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PRIME IDEAL CHARACTERIZATION OF CHAIN BASED LATTICES

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Abstract

Epstein and Horn, in their paper 'Chain based lattices', characterized P_1-lattices, and P_2-lattices in terms of their prime ideals. But no such prime ideal characterization for P_0-lattices was given. Our main aim in this paper is to characterize P_0-lattices in terms of their prime ideals. We also give a necessary and sufficient condition for a P-algebra to be a P_0-lattice (and hence a P_2-lattice).

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0. Introduction

Traczyk (1963) introduced the concept of chain based lattices (P_0-lattices) as an abstraction of Post algebras. Epstein and Horn (1975) studied chain based lattices in detail and obtained a prime ideal characterization of these lattices in special cases, namely, P_1-lattices and P_2-lattices. But no such prime ideal characterization for P_0-lattices was given. In this paper, we give a prime ideal characterization for P_0-lattices. The main tool used in this paper is that every bounded distributive lattice is isomorphic with the lattice of all global sections of a sheaf of bounded distributive lattices over a Boolean space (See Maddana Swamy (1974) and Subrahmanyam (1978)).

Epstein and Horn (1975) (Theorem 7.3) proved that the prime ideals of a P_0-lattice of order n lie in disjoint maximal chains each with at most $n-1$ elements and they have shown that the converse is not true even in the case of a P-algebra. In this paper, we prove that a P-algebra L is a P_0-lattice (and hence a P_2-lattice) if and only if the prime ideals of L lie in disjoint maximal chains each with at most $n-1$ elements for some integer n and satisfy the continuity axiom (see Definition 3.3 and Theorem 3.5

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below). Further, we characterize P_1-lattices and P_2-lattices in terms of the stalks of the corresponding sheaves and such characterization for P_0-lattices is given by Maddana Swamy and Manikyamba (to appear) (Theorem 5.6).

Throughout this paper, by L, we mean a (nontrivial) bounded distributive lattice $(L, \vee, \wedge, 0, 1)$ and $B = B(L)$, the centre of L. The dual of L is denoted by L^d. For any $a \in B$, we write a' to denote the complement of a. For any $x \in L$, The principal ideal generated by x is denoted by $(x]$. For any $x, y \in L$, the largest element $z \in L$ such that $x \wedge z \leqslant y$ (if exists) is denoted by $x \to y$ and the largest element $a \in B$ such that $x \wedge a \leqslant y$ (if exists) is denoted by $x \Rightarrow y$. If, for every $x, y \in L$, $x \to y$ $(x \Rightarrow y)$ exists, then we say that L is a Heyting algebra (B-algebra). Further if, for any $x, y \in L$,

$$(x \to y) \vee (y \to x) = 1((x \Rightarrow y) \vee (y \Rightarrow x) = 1)$$

then L is called a L-algebra (P-algebra). For any $x \in L$, we write $!x$ to denote $1 \Rightarrow x$, (if it exists), and call it the pseudosupplement of x. We refer to Birkhoff (1967) and Epstein and Horn (1974) for the elementary properties of these types of lattices.

By a sheaf of bounded distributive lattices we mean a triple $(\mathscr{S}, \pi, X)$ satisfying the following:

(1) $\mathscr{S}$ and X are topological spaces.

(2) $\pi : \mathscr{S} \to X$ is a local homeomorphism.

(3) Each stalk $\pi^{-1}(p)$, $p \in X$, is a bounded distributive lattice.

(4) The maps $(x, y) \mapsto x \vee y$ and $(x, y) \mapsto x \wedge y$ from

$$\mathscr{S} \vee \mathscr{S} := \{(x, y) \in \mathscr{S} \times \mathscr{S} \mid \pi(x) = \pi(y)\}$$

into $\mathscr{S}$ are continuous.

(5) The maps $\hat{0} : p \mapsto 0(p)$ *and* $\hat{1} : p \mapsto 1(p)$ of X into $\mathscr{S}$ are continuous where $0(p)$ and $1(p)$ are the smallest and largest elements of $\pi^{-1}(p)$ respectively.

