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Semilinear elliptic equations involving power nonlinearities and Hardy potentials with boundary singularities

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Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ be a C^2 bounded domain and $\Sigma \subset \partial\Omega$ be a C^2 compact submanifold without boundary, of dimension k, $0 \leqslant k \leqslant N-1$. We assume that $\Sigma = \{0\}$ if $k = 0$ and $\Sigma = \partial\Omega$ if $k = N-1$. Let $d_\Sigma(x) = \mathrm{dist}\,(x, \Sigma)$ and $L_\mu = \Delta + \mu\, d_\Sigma^{-2}$, where $\mu \in \mathbb{R}$. We study boundary value problems $(P_\pm)$ $-L_\mu u \pm |u|^{p-1}u = 0$ in Ω and $\mathrm{tr}_{\mu,\Sigma}(u) = \nu$ on $\partial\Omega$, where $p > 1$, ν is a given measure on $\partial\Omega$ and $\mathrm{tr}_{\mu,\Sigma}(u)$ denotes the boundary trace of u associated to L_μ. Different critical exponents for the existence of a solution to $(P_\pm)$ appear according to concentration of ν. The solvability for problem (P_+) was proved in [**3**, **29**] in subcritical ranges for p, namely for p smaller than one of the critical exponents. In this paper, assuming the positivity of the first eigenvalue of $-L_\mu$, we provide conditions on ν expressed in terms of capacities for the existence of a (unique) solution to (P_+) in supercritical ranges for p, i.e. for p equal or bigger than one of the critical exponents. We also establish various equivalent criteria for the existence of a solution to (P_-) under a smallness assumption on ν.

Keywords: Hardy potentials; boundary singularities; capacities; critical exponents; removable singularity; Keller–Osserman estimates

2020 *Mathematics Subject Classification:* 35J60; 35J75; 35J10; 35J66

1. Introduction

1.1. A survey of the relevant literature

Let $N \geqslant 3$, $\Omega \subset \mathbb{R}^N$ be a C^2 bounded domain and $\Sigma \subset \partial\Omega$ be a C^2 compact submanifold in $\mathbb{R}^N$ without boundary, of dimension $0 \leqslant k \leqslant N-1$. We assume that $\Sigma = \{0\}$ if $k = 0$ and $\Sigma = \partial\Omega$ if $k = N-1$. Let $d_{\partial\Omega}(x) = \mathrm{dist}\,(x, \partial\Omega)$ and $d_\Sigma(x) = \mathrm{dist}\,(x, \Sigma)$. Two typical semilinear elliptic equations involving power nonlinearities and Hardy-type potentials are of the form

$$L_\mu u \pm |u|^{p-1}u = 0 \quad \text{in } \Omega, \tag{$E_\pm$}$$

where $p > 1$, $\mu \in \mathbb{R}$ is a parameter and

$$L_\mu u := \Delta u + \frac{\mu}{d_\Sigma^2} u.$$

The nonlinearity $|u|^{p-1}u$ in ($E_\pm$) is referred to as an *absorption* or a *source* depending whether the plus sign or minus sign appears in ($E_\pm$).

Boundary value problems for ($E_\pm$) with $\mu = 0$ became a central research subject in the area of partial differential equations with abundant literature. A rich theory has been developed for a boundary value problem with a power absorption in the case $\mu = 0$, namely for the problem

$$\begin{cases} -\Delta u + |u|^{p-1}u = 0 & \text{in } \ \Omega, \\ u = \nu & \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where ν is a measure on $\partial\Omega$. Throughout this paper, we denote by $\mathfrak{M}(\partial\Omega)$ and $\mathfrak{M}^+(\partial\Omega)$ the space of finite measures on $\partial\Omega$ and its positive cone respectively. The first study of (1.1) was carried out by Gmira and Véron in [**22**] where the existence of a solution is obtained for any $\nu \in \mathfrak{M}(\partial\Omega)$ in the subcritical case $1 < p < \frac{N+1}{N-1}$. In the supercritical case $p \geqslant \frac{N+1}{N-1}$, a breakthrough was achieved by Marcus and Véron [**32**], asserting that problem (1.1) possesses a solution if and only if ν is absolutely continuous with respect to the capacity $\mathrm{Cap}^{\partial\Omega}_{\frac{2}{p},p'}$, namely $\nu(E) = 0$ for any Borel set $E \subset \partial\Omega$ such that $\mathrm{Cap}^{\partial\Omega}_{\frac{2}{p},p'}(E) = 0$ [see (2.9) for the definition of the above capacities and see (2.10) for the meaning of the absolute continuity].

When $\mu \neq 0$, let $C_{\Omega,\Sigma}$ be the optimal Hardy constant defined by

$$C_{\Omega,\Sigma} := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_\Omega |u|^2 \, d_\Sigma^{-2} \, \mathrm{d}x} \tag{1.2}$$

and put

$$H := \frac{N-k}{2}.$$

It is known that $C_{\Omega,\Sigma} \in (0, H^2]$ (see e.g. [**7**, **26**] for $k = N - 1$ and [**14**] for $0 \leqslant k \leqslant N - 2$).

Consider the eigen problem

$$\lambda_{\mu,\Sigma} := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x - \mu \int_\Omega |u|^2 d_\Sigma^{-2} \, \mathrm{d}x}{\int_\Omega |u|^2 \, \mathrm{d}x}. \tag{1.3}$$

Note that $\lambda_{\mu,\Sigma} > -\infty$ if $\mu \leqslant H^2$, and $\lambda_{\mu,\Sigma} > 0$ if $\mu < C_{\Omega,\Sigma}$. Moreover, when $\mu < H^2$, problem (1.3) admits a minimizer $\phi_{\mu,\Sigma} \in H_0^1(\Omega)$ which satisfies $L_\mu \phi_{\mu,\Sigma} = \lambda_{\mu,\Sigma}\phi_{\mu,\Sigma}$ in Ω (see [**14**, corollary 1.3]). When $\mu = H^2$, there is no minimizer of problem (1.3) in $H_0^1(\Omega)$, but there exists a function $\phi_{\mu,\Sigma} \in H^1_{\mathrm{loc}}(\Omega)$ such that $L_\mu \phi_{\mu,\Sigma} =$

$\lambda_{\mu,\Sigma}\phi_{\mu,\Sigma}$ in Ω in the sense of distributions. In addition, by [3, proposition A.2] (see also [29, lemma 2.2]), for any $\mu \leqslant H^2$, there holds

$$\phi_{\mu,\Sigma} \approx d_{\partial\Omega}\, d_\Sigma^{-\alpha_-} \quad \text{in } \Omega \setminus \Sigma, \tag{1.4}$$

where

$$\alpha_\pm := H \pm \sqrt{H^2 - \mu}. \tag{1.5}$$

It is known that, when $\mu < C_{\Omega,\Sigma}$, there exists a Green function associated with $-L_\mu$, denoted by G_μ (see e.g. [2, 24] for more general potentials, and [17] for $\Sigma = \partial\Omega$ and $\mu \leqslant \frac{1}{4}$). In addition, by Ancona [2], there exists a Martin kernel associated with $-L_\mu$, denoted by K_μ, which is unique up to a normalization. Marcus and Nguyen [28] applied results for a class of more general Schrödinger operators in [24] to the model case L_μ and showed two-sided estimates of G_μ and K_μ. Recently, Barbatis *et al.* [3] followed a different approach to obtain the existence and sharp two-sided estimates of G_μ and K_μ for the whole range $\mu \leqslant H^2$ provided $\lambda_{H^2} > 0$. These estimates will be quoted in § 3.1.

