement II(a). We begin by showing this for statement II(b). Additionally, note that all countable locally finite groups are amenable, so we omit amenability as an assumption for a few of the results.

4.2.1. *Zero-entropy SFTs on locally finite groups.* We begin by showing that zero-entropy SFTs on locally finite groups consist of single fixed points.

LEMMA 4.12. (II(a) $\implies$ II(b)) *Let G be a countable locally finite group. Then if X is a non-empty G-SFT with $h(X) = 0$, then $X = \{x\}$ for some fixed point x.*

Proof. Let X be a G-SFT with $h(X) = 0$. Then, by assumption, $X = Y^{\uparrow G}$ for some $H \ll G$ and H-shift Y. By Proposition 3.7, we have $h(X) = h(Y) = 0$. Since H is finite,

$$0 = h(Y) = \frac{1}{|H|} \log(|Y|),$$

which implies that $|Y| = 1$. Then for any $C \in \mathscr{C}(H\backslash G)$ we have $|Y^C| = 1$, and therefore $|X| = |\kappa_C(Y^C)| = 1$ so $X = \{x\}$ for the only $x \in X$. Since X is shift invariant, it must be that x is a fixed point. □

To show the converse, recall the definition of the SFT $\mathcal{Q}_H$ from the beginning of §4.

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LEMMA 4.13. (II(b) $\Longrightarrow$ II(a)) *Let G be a countable amenable non-locally finite group. Then there exists a G-SFT X with zero topological entropy; however, $|X| > 1$.*

Proof. Let $H \leq G$ be an infinite, finitely generated subgroup. By Proposition 3.7, we have $h(2_H^{\uparrow G}) = h(2_H)$. Since G is countable and amenable, and $H \leq G$, it is the case that H is also countable and amenable, so let $\{F_n\}_{n=1}^{\infty}$ be a Følner sequence for H. Since 2_H contains exactly two points, 0^H and 1^H, it is clear to see that $\mathcal{L}_{F_n}(2_H) = \{0^{F_n}, 1^{F_n}\}$, and therefore $|\mathcal{L}_{F_n}(2_H)| = 2$. Additionally, it must be that $\lim_{n\to\infty} |F_n| = \infty$, because H is an infinite subgroup. Then

$$h(2_H^{\uparrow G}) = h(2_H) = \lim_{n\to\infty} |F_n|^{-1} \log(|\mathcal{L}_{F_n}(2_H)|) = \log(2) \lim_{n\to\infty} |F_n|^{-1} = 0.$$

Also, since $|2_H| = 2$, $|2_H^{\uparrow G}| > 1$, which gives the desired result. □

4.2.2. *Entropy minimality of SFTs on locally finite groups.* Recall that a G-SFT X is entropy minimal if for every G-shift $Y \subsetneq X$, we have $h(Y) < h(X)$. The following result shows that for a countable locally finite group, every SFT on the group is entropy minimal.

LEMMA 4.14. (II(a) $\Longrightarrow$ II(c)) *Let G be a countable locally finite group, and X be a G-SFT. Then X is entropy minimal.*

Proof. Since X is an SFT, let $F \Subset G$ be such that $X = \mathcal{X}^G[\mathcal{F}_F(X)]$. Let $Y \subsetneq X$ also be an SFT, and let $E \Subset G$ be such that $Y = \mathcal{X}^G[\mathcal{F}_E(Y)]$. Let $H = \langle F \cup E \rangle$, which is finite because E and F are finite and G is locally finite. Also we have $E, F \subset H$, and therefore $X = \mathcal{X}^G[\mathcal{F}_H(X)]$ and $Y = \mathcal{X}^G[\mathcal{F}_H(Y)]$. By Lemma 3.2, we obtain $X = (\mathcal{X}^H[\mathcal{F}_H(X)])^{\uparrow G}$ and $Y = (\mathcal{X}^H[\mathcal{F}_H(Y)])^{\uparrow G}$. Given that $Y \subsetneq X$, it must then be that $\mathcal{X}^H[\mathcal{F}_H(Y)] \subsetneq \mathcal{X}^H[\mathcal{F}_H(X)]$. With Proposition 3.10, this gives