We call $\mathscr{S}$ the sheaf space, X the base space and π the projection map. We write $\mathscr{S}_p$ for $\pi^{-1}(p)$ and call $\mathscr{S}_p$ the stalk of $\mathscr{S}$ at p. By a (global) section of the sheaf $(\mathscr{S}, \pi, X)$, we mean a continuous map $\sigma : X \to \mathscr{S}$ such that $\pi \circ \sigma = id_X$, identity map of X. For any two sections σ and τ, $\{p \in X \mid \sigma(p) = \tau(p)\}$ is open. For the preliminary results on sheaf theory, we refer to the pioneering work of Hoffmann (1972).

By Spec L we mean the set $Y = \mathscr{P}(L)$ of all prime ideals of L with the hull-kernel topology; the topology for which $\{Y_x \mid x \in L\}$ is a base, where for any $x \in L$, $Y_x = \{P \in Y \mid x \notin P\}$. Throughout this paper X denotes Spec B, which is a Boolean space; a compact, Hausdorff and totally disconnected space. Since $a \mapsto X_a$ is a Boolean isomorphism of B onto the Boolean algebra of all clopen subsets of X, we identify a and X_a and write simply a for X_a. We write Y^m to denote the subspace of Spec L consisting of all minimal prime ideals of L with the relative topology. We write, for any $x \in L$, Y^m_x for $Y^m \cap Y_x$. For any $p \in X$, let $\mathscr{S}_p$ be the quotient lattice L/θ_p, where θ_p is the congruence on L given by

$$(x, y) \in \theta_p \quad \text{if and only if } x \wedge a = y \wedge a \quad \text{for some } a \in B - p$$

and let $\mathscr{S}$ be the disjoint union of all $\mathscr{S}_p$, $p \in X$. For each $x \in L$, define $\hat{x} : X \to \mathscr{S}$ by $\hat{x}(p) = \theta_p(x)$, the congruence class of θ_p containing x. Topologize $\mathscr{S}$ with the largest topology such that each $\hat{x}$, $x \in L$, is continuous. In this topology $\{\hat{x}(U) \mid U$ is a neighbourhood of $p\}$ forms a basis for the neighbourhoods of $\hat{x}(p)$. Define $\pi : \mathscr{S} \to X$ by $\pi(s) = p$ if $s \in \mathscr{S}_p$. The following theorem is the main tool used in this paper and is due to Subrahmanyam (1978) (see also Maddana Swamy (1974)).

THEOREM 0.1.
(1) $(\mathscr{S}, \pi, X)$ *described above is a sheaf of bounded distributive lattices in which each stalk* $\mathscr{S}_p$ *has exactly two complemented elements, namely* $\hat{0}(p)$ *and* $\hat{1}(p)$.
(2) *For any* $a \in B$, $p \in X$, $\hat{a}(p) = \hat{1}(p)$ *if* $p \in a$ *and* $\hat{a}(p) = \hat{0}(p)$ *if* $p \notin a$.
(3) *For any* $x, y \in L$ *and* $a \in B$, $\hat{x}|_a = \hat{y}|_a$ *if and only if* $x \wedge a = y \wedge a$.
(4) $x \mapsto \hat{x}$ *is an isomorphism of* L *onto the lattice* $\Gamma(X, \mathscr{S})$ *of all global sections of the sheaf* $(\mathscr{S}, \pi, X)$. *We identify* $\hat{x}$ *with* x *and write simply* x *for* $\hat{x}$.
(5) *For any prime ideal* P *of* L, *there exists a unique* $p \in X$ *such that* $P_p := \{x(p) \mid x \in P\}$ *is a prime ideal of* $\mathscr{S}_p$. *On the other hand, if* Q *is a prime ideal of* $\mathscr{S}_p$ *where* $p \in X$, *then* $\{x \in L \mid x(p) \in Q\}$ *is a prime ideal of* L. *This correspondence exhibits the set of all prime ideals of* L *as the disjoint union of the sets of prime ideals of the stalks. Moreover, if* P *and* Q *are prime ideals of distinct stalks* $\mathscr{S}_p$ *and* $\mathscr{S}_q$, *the* P *and* Q *are incomparable when they are regarded as prime ideals of* L.