We denote by $\mathfrak{M}(\Omega;\phi_{\mu,\Sigma})$ the space of measures τ such that $\int_\Omega \phi_{\mu,\Sigma}\,\mathrm{d}|\tau| < +\infty$ and by $\mathfrak{M}^+(\Omega;\phi_{\mu,\Sigma})$ the positive cone of $\mathfrak{M}(\Omega;\phi_{\mu,\Sigma})$. The Green operator and the Martin operator are respectively defined by

$$\mathbb{G}_\mu[\tau](x) := \int_\Omega G_\mu(x,y)\,\mathrm{d}\tau(y), \quad \tau \in \mathfrak{M}(\Omega;\phi_{\mu,\Sigma}),$$
$$\mathbb{K}_\mu[\nu](x) := \int_{\partial\Omega} K_\mu(x,y)\,\mathrm{d}\nu(y), \quad \nu \in \mathfrak{M}(\partial\Omega).$$

These operators are an important tool in the study of nonhomogenous linear equations involving $-L_\mu$. Main properties of the above operators were established in [3] and will be presented in subsections 3.2.

There is a vast literature on boundary value problems for ($E_\pm$). We list below some relevant works.

The extreme case $\Sigma = \{0\} \subset \partial\Omega$ was considered by Chen and Véron in [8] in which necessary and sufficient conditions in terms of suitable capacities for the existence of a solution to (E_+) with a prescribed boundary measure were established under the condition $\mu \leqslant H^2$.

In the other extreme case $\Sigma = \partial\Omega$, Marcus and Nguyen [28] introduced a notion of normalized boundary trace to study a boundary value problem for equation (E_+) with $0 < \mu < C_{\Omega,\partial\Omega}$. In this range of μ, they showed that if $1 < p < \frac{N-\alpha_-+1}{N-\alpha_--1}$ then the problem admits a unique solution for any $\nu \in \mathfrak{M}^+(\partial\Omega)$. Marcus and Moroz [27] extended the notion of normalized boundary trace and the results in [28] to the range $-\infty < \mu < 1/4$. Independently, under the assumption $\lambda_{\mu,\partial\Omega} > 0$, Gkikas and Véron [21] investigated a boundary value problem for (E_+) with a prescribed boundary trace defined in a dynamic way and obtained various existence results. Then it was shown in [18] that the two notions of boundary trace in [28] and in [21] coincide.

Afterwards, Marcus and Nguyen [29] generalized the notion of normalized boundary trace in [28] to the case $\Sigma \subsetneq \partial\Omega$ with dimension $0 \leqslant k \leqslant N-2$, under the restriction $\mu < \min\{C_{\Omega,\Sigma}, H - 1/4\}$. They proved the solvability for the boundary value problem for (E_+) with any prescribed normalized boundary trace $\nu \in \mathfrak{M}^+(\partial\Omega)$ in *subcritical ranges* for p, namely for $1 < p < \frac{N+1}{N-1}$ if ν has compact support in $\partial\Omega \setminus \Sigma$ or for $1 < p < \frac{N-\alpha_-+1}{N-\alpha_--1}$ if ν has compact support in Σ. They also showed that the problem has no solution either if $\nu = \delta_y$ (the Dirac measure concentrated at y) in *supercritical ranges* for p, namely $p \geqslant \frac{N+1}{N-1}$ if $y \in \partial\Omega \setminus \Sigma$ or $p \geqslant \frac{N-\alpha_-+1}{N-\alpha_-+1}$ if $y \in \Sigma$. Very recently, under the condition $\lambda_{\mu,\Sigma} > 0$, Barbatis *et al.* [**3**] obtained similar existence results in subcritical ranges for p and for the whole range $\mu \leqslant H^2$.

For boundary value problems with more general potentials singular on $\partial\Omega$, we refer to Marcus [**25**] and Bhakta *et al.* [**4**].

The case of source nonlinearity is sharply different from the case of absorption nonlinearity in the sense that existence results hold under a smallness condition of boundary data, while nonexistence results hold if boundary data are large enough, even in subcritical ranges of p. When $\mu = 0$, this phenomenon can be seen in [**6**]. When $\Sigma = \partial\Omega$, Bidaut-Véron *et al.* [**5**] established existence results for a boundary value problem with measure for (E_-) in a capacity framework under a smallness condition on boundary data. Afterwards, various necessary and sufficient conditions for the existence of a solution to (E_-) were obtained by Nguyen [**33**], Gkikas and Nguyen [**18**].

When $\Sigma \subset \Omega$, the corresponding boundary problems involving operator L_μ with an absorption and with a source were extensively studied by Gkikas and Nguyen in [**19**, **20**] respectively. See also the papers by Dávila and Dupaigne [**11**, **12**], Dupaigne and Nedev [**13**], Fall [**15**] and Chen and Zhou [**9**] for related results on semilinear equations with a source term.

1.2. Aim of the paper

Motivated by the above mentioned works, in the present paper, we aim to study boundary value problems for $(E_\pm)$, where $\Sigma \subset \partial\Omega$ of dimension $0 \leqslant k \leqslant N-2$, for $\mu \leqslant H^2$.

- First, we will establish removability results for equation (E_+) when $p \geqslant \frac{N-\alpha_-+1}{N-\alpha_--1}$ or $p \geqslant \frac{N+1}{N-1}$. We will also provide conditions in terms of suitable capacities for the existence of a solution to boundary value problems for (E_+).
- Then we will give various criteria for the existence of a solution to boundary value problems for (E_-).

The precise statement of these results will be presented in § 2.

2. Main results

2.1. Boundary trace, capacitary setting and main results

Throughout this paper, we assume that

$$0 \leqslant k \leqslant N-2, \quad \mu \leqslant H^2 \quad \text{and} \quad \lambda_{\mu,\Sigma} > 0. \tag{2.1}$$

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Under assumption (2.1), a theory for linear equations involving L_μ was established in [**3**, **29**], which forms a basis for the study of equation ($E_\pm$). We also note that the first and second inequalities in (2.1) imply that $\alpha_+ \geqslant H \geqslant 1$. Moreover $\alpha_+ = 1$ if and only if $k = N-2$ and $\mu = 1$; in this case, we have $\alpha_- = 1$.