$$\begin{aligned} h(Y) &= h(\mathcal{X}^H[\mathcal{F}_H(Y)]) = |H|^{-1} \log(|\mathcal{X}^H[\mathcal{F}_H(Y)]|) \\ &< |H|^{-1} \log(|\mathcal{X}^H[\mathcal{F}_H(X)]|) = h(\mathcal{X}^H[\mathcal{F}_H(X)]) = h(X), \end{aligned}$$

where the strict inequality follows from the fact that log is strictly increasing, and this implies that X is SFT-entropy minimal. Since X is an SFT, and SFT-entropy minimality and entropy minimality are equivalent for SFTs, we have that X is entropy minimal. □

For the converse result about entropy minimality, we again use the SFT 2_H.

LEMMA 4.15. (II(c) $\Longrightarrow$ II(a)) *Let G be a countable amenable non-locally finite group. Then there exists a G-SFT X which is not entropy minimal.*

Proof. Let $H \leq G$ be an infinite, finitely generated subgroup. We have that $2_H^{\uparrow G}$ is a G-SFT, and $h(2_H^{\uparrow G}) = 0$, an argument for which can be found in Lemma 4.13. It is clear that $\{0^G\} \subset 2_H^{\uparrow G}$, and $\{0^G\}$ is clearly a G-shift, as it is conjugate to the full G-shift on one symbol. Additionally, $h(\{0^G\}) = 0$, and therefore $2_H^{\uparrow G}$ is not entropy minimal. □

4.2.3. *The set of SFT entropies for locally finite groups.* Next, we establish the set of all entropies that SFTs can obtain for locally finite groups. The following result shows that

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II(a) implies II(d). The first part of the implication is trivial; if G is locally finite, then every finitely generated subgroup is finite, and therefore G is locally non-torsion. The second part of the implication is given below.

LEMMA 4.16. (II(a) $\Longrightarrow$ II(d)) *Let G be a countable locally finite group. Then*

$$\mathcal{E}(G) = \left\{ \frac{\log(n)}{|H|} : H \ll G, n \in \mathbb{N} \right\} \subset \mathbb{Q}^{+}_{\log}.$$

Proof. First, consider the case when G is finite. Let X be a G-SFT. Then $h(X) = (\log(|X|)/|G|) \in \mathbb{Q}^{+}_{\log}$, and so

$$\mathcal{E}(G) \subset \left\{ \frac{\log(n)}{|H|} : H \ll G, n \in \mathbb{N} \right\},$$

since $G \ll G$. Now let $H \ll G$ and $n \in \mathbb{N}$. Since G and H are finite, let $m = |G|/|H| \in \mathbb{N}$. Let $\mathcal{A}$ be a finite alphabet with $|\mathcal{A}| = n^m$. Then, let $X = \{a^G : a \in \mathcal{A}\}$, which is a G-SFT, and $|X| = n^m$. Then

$$h(X) = \frac{\log(|X|)}{|G|} = \frac{\log(n^m)}{|G|} = \frac{m\log(n)}{|G|} = \frac{\log(n)}{|H|},$$

and therefore,

$$\left\{ \frac{\log(n)}{|H|} : H \ll G, n \in \mathbb{N} \right\} \subset \mathcal{E}(G),$$

which gives the desired result.

If G is infinite, then G must be infinitely generated, and so by Corollary 3.15,

$$\mathcal{E}(G) = \bigcup_{F \Subset G} \mathcal{E}(\langle F \rangle).$$

Since G is locally finite, $H \ll G$ if and only if H is finitely generated, which gives

$$\mathcal{E}(G) = \bigcup_{H \ll G} \mathcal{E}(H) = \bigcup_{H \ll G} \left\{ \frac{\log(n)}{|K|} : K \ll H, n \in \mathbb{N} \right\}$$
$$= \left\{ \frac{\log(n)}{|H|} : H \ll G, n \in \mathbb{N} \right\}. \qquad \square$$

Many locally finite groups do not satisfy $\mathcal{E}(G) = \mathbb{Q}^{+}_{\log}$, due to the lack of subgroups of certain orders. For example, $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ is locally finite, but only has subgroups of order 2^n. However, there are locally finite groups which do attain $\mathcal{E}(G) = \mathbb{Q}^{+}_{\log}$, with the most prominent example likely being Hall's universal group $\mathbb{U}$ [10], which has the property that every countable locally finite group can be embedded within it, which includes all finite groups. As such, it has finite subgroups of every order, and so $\mathcal{E}(\mathbb{U}) = \mathbb{Q}^{+}_{\log}$.