Throughout this paper, by stalk $\mathscr{S}_p$, $p \in X$, we mean the stalk of the sheaf $(\mathscr{S}, \pi, X)$ described above.

For any $x, y \in L$, we write $(x, y)^*_L$ for the ideal $\{z \in L \mid x \wedge z \leqslant y\}$ of L and by $(x, y)^*_B$, we mean the ideal $(x, y)^*_L \cap B$ of B. We write $(x)^*_L$ for $(x, 0)^*_L$. L is said to be dense if $(x)^*_L = \{0\}$ for all $0 \neq x \in L$. Following Cignoli (1971, 1978), L is said to be B-normal if, for any $x, y \in L$, $(x \wedge y)^*_B = (x)^*_B \vee (y)^*_B$, where $\vee$ denotes the join operation in the lattice of ideals of B and L is said to be B-completely normal if, for any $x, y \in L$, $(x, y)^*_B \vee (y, x)^*_B = B$.

Since, for any $p \in X$, the stalk $\mathscr{S}_p$ is dense if and only if ($(x \wedge y)^*_B \nsubseteq p \Leftrightarrow (x)^*_B \nsubseteq p$ or $(y)^*_B \nsubseteq p$ for all $x, y \in L$), the following theorem is a consequence of the results of Cignoli (1971).

THEOREM 0.2. *The following are equivalent.*
(1) L *is* B-*normal.*
(2) *For any* $x, y \in L$ *such that* $x \wedge y = 0$, $(x)^*_B \vee (y)^*_B = B$.
(3) *For any* $p \in X$, *the ideal* (p) *of* L *generated by* p *is prime.*
(4) *For any* $a, b \in B$, $a \leqslant b$, $[a, b]_L := \{x \in L \mid a \leqslant x \leqslant b\}$ *is* B-*normal.*
(5) *For any* $a \in B$, $[0, a]_L$ *is* B-*normal.*
(6) *Each stalk* $\mathscr{S}_p$, $p \in X$, *is dense.*

Following Cornish (1972), L is said to be normal if every prime ideal of L contains a unique minimal prime ideal and L is said to be relatively normal if each interval $[x, y]$ in L is normal. It can be observed that every B-completely normal lattice is relatively normal. But the converse is not true (see Theorem 0.4 below). If both L and L^d are normal (relatively normal), then we say that L is doubly normal (doubly relatively normal). The following is a routine verification by using the results of Cornish (1972).

THEOREM 0.3. *The following are equivalent.*
(1) *L is doubly relatively normal.*
(2) *L is relatively normal and L^d is normal.*
(3) *L is normal and L^d is relatively normal.*
(4) Spec *L is a disjoint union of maximal chains.*

Cignoli (1978) proved that every B-completely normal lattice is isomorphic with the lattice of all global sections of a sheaf of chains over a Boolean space. If L is B-completely normal, then our stalks $\mathscr{S}_p$ turn out to be chains (see Theorem 0.4 below) and our sheaf $(\mathscr{S}, \pi, X)$ coincides with that of Cignoli (1978). In the following theorem, the equivalence of (1), (2) and (4) is proved by Cignoli (1978) (Theorem 2.1), (2) $\Leftrightarrow$ (6) is proved by Subrahmanyam (1978) and the equivalence of (1), (3) and (5) follows from Theorem 0.3 and the fact that a dense distributive lattice has a unique minimal prime ideal.