First we focus on the equation with an absorption power nonlinearity

$$-L_\mu u + |u|^{p-1}u = 0 \quad \text{in } \Omega. \tag{E_+}$$

Before stating the main results for equation (E_+), we introduce some notations. For any $\beta > 0$, we set

$$\Sigma_\beta := \{x \in \mathbb{R}^N \setminus \Sigma : \, d_\Sigma(x) < \beta\} \quad \text{and} \quad \Omega_\beta := \{x \in \Omega : d_{\partial\Omega}(x) < \beta\}. \tag{2.2}$$

It is well known that (see appendix A.1) there is a small enough number $\beta_0 > 0$ such that for any $x \in \Omega_{\beta_0}$ there exists a unique $\xi_x \in \partial\Omega$ satisfying $d_{\partial\Omega}(x) = |x - \xi_x|$. Now set

$$\tilde{d}_\Sigma(x) := \sqrt{|\mathrm{dist}^{\partial\Omega}(\xi_x, \Sigma)|^2 + |x - \xi_x|^2}, \tag{2.3}$$

where dist $^{\partial\Omega}$ denotes the geodesic distance on $\partial\Omega$.

Let $\beta_3 > 0$ be the constant in proposition A.1. (One may choose $\beta_3 < \beta_0$.) Let η_{β_3} be a smooth cut-off function such that $0 \leqslant \eta_{\beta_3} \leqslant 1$ such that $\eta_{\beta_3} = 1$ in $\overline{\Sigma_{\frac{\beta_3}{4}}}$ with compact support in $\Sigma_{\frac{\beta_3}{2}}$. We define

$$W(x) := \begin{cases} (d_{\partial\Omega}(x) + \tilde{d}_\Sigma(x)^2)\tilde{d}_\Sigma(x)^{-\alpha_+}, & \text{if } \mu < H^2, \\ (d_{\partial\Omega}(x) + \tilde{d}_\Sigma(x)^2)\, d_\Sigma(x)^{-H} |\ln \tilde{d}_\Sigma(x)|, & \text{if } \mu = H^2, \end{cases} \qquad x \in \Omega \cap \Sigma_{\beta_3},$$

where α_+ is defined in (1.5), and define

$$\tilde{W}(x) := (1 - \eta_{\beta_3}(x)) + \eta_{\beta_3}(x) W(x), \quad x \in \Omega. \tag{2.4}$$

In the particular case $\mu = 0$ and $\Sigma = \partial\Omega$, we have $\alpha_+ = 1$, whence $\tilde{W}(x) \approx 1$. We note that $\tilde{W}$ is an appropriate function to describe the boundary behaviour in a normalization sense of solutions to ($E_\pm$). For more detail, see (4.1) and (4.2) (see also [**3**, lemma 6.8]).

Our first theorem provides a removability result when $p \geqslant \frac{\alpha_+ + 1}{\alpha_+ - 1}$ in the sense that if a nonnegative solution 'vanishes' on $\partial\Omega \setminus \Sigma$ as in (2.5), then it must be identically zero.

THEOREM 2.1. *Assume $\mu \leqslant H^2$ if $k < N-2$ or $\mu < H^2$ if $k = N-2$, and $p \geqslant \frac{\alpha_+ + 1}{\alpha_+ - 1}$. We additionally assume that Ω is a C^3 open bounded domain. If $u \in C^2(\Omega)$ is a nonnegative solution of (E_+) such that*

$$\lim_{x \in \Omega,\, x \to \xi} \frac{u(x)}{\tilde{W}(x)} = 0, \quad \forall \xi \in \partial\Omega \setminus \Sigma, \tag{2.5}$$

locally uniformly in $\partial\Omega \setminus \Sigma$, then $u \equiv 0$ in Ω.

We remark that if $k = 0$ the result in the theorem 2.1 coincides with the result in [**8**, theorem J with $h = 0$]. In addition, when $p \geqslant \frac{\alpha_+ + 1}{\alpha_+ - 1}$, boundary behaviour of solutions on Σ is not imposed. However, when $\frac{N-\alpha_-+1}{N-\alpha_--1} \leqslant p < \frac{\alpha_++1}{\alpha_+-1}$, zero boundary condition on Σ is additionally required for the removability of isolated singularities, as stated in the following theorem.

THEOREM 2.2. *Assume $k \geqslant 1$, $\mu \leqslant H^2$, $z \in \Sigma$ and $\frac{N-\alpha_-+1}{N-\alpha_--1} \leqslant p < \frac{\alpha_++1}{\alpha_+-1}$ if $\alpha_+ > 1$ or $\frac{N}{N-2} \leqslant p$ if $\alpha_+ = 1$. We additionally assume that Ω is a C^3 bounded domain. If $u \in C^2(\Omega)$ is a nonnegative solution of (E_+) such that*

$$\lim_{x\in\Omega,\ x\to\xi} \frac{u(x)}{\tilde{W}(x)} = 0 \quad \forall \xi \in \partial\Omega \setminus \{z\}, \tag{2.6}$$

locally uniformly in $\partial\Omega \setminus \{z\}$, then $u \equiv 0$.

Next, we discuss existence results for a boundary value problem for (E_+). In order to formulate the boundary value problem for (E_+), we use a notion of boundary trace, introduced in [**3**], the definition of which is recalled below.

A family $\{\Omega_n\}$ is called a C^2 *exhaustion* of Ω if Ω_n is a C^2 bounded domain, $\Omega_n \Subset \Omega_{n+1} \Subset \Omega$ for any $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}}\Omega_n = \Omega$.

Let $x_0 \in \Omega$ be a fixed reference point.

DEFINITION 2.3 Boundary trace. *We say that a function $u \in W^{1,\kappa}_{\mathrm{loc}}(\Omega)$ ($\kappa > 1$) possesses a boundary trace if there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that for any C^2 exhaustion $\{\Omega_n\}$ of Ω containing x_0, there holds*

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \phi u \,\mathrm{d}\omega^{x_0}_{\Omega_n} = \int_{\partial\Omega} \phi \,\mathrm{d}\nu \quad \forall \phi \in C(\overline{\Omega}).$$

The boundary trace of u is denoted by $\mathrm{tr}_{\mu,\Sigma}(u)$. Here $\omega^{x_0}_{\Omega_n}$ is the L_μ-harmonic measure on $\partial\Omega_n$ relative to x_0 (see § 4.1).