A direct converse of the previous lemma has been elusive to the author, which is the reason for the additional statement that G is locally non-torsion in II(d). The following lemma gives the most general form of a converse that has been found by the author.

LEMMA 4.17. *Let G be a countable amenable group such that $\mathcal{E}(G) \subset \mathbb{Q}^{+}_{\log}$. Then G is periodic.*

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Proof. We proceed by the contrapositive. Let G be a group which is not periodic, meaning that there exists $h \in G$ whose order is infinite. Let $H = \langle h \rangle$ so that H is isomorphic to $\mathbb{Z}$, and define $\mathbf{F} = \{1^{\{e,h\}}\} \subset \{0, 1\}^{\{e,h\}}$, and let $X = \mathcal{X}^H[\mathbf{F}]$. Since G is amenable, and $H \leq G$, it must be that H is amenable. Then X is conjugate to the well-known golden mean shift on $\mathbb{Z}$, so $h(X) = \log(\varphi)$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. X is also clearly an SFT, so by Lemma 3.10 the G-shift $X^{\uparrow G}$ is an SFT, and by Proposition 3.7 we have $h(X^{\uparrow G}) = h(X) = \log(\varphi)$. It is an elementary number theory exercise to show that φ^n is irrational for all $n \in \mathbb{N}$, and so it must be that for any $n, m \in \mathbb{N}$, we have $\varphi^m \neq n$. Therefore for all $n, m \in \mathbb{N}$, it is the case that $\log(\varphi) \neq (\log(n)/m)$, so $\log(\varphi) \notin \mathbb{Q}^+_{\log}$. But $\log(\varphi) \in \mathcal{E}(G)$, and therefore $\mathcal{E}(G) \not\subset \mathbb{Q}^+_{\log}$. □

It remains to show that periodic but not locally finite groups have SFTs with entropy outside of $\mathbb{Q}^+_{\log}$; however, it is in general quite difficult to construct SFTs on such groups in a manner conducive to computing its topological entropy. As a result, we instead add the statement that G is locally non-torsion, which removes the need to consider such groups.

LEMMA 4.18. (II(d) $\Longrightarrow$ II(a)) *Let G be a countable amenable group which is locally non-torsion, and $\mathcal{E}(G) \subset \mathbb{Q}^+_{\log}$. Then G is locally finite.*

Proof. By Lemma 4.17, G is periodic. Let $F \Subset G$, and consider $H = \langle F \rangle$. Since G is periodic, H is periodic. Since G is locally non-torsion, H is finite or not periodic, and therefore H must be finite. Since $F \Subset G$ was arbitrary, G is locally finite. □

The author suspects that if $\mathcal{E}(G) \subset \mathbb{Q}^+_{\log}$, then $\mathcal{E}(G)$ must be locally finite. This would allow for statement II(d) to have the locally non-torsion assumption removed, and only leave $\mathcal{E}(G) \subset \mathbb{Q}^+_{\log}$.

4.2.4. *Measures of maximal entropy for SFTs on locally finite groups.* Finally, we show that every SFT on a countable locally finite group has a unique measure of maximal entropy, and that if every SFT on a countable amenable group has a unique measure of maximal entropy, then the group must be locally finite. First, we require a simple but powerful result about the topological structure of SFTs on countable locally finite groups.

LEMMA 4.19. *Let G be a countable locally finite group, and let X be a G-SFT. Then there exists a sequence $\{H_n\}_{n=1}^{\infty}$ with $H_n \leq H_{n+1} \ll G$ for all n, such that $G = \bigcup_{n \in \mathbb{N}} H_n$, and there exist H_n-SFTs Y_n such that $X = Y_n^{\uparrow G}$ for all n. Furthermore, the set*

$$\mathfrak{B}[\{Y_n\}] = \{[y] \cap X : n \in \mathbb{N}, y \in Y_n\}$$

is a basis for the subspace topology on X.