THEOREM 0.4. *The following are equivalent.*
(1) *L is B-completely normal.*
(2) *For any $x, y \in L$, there exists $a \in B$ such that $x \wedge a \leqslant y$ and $y \wedge a' \leqslant x$.*
(3) *L is relatively normal and L^d is B-normal.*
(4) *L is B-normal and L^d is relatively normal.*
(5) *L^d is B-completely normal.*
(6) *Each stalk $\mathscr{S}_p$, $p \in X$, is a chain.*

1. P_0-lattices

DEFINITION 1.1. (Traczyk (1963).) If there is a chain $0 = e_0 \leqslant e_1 \leqslant \dots \leqslant e_{n-1} = 1$ in L such that L is generated by $B \cup \{e_0, e_1, \dots, e_{n-1}\}$, then we say that $\langle L; e_0, e_1, \dots, e_{n-1}\rangle$ is a P_0-lattice. In this case $\{e_0, e_1, \dots, e_{n-1}\}$ is called a *chain base* for L.

From Theorem 0.4 above and from Theorem 5.6 of Maddana Swamy and Manikyamba (to appear), it follows that every P_0-lattice is B-completely normal.

DEFINITION 1.2. Let $S \subseteq L$. Then we say that the set $\mathscr{P}(L)$ of prime ideals of L is determined by S if $0, 1 \in S$ *and for each* $P \in \mathscr{P}(L)$, there exists $s \in S$ and a minimal prime ideal $P_0 \subseteq P$ such that $P = (s, P_0]$, the ideal of L generated by s and P_0.

THEOREM 1.3. *Let L be a B-completely normal lattice, S a finite subset of L and $0, 1 \in S$. Then the following are equivalent.*

(1) *S together with B generate L.*

(2) *$s \mapsto s(p)$ is a surjective map of S onto $\mathscr{S}_p$ for each $p \in X$.*

(3) *$\mathscr{P}(L)$ is determined by S.*

PROOF. (1) $\Leftrightarrow$ (2) is an imitation of the proof of Theorem 5.6 of Maddana Swamy and Manikyamba (to appear).

(2) $\Rightarrow$ (3) : Let $P \in \mathscr{P}(L)$ and P_0 be the unique minimal prime ideal of L contained in P. Write $p = P_0 \cap B \in X$. Then it can be seen that $P_p := \{x(p) \mid x \in P\}$ is a prime ideal of $\mathscr{S}_p$. Since $\mathscr{S}_p$ is a finite chain, there exists $s \in S$ such that $P_p = (s(p)]$. We show that $P = (s, P_0]$. If $x \in P$, then $x(p) \leqslant s(p)$, so that there exists $a \in B - p$ such that $x \wedge a \leqslant s$. Now $a' \in p$ and $x \leqslant s \vee a' \in (s, P_0]$ and hence $P \subseteq (s, P_0]$ and the other inequality follows from the fact that $\{y \in L \mid y(p) \in P_p\} = P$.

(3) $\Rightarrow$ (2): Let $x \in L$ and $p \in X$. We may assume that $x(p) \neq 1(p)$. Then $(x(p)] \in \mathscr{P}(\mathscr{S}_p)$ so that $P = \{y \in L \mid y(p) \leqslant x(p)\} \in \mathscr{P}(L)$. Hence $P = (s, P_0]$ for some $s \in S$, where P_0 is the unique minimal prime ideal contained in P. Also, since $s \in P$, $s(p) \leqslant x(p) \neq 1(p)$ and hence $Q := \{y \in L \mid y(p) \leqslant s(p)\} \in \mathscr{P}(L)$. Now $Q \subseteq P$ and hence $P_0 \subseteq Q$. Since $s \in Q$, we have $P \subseteq Q$ and hence $P = Q$. Therefore $x \in Q$, so that $x(p) \leqslant s(p)$. Hence $x(p) = s(p)$. Thus (2) follows.