It is known by [**3**, lemmas 8.1 and 8.2] that

$$\mathrm{tr}_{\mu,\Sigma}(\mathbb{K}_\mu[\nu]) = \nu \quad \forall \nu \in \mathfrak{M}(\partial\Omega) \quad \text{and} \quad \mathrm{tr}_{\mu,\Sigma}(\mathbb{G}_\mu[\tau]) = 0 \quad \forall \tau \in \mathfrak{M}(\Omega;\phi_{\mu,\Sigma}). \tag{2.7}$$

For $q \in [1,+\infty)$, denote by $L^q(\Omega;\phi_{\mu,\Sigma})$ the weighted Lebesgue space

$$L^q(\Omega;\phi_{\mu,\Sigma}) := \Bigg\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \|u\|_{L^q(\Omega;\phi_{\mu,\Sigma})} := \left(\int_\Omega |u|^q \phi_{\mu,\Sigma} \,\mathrm{d}x \right)^{\frac{1}{q}} < +\infty \Bigg\}.$$

Let $H^1(\Omega;\phi^2_{\mu,\Sigma})$ be the weighted Sobolev space

$$H^1(\Omega;\phi^2_{\mu,\Sigma}) := \Bigg\{ u \in H^1_{\mathrm{loc}}(\Omega) : \ \|u\|_{H^1(\Omega;\phi^2_{\mu,\Sigma})} := \left(\int_\Omega |u|^2 \phi^2_{\mu,\Sigma} \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \phi^2_{\mu,\Sigma} \,\mathrm{d}x \right)^{\frac{1}{2}} < +\infty \Bigg\}.$$

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We also denote by $H^1_0(\Omega;\phi^2_{\mu,\Sigma})$ the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{H^1(\Omega;\phi^2_{\mu,\Sigma})}$. It is worth mentioning here that $H^1_0(\Omega;\phi^2_{\mu,\Sigma}) = H^1(\Omega;\phi^2_{\mu,\Sigma})$ (see [**3**, theorem 4.5]).

Weak solutions of the boundary value problem for (E_+) with prescribed boundary trace are defined below.

DEFINITION 2.4. *Let $p > 1$. We say that u is a weak solution of*

$$\begin{cases} -L_\mu u + |u|^{p-1}u = 0 & \text{in } \Omega, \\ \mathrm{tr}_{\mu,\Sigma}(u) = \nu. \end{cases} \tag{P$_+$}$$

if $u \in L^1(\Omega;\phi_{\mu,\Sigma})$, $|u|^p \in L^1(\Omega;\phi_{\mu,\Sigma})$ and

$$-\int_\Omega u L_\mu \zeta \,\mathrm{d}x + \int_\Omega |u|^{p-1}u\zeta \,\mathrm{d}x = -\int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta \,\mathrm{d}x \quad \forall \zeta \in \mathbf{X}_\mu(\Omega),$$

where

$$\mathbf{X}_\mu(\Omega) := \{\zeta \in H^1_{\mathrm{loc}}(\Omega) : \phi^{-1}_{\mu,\Sigma}\zeta \in H^1(\Omega;\phi^2_{\mu,\Sigma}),\ \phi^{-1}_{\mu,\Sigma} L_\mu\zeta \in L^\infty(\Omega)\}. \tag{2.8}$$

We remark that in light of [**3**, theorem 2.12], a function u is a weak solution to problem (P$_+$) if and only if

$$u + \mathbb{G}_\mu[|u|^{p-1}u] = \mathbb{K}_\mu[\nu] \quad \text{a.e. in } \Omega.$$

It was known by [**3**, theorem B.4 (b)] (see also [**29**, theorem 1.18]) that, in the subcritical case $1 < p < \frac{N-\alpha_-+1}{N-\alpha_--1}$, problem (P$_+$) admits a unique weak solution for any $\nu \in \mathfrak{M}(\partial\Omega)$ with support in Σ. The supercritical case $p \geqslant \frac{N-\alpha_-+1}{N-\alpha_--1}$ is more challenging. In order to treat this case, we will make use of appropriate capacities.

For $\theta \in \mathbb{R}$, we define the Bessel kernel of order α in $\mathbb{R}^d$ by $\mathcal{B}_{d,\theta}(\xi) := \mathcal{F}^{-1}((1+|.|^2)^{-\frac{\theta}{2}})(\xi)$, where $\mathcal{F}$ is the Fourier transform in the space $\mathcal{S}'(\mathbb{R}^d)$ of moderate distributions in $\mathbb{R}^d$. For $\kappa > 1$, the Bessel space $L_{\theta,\kappa}(\mathbb{R}^d)$ is defined by

$$L_{\theta,\kappa}(\mathbb{R}^d) := \{f = \mathcal{B}_{d,\theta} * g : g \in L^\kappa(\mathbb{R}^d)\},$$

with norm

$$\|f\|_{L_{\theta,\kappa}} := \|g\|_{L^\kappa} = \|\mathcal{B}_{d,-\theta} * f\|_{L^\kappa}.$$

The Bessel capacity is defined for compact set $A \subset \mathbb{R}^d$ by

$$\mathrm{Cap}^{\mathbb{R}^d}_{\theta,\kappa}(A) := \inf\{\|f\|^\kappa_{L_{\theta,\kappa}} :\ g \in L^\kappa_+(\mathbb{R}^d),\ f = \mathcal{B}_{d,\theta} * g \geqslant \mathbb{1}_A\},$$

and is extended to open sets and arbitrary sets in $\mathbb{R}^d$ in the standard way. Here $\mathbb{1}_A$ denotes the indicator function of A.

We denote by $B^d(x,r)$ the open ball of centre $x \in \mathbb{R}^d$ and radius $r > 0$ in $\mathbb{R}^d$.

Using the Bessel capacities, we are able to define capacities for subsets of $\partial\Omega$ as follows. If $\Gamma \subset \partial\Omega$ is a C^2 submanifold without boundary, of dimension d with $1 \leqslant d \leqslant N-1$ then there exist open sets $O_1,\ldots,O_m$ in $\mathbb{R}^N$, diffeomorphisms T_i :

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$O_i \to B^d(0,1) \times B^{N-d-1}(0,1) \times (-1,1)$, $i = 1, \ldots, m$, and compact sets $K_1, ..., K_m$ in Γ such that

(i) $K_i \subset O_i$, $1 \leqslant i \leqslant m$ and $\Gamma = \cup_{i=1}^m K_i$;

(ii) $T_i(O_i \cap \Gamma) = B^d(0,1) \times \{(x_{d+1}, \ldots, x_{N-1}) = 0_{\mathbb{R}^{N-d-1}}\} \times \{x_N = 0\}$, $T_i(O_i \cap \Omega) = B^d(0,1) \times B^{N-d-1}(0,1) \times (0,1)$;

(iii) For any $x \in O_i \cap \Omega$, there exists $y \in O_i \cap \Gamma$ such that $d_\Gamma(x) = |x-y|$ (here $d_\Gamma(x)$ denotes the distance from x to Γ).

We then define the $\mathrm{Cap}^{\Gamma}_{\theta,\kappa}-$capacity of a compact set $E \subset \Gamma$ by

$$\mathrm{Cap}^{\Gamma}_{\theta,\kappa}(E) := \sum_{i=1}^m \mathrm{Cap}^{\mathbb{R}^d}_{\theta,\kappa}(\tilde{T}_i(E \cap K_i)), \tag{2.9}$$

where $T_i(E \cap K_i) = \tilde{T}_i(E \cap K_i) \times \{(x_{d+1}, \ldots, x_{N-1}) = 0_{\mathbb{R}^{N-d-1}}\} \times \{x_N = 0\}$.

We remark that the definition of the capacities does not depend on O_i, $i = 1, \ldots, m$.