Proof. First, since X is a G-SFT, by Proposition 4.2, there exist $H_1 \ll G$ and H_1-SFT Y_1 such that $X = Y_1^{\uparrow G}$. Then, since G is countable, let $G = \{g_n : n \in \mathbb{N}\}$ be an enumeration of G. Define, for $n \geq 2$,

$$H_n = \langle H_1 \cup \{g_i : i < n\} \rangle.$$

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Since H_1 and $\{g_i : i < n\}$ are both finite, H_n is finitely generated, and therefore finite. Furthermore, for any $g \in G$, there is some $n \in \mathbb{N}$ for which $g = g_n$, and clearly $g_n \in H_{n+1}$. Also, $H_n \leq H_{n+1}$.

Now, for each $n \geq 2$, let $Y_n = Y_1^{\uparrow H_n}$. By Lemma 3.8, we obtain $X = Y_1^{\uparrow G} = (Y_1^{\uparrow H_n})^{\uparrow G} = Y_n^{\uparrow G}$.

Finally, let $\mathfrak{B}$ be the standard basis of all cylinder sets for X. To show that $\mathfrak{B}[\{Y_n\}]$ is a basis for the topology on X, first note that $\mathfrak{B}[\{Y_n\}] \subset \mathfrak{B}$, and therefore it suffices to show that any set in $\mathfrak{B}$ can be constructed by sets in $\mathfrak{B}[\{Y_n\}]$. Let $w \in \mathcal{L}(X)$ so that $[w] \cap X$ is non-empty, and let F be the shape of w. Since $G = \bigcup_{n \in \mathbb{N}} H_n$ and $H_n \leq H_{n+1}$, it follows there exists $N \in \mathbb{N}$ such that $F \subset H_N$. Then it is clear that

$$[w] \cap X = \bigcup_{z \in [w]_{H_N} \cap Y_N} [z] \cap X,$$

which implies that $\tau(\mathfrak{B})$, the topology generated by $\mathfrak{B}$, is contained in $\tau(\mathfrak{B}[\{Y_n\}])$, so $\mathfrak{B}[\{Y_n\}]$ is a basis for the topology on X. □

LEMMA 4.20. (II(a) $\Longrightarrow$ II(e)) *Let G be a countable locally finite group. Then for any G-SFT X, there exists a unique measure of maximal entropy.*

Proof. Since shift actions of countable amenable groups are expansive, the map $\mu \mapsto h_\mu(X)$ is upper semi-continuous [7, Theorem 2.1], and so X has a measure of maximal entropy $\mu \in \mathcal{M}(X)$ such that $h_\mu(X) = h(X)$.

By Lemma 4.19, there exist $\{H_n\}_{n=1}^{\infty}$ and H_n-SFTs Y_n such that $\mathfrak{B}[\{Y_n\}]$ is a basis for the topology on X, and therefore also generates the Borel σ-algebra on X. Furthermore, since $X = Y_n^{\uparrow G}$, Lemma 3.7 gives that

$$h(X) = h(Y_1) = h(Y_n) = \frac{\log(|Y_1|)}{|H_1|} = \frac{\log(|Y_n|)}{|H_n|}$$

for all $n \in \mathbb{N}$. Also note that $\{H_n\}$ is a Følner sequence for G, and therefore

$$h_\mu(X) = \inf_n \frac{H_\mu(X, H_n)}{|H_n|}.$$

As such, we obtain

$$h_\mu(X) \leq \frac{H_\mu(X, H_n)}{|H_n|}$$

for all $n \in \mathbb{N}$. But

$$H_\nu(X, H_n) \leq \log(|\mathcal{L}_{H_n}(X)|) = \log(|Y_n|)$$

for any $\nu \in \mathcal{M}(X)$, and therefore

$$\frac{\log(|Y_n|)}{|H_n|} = h(X) = h_\mu(X) \leq \frac{H_\mu(X, H_n)}{|H_n|} \leq \frac{\log(|Y_n|)}{|H_n|},$$

so all of these quantities must be equal, which further implies that for every n and $y \in Y_n$, we have $\mu[y] = 1/|Y_n|$. This is true for any $n \in \mathbb{N}$, and therefore any measure of maximal entropy must take these specific values for every element of $\mathfrak{B}[\{Y_n\}]$. By the Carathéodory