THEOREM 1.4. *Let L be a B-completely normal lattice. Then the following are equivalent.*

(1) *L is P_0-lattice.*

(2) *$\mathscr{P}(L)$ is determined by a finite subset of L.*

(3) *$\mathscr{P}(L)$ is determined by a finite chain in L.*

PROOF. (1) $\Leftrightarrow$ (3) follows from the above theorem and Theorem 5.6 of Maddana Swamy and Manikyamba (to appear) and (3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) : Let S be a finite subset of L which determines $\mathscr{P}(L)$. For each $p \in X$, since $s \mapsto s(p)$ is a surjective map of S onto $\mathscr{S}_p$, there exists a partition $\alpha_p = \{A_{1p}, A_{2p}, \ldots, A_{n_p p}\}$ of S such that, for $1 \leqslant i \leqslant n_p$, $x(p) = y(p)$ for all $x, y \in A_{ip}$ and $x(p) < y(p)$ whenever $x \in A_{ip}$ and $y \in A_{i+1\,p}$. Hence there exists $a_p \in B - p$ such that, for $1 \leqslant i \leqslant n_p$, $x \wedge a_p = y \wedge a_p$ for all $x, y \in A_{ip}$ and $x \wedge a_p \leqslant y \wedge a_p$ whenever $x \in A_{ip}$ and $y \in A_{i+1\,p}$. By the usual compactness argument in the Boolean space X, there exist a partition $\{a_1, \ldots, a_k\}$ of B and partitions $\alpha_j = \{A_{1j}, A_{2j}, \ldots, A_{n_j j}\}$, $1 \leqslant j \leqslant k$, of S such that, for $1 \leqslant i \leqslant n_j$ and $1 \leqslant j \leqslant k$, $x \wedge a_j = y \wedge a_j$ for all $x, y \in A_{ij}$ and

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$x \wedge a_j \leqslant y \wedge a_j$ whenever $x \in A_{ij}$ and $y \in A_{i+1j}$. Put $n = \max_{1 \leqslant j \leqslant k} n_j$. If, for any j, $n_j < n$, then we write $A_{ij} = A_{n_j j}$ for all $n_j < i \leqslant n$. Choose $x_{ij} \in A_{ij}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$ and write $e_i = \bigvee_{j=1}^{k} (x_{ij} \wedge a_j)$ for $1 \leqslant i \leqslant n$. For any $p \in X$, $p \in a_j$ for exactly one j and hence $e_i(p) = x_{ij}(p) \leqslant x_{i+1j}(p) = e_{i+1}(p)$, which shows that $\{e_1, \ldots, e_n\}$ is a chain in L. Now, let $x \in L$ and $p \in X$. Then there exists $s \in S$ such that $x(p) = s(p)$. Choose j, $1 \leqslant j \leqslant k$, such that $p \in a_j$. Then $s \in A_{ij}$ for some i, $1 \leqslant i \leqslant n$, so that $x(p) = s(p) = x_{ij}(p) = e_i(p)$. Hence, by Theorem 1.3 above, it follows that $\{0, e_0, e_1, \ldots, e_n, 1\}$ determines $\mathscr{P}(L)$.

2. P_1-lattices

DEFINITION 2.1. (Epstein and Horn (1975).) A P_1-lattice is a P_0-lattice $\langle L; e_0, e_1, \ldots, e_{n-1} \rangle$ such that $e_{i+1} \to e_i = e_i$ for $0 \leqslant i \leqslant n-2$.

DEFINITION 2.2. If $C = \{x_0 < x_1 < \ldots < x_m\}$ is a finite chain and n is a positive integer, then by the nth element of C we mean x_n if $n < m$ and x_m if $n \geqslant m$.

THEOREM 2.3. *Let L be a B-completely normal lattice and each maximal chain of prime ideals of L contains at most $n-1$ elements. Then L is a P_1-lattice if and only if, for any $x \in L$ and $0 \leqslant i \leqslant n-1$, $G_{ix} := \{p \in X \mid x(p)$ is the ith element of $\mathscr{S}_p\}$ is open.*

PROOF. (Thanks to the referee for suggesting this proof which is simpler than the original one.) Suppose L is a P_1-lattice. Then by Theorems 7.5 and 3.3 of Epstein and Horn (1975), there exist $e_0, e_1, \ldots, e_{n-1}$ such that $\langle L; e_0, e_1, \ldots, e_{n-1} \rangle$ is a P_1-lattice. Now $e_i(p) = e_{i+1}(p)$ implies $e_i(p) = 1(p)$ *because, if* $b \in B - p$ is such that $b \wedge e_{i+1} \leqslant e_i$, then $b \leqslant e_{i+1} \to e_i = e_i$. Therefore, by Definition 2.2, $e_i(p)$ is the ith element of $\mathscr{S}_p$. Hence G_{ix} is open.