In the sequel, we will say that $\nu \in \mathfrak{M}^+(\partial\Omega)$ is *absolutely continuous* with respect to a capacity $\mathscr{C}$ if

$$\forall \text{ Borel set } E \subset \partial\Omega \text{ such that } \mathscr{C}(E) = 0 \Longrightarrow \nu(E) = 0. \tag{2.10}$$

Our next main result gives a sufficient condition in terms of appropriate capacities for the solvability of problem (P_+) in the range $\frac{N-\alpha_-+1}{N-\alpha_--1} \leqslant p < \frac{\alpha_++1}{\alpha_+-1}$ when the boundary trace is supported in Σ.

THEOREM 2.5. *Assume* $k \geqslant 1$, $\mu \leqslant H^2$, $\frac{N-\alpha_-+1}{N-\alpha_--1} \leqslant p < \frac{\alpha_++1}{\alpha_+-1}$ *if* $\alpha_+ > 1$ *or* $\frac{N}{N-2} \leqslant p$ *if* $\alpha_+ = 1$. *Let* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *with compact support in* Σ. *If* ν *is absolutely continuous with respect to* $\mathrm{Cap}^{\Sigma}_{\vartheta,p'}$, *where* $p' = \frac{p}{p-1}$, *then problem* (P_+) *admits a unique weak solution.*

When ν has support in $\partial\Omega \setminus \Sigma$, we provide a necessary and sufficient condition on the boundary trace for the existence of a solution to problem (P_+) in the supercritical range $p \geqslant \frac{N+1}{N-1}$.

THEOREM 2.6. *Assume* $\mu \leqslant H^2$, $p \geqslant \frac{N+1}{N-1}$ *and* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *with compact support in* $\partial\Omega \setminus \Sigma$. *Then problem* ($P_+$) *admits a unique weak solution if and only if* $\mathbb{1}_F\nu$ *is absolutely continuous with respect to* $\mathrm{Cap}^{\partial\Omega}_{\frac{2}{p},p'}$ *for any compact set* $F \subset \partial\Omega \setminus \Sigma$.

Next we investigate boundary value problem for (E_-) of the form

$$\begin{cases} -L_\mu u - |u|^{p-1}u = 0 & \text{in } \Omega, \\ \mathrm{tr}_{\mu,\Sigma}(u) = \sigma\nu, \end{cases} \tag{P^σ_-}$$

where $\sigma > 0$ is a parameter and $\nu \in \mathfrak{M}^+(\partial\Omega)$.

Weak solutions to problem (P^σ_-) are defined similarly as in definition 2.4 with obvious modifications.

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In the following theorems, for any $\nu \in \mathfrak{M}^+(\partial\Omega)$, we extend it to be a measure defined on $\overline{\Omega}$ by setting $\nu(\Omega) = 0$ and use the same notation ν for the extension.

When ν is concentrated on Σ, various equivalent criteria for the existence of a weak solution to problem (P^σ_-) are described in the following result.

Theorem 2.7. *Assume that $\mu < \frac{N^2}{4}$ and*

$$1 < p < \frac{\alpha_- + 1}{\alpha_- - 1} \text{ if } \alpha_- > 1 \quad \text{or} \quad p > 1 \text{ if } \alpha_- \leqslant 1. \tag{2.11}$$

Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ with compact support in Σ. Then the following statements are equivalent.

1. *Problem (P^σ_-) has a positive weak solution for $\sigma > 0$ small.*
2. *For any Borel set $E \subset \overline{\Omega}$, there holds*

$$\int_E \mathbb{K}_\mu[\mathbb{1}_E \nu]^p \phi_{\mu,\Sigma} \, \mathrm{d}x \leqslant C\, \nu(E). \tag{2.12}$$

3. *The following inequality holds*

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^p] \leqslant C\, \mathbb{K}_\mu[\nu] < +\infty \quad \text{a.e. in } \Omega. \tag{2.13}$$

Assume, in addition, that $k \geqslant 1$ and

$$\begin{cases} \max\left\{1, \dfrac{N-k-\alpha_- + 1}{N-1-\alpha_-}\right\} < p < \dfrac{\alpha_+ + 1}{\alpha_+ - 1} & \text{if } \alpha_+ > 1 \\ \text{or} \quad \max\left\{1, \dfrac{N-k}{N-2}\right\} < p & \text{if } \alpha_+ = 1. \end{cases} \tag{2.14}$$

Put

$$\vartheta := \frac{\alpha_+ + 1 - p(\alpha_+ - 1)}{p}. \tag{2.15}$$

Then any of the above statements is equivalent to the following statement

4. *For any Borel set $E \subset \Sigma$, there holds*

$$\nu(E) \leqslant C\, \mathrm{Cap}^{\Sigma}_{\vartheta,p'}(E).$$

We remark that the case $\Sigma = \{0\}$ and $\mu = \frac{N^2}{4}$ is treated in § 6.3 with slightly modified capacities; see in particular remark 6.15.

When ν is concentrated on $\partial\Omega \setminus \Sigma$, we obtain necessary and sufficient conditions for the existence of a weak solution of (P^σ_-) for the whole range $\mu \leqslant \frac{N^2}{4}$.

Theorem 2.8. *Assume that $\mu \leqslant \frac{N^2}{4}$, p satisfies (P^σ_-) and $\nu \in \mathfrak{M}^+(\partial\Omega)$ with compact support in $\partial\Omega \setminus \Sigma$. Then the following statements are equivalent.*

1. *Equation (P^σ_-) has a positive solution for $\sigma > 0$ small.*

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2. *For any Borel set* $E \subset \overline{\Omega}$, (2.12) *holds.*

3. *Estimate* (2.13) *holds.*

4. *For any Borel set* $E \subset \partial\Omega$, *there holds* $\nu(E) \leqslant C\,\mathrm{Cap}^{\partial\Omega}_{\frac{2}{p},p'}(E)$.

We note that the case $p \geqslant \frac{\alpha_-+1}{\alpha_--1}$ (if $\alpha_- > 1$) is still open and requires a different method.

2.2. Proof strategies and comparison with relevant works in the literature

The distinctive feature of the problems (P_+) and (P_-^σ) is characterized by the interplay between the concentration of Σ, the type of nonlinearity, the exponent p and the parameter μ. By employing a fine analysis in capacitary setting, we are able to obtain existence and nonexistence results in the supercritical ranges for p and the critical case for the parameter μ, which justifies the novelty of our paper in comparison with related works in the literature. This is discussed in more detail below.