Extension Theorem, there exists a unique Borel probability measure with these properties, and therefore there exists only one measure of maximal entropy. □

Though the previous proof does not explicitly mention how to construct the measure of maximal entropy, its construction is fairly simple. For a countable locally finite group G and G-SFT X, take some $H \ll G$ and H-SFT Y such that $X = Y^{\uparrow G}$. Let ν be a measure on Y defined by $\nu(y) = 1/|Y|$ for all $y \in Y$. Then for any $C \in \mathscr{C}(H\backslash G)$, the pushforward measure $\mu = (\nu)^C \circ \kappa_C^{-1}$ is an invariant measure of maximal entropy for X. Informally, μ is the uniform measure on X, which is obtained as the push forward of a product measure under a construction function. It can also be shown that μ is independent of choice of H and Y for which $X = Y^{\uparrow G}$.

For the converse result, we give an SFT on any non-locally finite group which has multiple measures of maximal entropy.

LEMMA 4.21. ((e) $\Longrightarrow$ (a)) *Let G be a countable amenable non-locally finite group. Then there exists a G-SFT X which has multiple measures of maximal entropy.*

Proof. Since $h(2_H^{\uparrow G}) = 0$, the Variational Principle gives that for all $\mu \in \mathcal{M}(X)$, we have $0 \leq h_\mu(2_H^{\uparrow G}) \leq h(2_H^{\uparrow G}) = 0$, and so $h_\mu(2_H^{\uparrow G}) = h(2_H^{\uparrow G})$. This means every measure $\mu \in \mathcal{M}(X)$ is a measure of maximal entropy.

Since 0^G and 1^G are both elements of $2_H^{\uparrow G}$, the two Dirac measures δ_{0^G} and δ_{1^G} are distinct, and since both 0^G and 1^G are fixed points they are both invariant, and therefore contained within $\mathcal{M}(2_H^{\uparrow G})$. As such, $2_H^{\uparrow G}$ has at least two measures of maximal entropy. □

5. *Final remarks*

The main results of this paper give that the class of locally finite groups presents interesting dynamical behaviors that are unexpected in general. Combined with the converse results, which show these interesting behaviors are unique to locally finite groups, this gives insights into the types of groups where interesting behavior is possible. As mentioned in §1, Theorem II(b) gives that the only groups for which there are only trivial zero-entropy dynamics are precisely the locally finite groups, and this property indirectly answers in the affirmative Question 3.19 of Barbieri [1]: 'Does there exist an amenable group G and a G-SFT which does not contain a zero-entropy G-SFT?' For any countable locally finite group G, take any finite $H \ll G$ with $|H| > 1$, and pick any H-SFT Y which does not contain any fixed points. Then $X = Y^{\uparrow G}$ also does not contain a fixed point, and therefore contains no zero-entropy SFTs. This answer to the question leads to the following refinement of the question, as infinite locally finite groups are necessarily infinitely generated.

Question. Do there exist an infinite, *finitely generated* amenable group G and a G-SFT which does not contain a zero-entropy G-SFT?

Theorem II(d) also aids in the overall classification of the possible sets which are attainable as the set of entropies of SFTs on a specific amenable group. In the case that

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G is locally finite, $\mathcal{E}(G) \subset \mathbb{Q}^+_{\log}$ (and, in particular, an exact form for $\mathcal{E}(G)$ is known). In the case that G is not periodic, then it contains an element of infinite order (and therefore a subgroup isomorphic to $\mathbb{Z}$), and thus, by Lemma 3.14, $\mathcal{E}(\mathbb{Z}) \subset \mathcal{E}(G)$. Although more research is needed to classify $\mathcal{E}(G)$ exactly for these types of groups (such as the work of Barbieri [1]), at least it is known that $\mathbb{Z}$-SFT entropies are attainable. The remaining class of groups are the finitely generated amenable torsion groups. We have shown in Lemma 4.17 that $\mathcal{E}(G) \subset \mathbb{Q}^+_{\log}$ does imply that the G is periodic; however, it is unclear whether the following question can be answered in the affirmative.