Conversely suppose G_{ix} is open for all $x \in L$ and $0 \leqslant i \leqslant n-1$. Hence there exists a chain

$$0 = e_0 \leqslant e_1 \leqslant \ldots \leqslant e_{n-1} = 1$$

in L such that $e_i(p)$ is the ith element of $\mathscr{S}_p$ for all $p \in X$ and $0 \leqslant i \leqslant n-1$. Since, for any $x \in L$ and $p \in X$, $x(p) = e_i(p)$ for some i, $0 \leqslant i \leqslant n-1$, by Theorem 1.3 above, $\langle L; e_0, e_1, \ldots, e_{n-1} \rangle$ is a P_0-lattice. Suppose $x \wedge e_{i+1} \leqslant e_i$. Then either $x(p) \leqslant e_i(p)$ or $e_{i+1}(p) \leqslant e_i(p)$ and in the later case, $e_i(p) = 1(p)$. Hence $x(p) \leqslant e_i(p)$ for all $p \in X$, so that $e_{i+1} \to e_i = e_i$. Thus $\langle L; e_0, e_1, \ldots, e_{n-1} \rangle$ is a P_1-lattice.

3. P_2-lattices

DEFINITION 3.1. (Epstein and Horn (1975).) A P_2-lattice is a P_1-lattice $\langle L; e_0, e_1, \ldots, e_{n-1} \rangle$ such that $!e_i$ exists for all i.

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The following theorem is a consequence of Defintion 4.3 of Epstein and Horn (1975) and Theorem 4.3 of Maddana Swamy and Manikyamba (to appear).

THEOREM 3.2. *L is a P_2-lattice if and only if L is a P_1-lattice and for any $x \in L$, $\{p \in X \mid x(p) < 1(p)\}$ is open.*

DEFINITION 3.3. Let L be a B-completely normal lattice and let each maximal chain of prime ideals of L be finite. For each $p \in X$, let $n(p)$ be the number of prime ideals of L which contain p. We say that $\mathscr{P}(L)$ satisfies the continuity axiom if $p \mapsto n(p)$ is a continuous map from X into Z with the discrete topology (where Z is the set of all integers).

Observe that, if L is a Store lattice B-completely normal, then $p \mapsto (p)$, the ideal in L generated by p, is a homeomorphism (see Maddana Swamy and Manikyamba (1979), Theorem 4) of X onto Y^m.

REMARK 3.4. Epstein and Horn (1975) gave an example of a P-algebra L which is not a P_0-lattice. We observe that it is only because $\mathscr{P}(L)$ does not satisfy the continuity axiom.

THEOREM 3.5. *Let L be a P-algebra. Then L is a P_2-lattice if and only if $\mathscr{P}(L)$ satisfies the continuity axiom and there exists an integer n such that each maximal chain in $\mathscr{P}(L)$ has at most $n-1$ elements.*

PROOF. Suppose $\langle L; e_0, e_1, \ldots, e_{n-1}\rangle$ is a P_2-lattice. Suppose $n(p) = i$. Then