To establish the removability results (theorems 2.1 and 2.2), we treat the cases $p \geqslant \frac{\alpha_-+1}{\alpha_--1}$ and $p \geqslant \frac{N-\alpha_-+1}{N-\alpha_--1}$ separately. When $p \geqslant \frac{\alpha_-+1}{\alpha_--1}$, we provide a proof the heart of which is the assertion that all nonnegative solutions u of problem (E_+)–(2.5) are dominated by $\tilde{W}$ in light of Keller–Osserman type estimates (see proposition 5.2), hence are uniformly bounded in $L^p(\Omega;\phi_{\mu,\Sigma})$. Consequently, thanks to the representation theorem (see theorem 3.3), these solutions admit boundary traces concentrated on Σ with uniformly bounded total mass. Therefore, by contradiction, if there is a nontrivial nonnegative solution with positive boundary trace then there is a sequence of solutions whose total mass are unbounded, which clearly contradicts the above assertion. In the larger range $p \geqslant \frac{N-\alpha_-+1}{N-\alpha_--1}$, the above assertion is no longer valid and we focus on solutions with possible isolated boundary singularities concentrated at a point on Σ depicted by (2.6). We offer a proof, which relies on a combination of localization techniques, Keller–Osserman type estimates and weak formulation for nonhomogeneous linear equations, to show the removability of isolated singularities. Our results are new and cover [**29**, theorem 1.17].

We prove the solvability for problem (P_+) (theorems 2.5 and 2.6) by extending the method in [**31**]. When the boundary trace ν has support in Σ, a crucial ingredient in the proof of theorem 2.5 is the equivalence between the quantities $\|\mathbb{K}_\mu[\nu]\|_{L^p(\Omega;\phi_{\mu,\Sigma})}$ and $\|\nu\|_{B^{-\vartheta,p}(\Sigma)}$, where $B^{-\vartheta,p}(\Sigma)$ is the dual of an appropriate Besov space (see theorem 5.4). This allows us to utilize an approximation argument to prove the existence of a (unique) weak solution to problem (P_+). When ν has support in $\partial\Omega \setminus \Sigma$, we construct the Poisson kernel associated to $-L_\mu$ and adapt the idea in [**32**] to prove theorem 2.6. In this case, the effect of the potential d_Σ^{-2} is not pivotal as it can be seen that the critical exponent and the involved capacities are the same as in the free potential case. To our knowledge, theorems 2.5 and 2.6 are the first existence results for problem (P_+) expressed in terms

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of capacities in case $1 \leqslant k \leqslant N-2$, which complements or extends the results in [**21**, **29**, **32**].

The source case is sharply different from the absorption case in several aspects due to the distinct effect of the source nonlinearity and hence require a completely different approach. Theorems 2.7 and 2.8 provide various necessary and sufficient conditions for the existence of a weak solution to problems with source nonlinearity (P_-^σ) for $\operatorname{supp}\nu \subset \Sigma$ and $\operatorname{supp}\nu \subset \partial\Omega \setminus \Sigma$ respectively. The proofs are in the spirit of [**5**], requiring several sharp estimates to adapt nontrivially an abstract result in [**23**] to our setting. Our theorems extend the existence results in [**5**, **6**, **18**, **33**] and can be regarded as a counterpart of the results in [**20**].

It is worth pointing out that the optimal Hardy constant $C_{\Omega,\Sigma}$ defined in (1.2), as well as the asymptotic behaviour of the first eigenfunction $\phi_{\mu,\Sigma}$ in (1.4), the Green function and the Martin kernel, are different from those in the case where the potentials blow up on subsets of Ω. As a result, the critical exponents for the existence of a solution to (P_+) and (P_-^σ) and the employed capacities are different from those in the [**19**, **20**].

Organization of the paper. In § 3, we quote two-sided estimates of the Green function and the Martin kernel from [**3**], recall the representation theorem and results for linear and semilinear equations with an absorption established in [**3**]. In § 4, we give the definition of the L_μ-harmonic measures and show identities regarding the Poisson kernel and Martin kernel. Section 5 is devoted to the derivation of various results for equation (E_+) such as a prior estimate, removable singularities (theorems 2.1 and 2.2) and existence results (theorems 2.5 and 2.6). In § 6, we demonstrate necessary and sufficient conditions for the existence of a weak solution to (P_-^σ) (theorems 2.7 and 2.8). Finally, in appendix A, we provide the local representation of Σ and Ω and construct a barrier function for solutions under assumption that Ω is a C^3 bounded domain.

Notations. We denote by c, c_1, C... the constants which depend on initial parameters and may change from one appearance to another. The notation $A \gtrsim B$ (resp. $A \lesssim B$) means $A \geqslant c\,B$ (resp. $A \leqslant c\,B$) where c is a positive constant depending on some initial parameters. If $A \gtrsim B$ and $A \lesssim B$, we write $A \approx B$. *Throughout the paper, most of the implicit constants depend on some (or all) of the initial parameters such as $N, \Omega, \Sigma, k, \mu$ and we will omit these dependencies in the notations (except when it is necessary).* For a set $D \subset \mathbb{R}^N$, $\mathbb{1}_D$ denotes the indicator function of D.

3. Preliminaries

3.1. Two-sided estimates on Green function and Martin kernel

In this subsection, we recall sharp two-sided estimates on the Green function G_μ and the Martin kernel K_μ associated to $-L_\mu$ in Ω, as well as the representation formula for nonnegative L_μ-harmonic functions.

Proposition 3.1 [**3**, proposition 5.3]. *Assume that $\mu \leqslant H^2$ and $\lambda_{\mu,\Sigma} > 0$.*

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(i) *If* $\mu < H^2$ *or* $\mu = H^2$ *and* $k > 0$ *then for any* $x, y \in \Omega$ *and* $x \neq y$*, there holds*

$$G_\mu(x,y) \approx \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{d_{\partial\Omega}(x)\, d_{\partial\Omega}(y)}{|x-y|^N}\right\} \times \left(\frac{(d_\Sigma(x)+|x-y|)\,(d_\Sigma(y)+|x-y|)}{d_\Sigma(x)\, d_\Sigma(y)}\right)^{\alpha_-}. \tag{3.1}$$

(ii) *If* $k = 0$ *and* $\mu = H^2$ *then for any* $x, y \in \Omega$ *and* $x \neq y$*, there holds*

$$G_\mu(x,y) \approx \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{d_{\partial\Omega}(x)\, d_{\partial\Omega}(y)}{|x-y|^N}\right\} \times \left(\frac{(|x|+|x-y|)\,(|y|+|x-y|)}{|x||y|}\right)^{-\frac{N}{2}} + \frac{d_{\partial\Omega}(x)\, d_{\partial\Omega}(y)}{(|x||y|)^{\frac{N}{2}}}\left|\ln\left(\min\left\{|x-y|^{-2}, (d_{\partial\Omega}(x)\, d_{\partial\Omega}(y))^{-1}\right\}\right)\right|. \tag{3.2}$$

Proposition 3.2 [3, theorem 2.8]. *Assume that* $\mu \leqslant H^2$ *and* $\lambda_{\mu,\Sigma} > 0$*.*

(i) *If* $\mu < H^2$ *or* $\mu = H^2$ *and* $k > 0$ *then*

$$K_\mu(x,\xi) \approx \frac{d_{\partial\Omega}(x)}{|x-\xi|^N}\left(\frac{(d_\Sigma(x)+|x-\xi|)^2}{d_\Sigma(x)}\right)^{\alpha_-} \quad \textit{for all } x \in \Omega,\ \xi \in \partial\Omega. \tag{3.3}$$