Question. If G is a countable amenable group such that $\mathcal{E}(G) \subset \mathbb{Q}^+_{\log}$, then must it be the case that G is locally finite? If not, is

$$\mathcal{E}(G) = \left\{ \frac{\log(n)}{|H|} : H \ll G, n \in \mathbb{N} \right\}$$

sufficient to conclude that G is locally finite?

Answering either of these questions in the affirmative would permit the locally non-torsion statement II(d) to be dropped, leading to a strictly stronger result. Following the method used in proving that $\mathcal{E}(G) \subset \mathbb{Q}^+_{\log}$ implies periodicity, it would suffice to produce, for any finitely generated, amenable, torsion group H, an H-SFT with entropy outside of $\mathbb{Q}^+_{\log}$. Then, for any amenable torsion group G which is not locally finite, it must contain a finitely generated torsion subgroup H (potentially the whole group), and this SFT can be defined on H, and then freely extended to G with the same entropy. Defining SFTs is not difficult in general; the primary difficulty is in computing their entropy, especially when arbitrary finitely generated groups are considered.

Including strengthening statement II(d), there are likely other statements that could be added to Theorems I and II. The types of dynamical properties explored in this work are by no means exhaustive, so future work may be able to add to these theorems, and any such work will likely use free extensions extensively as they have been used here. In addition to extending these theorems, expanding the theory of free extensions may be fruitful in the study of shifts on groups. For instance, while the forward direction of Lemma 3.10 is true in greater generality using the more general embeddings of Barbieri [1], the reverse direction for the specific case of free extensions is, to the knowledge of the author, a new result. Lemmas 3.14, 3.16 and 3.17 indicate that the study of SFTs, sofic shifts and strongly irreducible shifts may be reduced to the study of such shifts on finitely generated groups. Furthermore, it is possible to take a minimal such finitely generated subgroup, so that the shift may not be further reduced from the perspective of free extensions. Such shifts may be considered *intrinsic* to the group, in the sense that they do not arise as the free extension of any shift on a proper subgroup.

Given that Lemmas 3.14, 3.16 and 3.17 give strong connections between free extensions, and the finite type, sofic and strongly irreducible properties, along with Lemmas 3.10 and 3.11 giving that the finite type and strongly irreducible properties transfer readily between a free extension and its base shift, there is some indication that a similar result may exist for sofic shifts. Lemmas 3.12 and 3.10 readily give that if a shift is sofic, then any free extension of it is also sofic; however, the converse result is not so simple. Jeandel first

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posed whether the free $\mathbb{Z}^2$-extension of a $\mathbb{Z}$-shift X being sofic implies that X is sofic, which has remained open since at least 2011 [**19**]. We may say that a group G *has property S* if, for any subgroup $H \leq G$ and H-shift Y, the G-shift $Y^{\uparrow G}$ being sofic implies that Y is sofic. Jeandel's question may then be posed more generally for all groups as follows.

Question. Which groups have property S?

By Theorem I and Lemma 3.10, we have that any locally finite group has property S, and so there are groups with this property. However, not all groups have property S, as Barbieri, Sablik and Salo have shown that a certain class of non-amenable G do not have property S [**2**]. It remains to be seen whether groups such as $\mathbb{Z}$ and $\mathbb{Z}^2$ have property S, and perhaps whether amenable or sofic groups have property S.

Lastly, the mere existence of the two main theorems suggests that it may be possible to classify other dynamical properties by properties of the group, such as property S. To the knowledge of the author, these results may be the only results in symbolic dynamics that give implications about the group only from dynamical properties of the group, let alone a complete characterization of the group by dynamical properties. By assuming additional structure on the group, it may be possible to characterize other dynamical properties by this structure, and derive similar theorems to Theorems I and II for other classes of groups.

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