$$e_0(p) < e_1(p) < \ldots < e_i(p) = 1.$$

Now for any $x \in L$, it is clear that $x(p) = 1(p)$ if and only if $!x \in B-p$. Therefore $!e_{i-1} \in p$ and $!e_i \in B-p$. Let $a = !e_i - !e_{i-1}$. Then $p \in a$ and for all $q \in a$, $!e_i - !e_{i-1} \in B-q$, so that $e_{i-1}(q) < e_i(q) = 1(q)$. Thus $n(q) = i$ for all $q \in a$ and the continuity axiom is proved. Conversely suppose $\mathscr{P}(L)$ satisfies the continuity axiom and each maximal chain in $\mathscr{P}(L)$ has at most $n-1$ elements. Since $p \mapsto (p)$ is a homeomorphism of X onto Y^m and since for any $p \in X$, $|\mathscr{S}_p| = n(p)+1$, to each $p \in X$, there exists $a \in B-p$ such that $|\mathscr{S}_p| = |\mathscr{S}_q|$ for all $q \in a$. Hence there exists a partition $\{a_1, \ldots, a_k\}$ of X such that $|\mathscr{S}_p| = |\mathscr{S}_q|$ for all $p, q \in a_i$ and $1 \leqslant i \leqslant k$. Hence $L = \prod_{i=1}^{k} \Gamma(a_i, \mathscr{S})$, where $\Gamma(a_i, \mathscr{S})$ is the lattice of all sections of the clopen set a_i into $\mathscr{S}$. By Theorems 16 and 17 of Epstein (1960), $\Gamma(a_i, \mathscr{S})$ is a Post algebra and hence by Lemma 4.9 of Epstein and Horn (1975), it follows that L is a P_2-lattice.

The following lemma is due to Maddana Swamy and Manikyamba (to appear).

LEMMA 3.6. *L is a B-algebra if and only if $\{p \in X \mid x(p) \leqslant y(p)\}$ is clopen for every $x, y \in L$.*

THEOREM 3.7. *Let L be a B-completely normal lattice satisfying the continuity axiom and suppose each maximal chain in $\mathscr{P}(L)$ has finite length. Then the following are equivalent.*

(1) *L is a P_0-lattice.*
(2) *L is a P_1-lattice.*
(3) *L is a P_2-lattice.*
(4) *L is a B-algebra.*
(5) *L is a P-algebra.*

PROOF. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear and the equivalence of (4) and (5) is immediate by the definition, since L is B-completely normal. Further, (5) $\Rightarrow$ (3) is proved in the above theorem. Now we are left with the proof of (1) $\Rightarrow$ (4).

Suppose $\langle L; e_0, e_1, \ldots, e_{n-1}\rangle$ is a P_0-lattice. Let $x, y \in L$ and $p \in X$ such that $x(p) < y(p)$. Now, write

$$\mathscr{S}_p = \{0(p) = e_0(p) \leqslant e_1(p) \leqslant \ldots \leqslant e_{n-1}(p) = 1(p)\}.$$

Choose integers $i_1 < i_2 < \ldots < i_k$ such that

$$e_0(p) = \ldots = e_{i_1-1}(p) < e_{i_1}(p) = e_{i_1+1}(p) = \ldots = e_{i_2-1}(p) < e_{i_2}(p) = \ldots.$$

Since $\mathscr{P}(L)$ satisfies the continuity axiom, there exists $a \in B - p$ such that $|\mathscr{S}_q| = |\mathscr{S}_p|$ for all $q \in a$. Also, for each i_j, there exists $a_{i_j} \in B - p$ such that $e_{i_j} \wedge a_{i_j} = e_{i_{j+1}-1} \wedge a_{i_j}$ and hence there exists $b \in B - p$ such that $e_{i_j} \wedge b = e_{i_{j+1}-1} \wedge b$ for all i_j. Now there exists $j < k$ such that $x(p) = e_{i_j}(p)$ and $y(p) = e_{i_k}(p)$. Hence there exists $c \in B - p$ such that $x \wedge c = e_{i_j} \wedge c$ and $y \wedge c = e_{i_k} c$. Clearly $p \in a \wedge b \wedge c$ and, for any $q \in a \wedge b \wedge c$, if $x(q) = y(q)$, then $e_{i_j}(q) = e_{i_k}(q)$ which implies that $|\mathscr{S}_q| < |\mathscr{S}_q|$, which contradicts the fact that $p, q \in a$. Therefore $x(q) < y(q)$ for all $q \in a \wedge b \wedge c$. Hence $\{p \in X \mid x(p) < y(p)\}$ is open. Thus L is a B-algebra, by Lemma 3.6. This proves the theorem.

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