(ii) *If* $k = 0$ *and* $\mu = H^2$ *and then*

$$K_\mu(x,\xi) \approx \frac{d_{\partial\Omega}(x)}{|x-\xi|^N}\left(\frac{(|x|+|x-\xi|)^2}{|x|}\right)^{\frac{N}{2}} + \frac{d_{\partial\Omega}(x)}{|x|^{\frac{N}{2}}}\left|\ln\left(|x-\xi|\right)\right|, \quad \textit{for all } x \in \Omega,\ \xi \in \partial\Omega. \tag{3.4}$$

Recall that the Green operator and Martin operator are respectively defined by

$$\mathbb{G}_\mu[\tau](x) = \int_\Omega G_\mu(x,y)\, \mathrm{d}\tau(y), \quad \tau \in \mathfrak{M}(\Omega;\phi_{\mu,\Sigma}),$$
$$\mathbb{K}_\mu[\nu](x) = \int_{\partial\Omega} K_\mu(x,y)\, \mathrm{d}\nu(y), \quad \nu \in \mathfrak{M}(\partial\Omega).$$

A function $u \in L^1_{\mathrm{loc}}(\Omega)$ is called an L_μ*-harmonic function* in Ω if $L_\mu u = 0$ in the sense of distributions in Ω.

Next we state the representation theorem which provides a bijection between the class of positive L_μ-harmonic functions in Ω and the measure space $\mathfrak{M}^+(\partial\Omega)$.

Theorem 3.3 [3, theorem 2.9]. *For any* $\nu \in \mathfrak{M}^+(\partial\Omega)$*, the function* $\mathbb{K}_\mu[\nu]$ *is a positive* L_μ*-harmonic function in* Ω*. Conversely, for any positive* L_μ*-harmonic function* u *in* Ω*, there exists a unique measure* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *such that* $u = \mathbb{K}_\mu[\nu]$ *a.e. in* Ω*.*

3.2. Boundary value problems for linear equations and semilinear equations

We recall the existence, uniqueness and Kato-type inequalities for solutions to boundary value problems for linear equations.

THEOREM 3.4 [3, theorem 2.12]. *Let* $\tau \in \mathfrak{M}(\Omega; \phi_{\mu,\Sigma})$ *and* $\nu \in \mathfrak{M}(\partial\Omega)$. *Then there exists a unique weak solution* $u \in L^1(\Omega; \phi_{\mu,\Sigma})$ *of*

$$\begin{cases} -L_\mu u = \tau & \text{in } \Omega, \\ \operatorname{tr}_{\mu,\Sigma}(u) = \nu, \end{cases}$$

in the sense

$$-\int_\Omega u L_\mu \xi \, \mathrm{d}x = \int_\Omega \xi \, \mathrm{d}\tau - \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \xi \, \mathrm{d}x \quad \forall \xi \in \mathbf{X}_\mu(\Omega),$$

where $\mathbf{X}_\mu(\Omega)$ *has been defined in* (2.8). *Furthermore*

$$u = \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu] \quad \text{a.e. in } \Omega,$$

and for any $\zeta \in \mathbf{X}_\mu(\Omega)$, *there holds*

$$\|u\|_{L^1(\Omega;\phi_{\mu,\Sigma})} \leqslant \frac{1}{\lambda_\mu} \|\tau\|_{\mathfrak{M}(\Omega;\phi_{\mu,\Sigma})} + C\|\nu\|_{\mathfrak{M}(\partial\Omega)},$$

where $C = C(N, \Omega, \Sigma, \mu)$. *In addition, if* $\mathrm{d}\tau = f\mathrm{d}x + \mathrm{d}\rho$ *with* $\rho \in \mathfrak{M}(\Omega; \phi_{\mu,\Sigma})$ *and* $f \in L^1(\Omega; \phi_{\mu,\Sigma})$, *then, for any* $0 \leqslant \zeta \in \mathbf{X}_\mu(\Omega)$, *the following Kato-type inequalities are valid*

$$\begin{aligned} -\int_\Omega |u| L_\mu \zeta \, \mathrm{d}x &\leqslant \int_\Omega \operatorname{sign}(u) f \zeta \, \mathrm{d}x + \int_\Omega \zeta \, \mathrm{d}|\rho| - \int_\Omega \mathbb{K}_\mu[|\nu|] L_\mu \zeta \, \mathrm{d}x, \\ -\int_\Omega u^+ L_\mu \zeta \, \mathrm{d}x &\leqslant \int_\Omega \operatorname{sign}^+(u) f \zeta \, \mathrm{d}x + \int_\Omega \zeta \, \mathrm{d}\rho^+ - \int_\Omega \mathbb{K}_\mu[\nu^+] L_\mu \zeta \, \mathrm{d}x. \end{aligned} \tag{3.5}$$

Here $u^+ = \max\{u, 0\}$.

PROPOSITION 3.5 [3, theorem 9.7]. *Let* $\nu \in \mathfrak{M}(\partial\Omega)$ *and* $g \in C(\mathbb{R})$ *be a nondecreasing function such that* $g(0) = 0$ *and* $g(\mathbb{K}_\mu[\nu_+]), g(\mathbb{K}_\mu[\nu_-]) \in L^1(\Omega; \phi_{\mu,\Sigma})$. *Then there exists a unique weak solution* $u \in L^1(\Omega; \phi_{\mu,\Sigma})$ *of*

$$\begin{cases} -L_\mu u + g(u) = 0 & \text{in } \Omega, \\ \operatorname{tr}_{\mu,\Sigma}(u) = \nu, \end{cases}$$

in the sense that $g(u) \in L^1(\Omega; \phi_{\mu,\Sigma})$ *and*

$$-\int_\Omega u L_\mu \zeta \, \mathrm{d}x + \int_\Omega g(u) \zeta \, \mathrm{d}x = -\int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta \, \mathrm{d}x \quad \forall \zeta \in \mathbf{X}_\mu(\Omega).$$

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4. L_μ-harmonic measures and Poisson kernel

4.1. L_μ-harmonic measures

Let $h \in C(\partial\Omega)$. Then by [3, lemma 6.8], there exists a unique solution v_h of the Dirichlet problem

$$\begin{cases} L_\mu v = 0 & \text{in } \Omega \\ v = h \quad \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Let $\tilde{W}$ be as in (2.4). The boundary value condition in (4.1) is understood as

$$\lim_{\operatorname{dist}(x,F)\to 0} \frac{v(x)}{\tilde{W}(x)} = h \quad \text{for every compact set} \quad F \subset \partial\Omega. \tag{4.2}$$

Let $z \in \Omega$ and set $\mathcal{L}_{\mu,z}(h) := u_h(z)$, then the mapping $h \mapsto \mathcal{L}_{\mu,z}(h)$ is a linear positive functional on $C(\partial\Omega)$. Thus, there exists a unique Borel measure on $\partial\Omega$, called *L_μ-harmonic measure on $\partial\Omega$ relative to z* and denoted by ω_Ω^z, such that

$$v_h(z) = \int_{\partial\Omega} h(y)\, \mathrm{d}\omega_\Omega^z(y).$$

Let $x_0 \in \Omega$ be a fixed reference point. Let $\{\Omega_n\}$ be a C^2 *exhaustion* of Ω, i.e. $\{\Omega_n\}$ is an increasing sequence of bounded C^2 domains such that

$$\overline{\Omega}_n \subset \Omega_{n+1}, \quad \cup_n \Omega_n = \Omega, \quad \mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega),$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.

Then $-L_\mu$ is uniformly elliptic and coercive in $H_0^1(\Omega_n)$ and its first eigenvalue $\lambda_{\mu,\Sigma}^{\Omega_n}$ in Ω_n is larger than its first eigenvalue $\lambda_{\mu,\Sigma}$ in Ω.

For $h \in C(\partial\Omega_n)$, the following problem

$$\begin{cases} -L_\mu v = 0 & \text{in } \Omega_n \\ v = h & \text{on } \partial\Omega_n, \end{cases}$$

admits a unique solution which allows to define the L_μ-harmonic measure $\omega_{\Omega_n}^{x_0}$ on $\partial\Omega_n$ by

$$v(x_0) = \int_{\partial\Omega_n} h(y)\, \mathrm{d}\omega_{\Omega_n}^{x_0}(y).$$

Let $G_\mu^{\Omega_n}(x,y)$ be the Green kernel of $-L_\mu$ on Ω_n. Then $G_\mu^{\Omega_n}(x,y) \uparrow G_\mu(x,y)$ for $x, y \in \Omega, x \neq y$.

Proposition 4.1 [3, proposition 7.7]. *Assume $x_0 \in \Omega_1$. Then for every $Z \in C(\overline{\Omega})$,*

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Z(x)\tilde{W}(x)\, \mathrm{d}\omega_{\Omega_n}^{x_0}(x) = \int_{\partial\Omega} Z(x)\, \mathrm{d}\omega_\Omega^{x_0}(x).$$

4.2. Poisson kernel

By the standard elliptic theory, we can easily show that for any $x \in \Omega$, $G_\mu(x,\cdot) \in C^{1,\gamma}(\overline{\Omega} \setminus (\Sigma \cup \{x\})) \cap C^2(\Omega \setminus \{x\})$ for all $\gamma \in (0,1)$. Therefore, we may define the *Poisson kernel* associated to $-L_\mu$ in $\Omega \times (\partial\Omega \setminus \Sigma)$ as

$$P_\mu(x,y) := -\frac{\partial G_\mu}{\partial \mathbf{n}}(x,y), \quad x \in \Omega,\ y \in \partial\Omega \setminus \Sigma, \tag{4.3}$$

where $\mathbf{n}$ is the unit outer normal vector of $\partial\Omega$. This kernel satisfies the following properties.

PROPOSITION 4.2. *Let $x_0 \in \Omega$ be a fixed reference point.*

(i) *Then there holds*

$$P_\mu(x,y) = P_\mu(x_0,y)K_\mu(x,y), \quad x \in \Omega,\ y \in \partial\Omega \setminus \Sigma. \tag{4.4}$$

(ii) *For any $h \in L^1(\partial\Omega; \mathrm{d}\omega_\Omega^{x_0})$ with compact support in $\partial\Omega \setminus \Sigma$, there holds*

$$\int_{\partial\Omega} h(y)\,\mathrm{d}\omega_\Omega^{x_0}(y) = \mathbb{P}_\mu[h\tilde{W}](x_0). \tag{4.5}$$

Here

$$\mathbb{P}_\mu[h\tilde{W}](x) := \int_{\partial\Omega} P_\mu(x,y)h(y)\tilde{W}(y)\,\mathrm{d}S_{\partial\Omega}(y), \quad x \in \Omega,$$

where $S_{\partial\Omega}$ is the $(N-1)$-dimensional surface measure on $\partial\Omega$.

Proof.

(i) We note that $P_\mu(\cdot,y)$ is L_μ-harmonic in Ω and

$$\lim_{x\in\Omega,\ x\to\xi} \frac{P_\mu(x,y)}{\tilde{W}(x)} = 0 \quad \text{for all } \xi \in \partial\Omega \setminus \{y\} \quad \text{and} \quad y \in \partial\Omega \setminus \Sigma.$$

Hence, $\frac{P_\mu(x,y)}{P_\mu(x_0,y)}$ is a kernel function with pole at y and basis at x_0 in the sense of [**3**, definition 2.7]. This, together with the fact that any kernel function with pole at y and basis at x_0 is unique (see [**3**, proposition 7.3]), implies (4.4).

(ii) Let $\zeta \in C(\partial\Omega)$ with compact support in $\partial\Omega \setminus \Sigma$ such that $\mathrm{dist}\,(\mathrm{supp}\,\zeta, \Sigma) = r > 0$. Let $Z \in C(\overline{\Omega})$ be such that $Z(y) = \zeta(y)$ for any $y \in \partial\Omega$ and $Z(y) = 0$ in $\Sigma_{\frac{r}{2}}$. Set $r_0 = \frac{1}{4}\min\{\beta_2, r\}$ where β_2 is the constant in (A.7). We consider

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a decreasing sequence of bounded C^2 domains $\{\Sigma_n\}$ such that

$$\Sigma \subset \Sigma_{n+1} \subset \overline{\Sigma}_{n+1} \subset \Sigma_n \subset \overline{\Sigma}_n \subset \Sigma_{\frac{r_0}{4}}, \quad \cap_n \Sigma_n = \Sigma. \tag{4.6}$$

Let ϕ_* be the unique solution of

$$\begin{cases} -L_\mu \phi_* = 0 & \text{in } \Omega \\ \phi_* = 1 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where the boundary condition in (4.7) is understood as

$$\lim_{\mathrm{dist}\,(x,F)\to 0} \frac{\phi_*(x)}{\tilde{W}(x)} = 1 \quad \text{for every compact set } F \subset \partial\Omega.$$

Then, by [3, lemma 6.8 and estimate (6.21)], there exist constants $c_1 = c_1(\Omega, \Sigma, \Sigma_n, \mu)$ and $c_2 = c_2(\Omega, \Sigma, N, \mu)$ such that $0 < c_1 \leqslant \phi_*(x) \leqslant c_2 d_\Sigma(x)^{-\alpha_+}$ for all $x \in \Omega \setminus \Sigma_n$. By the standard elliptic theory, $\phi_* \in C^2(\Omega) \cap C^{1,\gamma}(\overline{\Omega} \setminus \Sigma)$ for any $0 < \gamma < 1$.

Now, for any $\eta \in C(\partial\Omega)$, we can easily show that u_η is a solution of

$$\begin{cases} -L_\mu v = 0 & \text{in } \Omega \setminus \Sigma_n \\ v = \eta & \text{on } \partial(\Omega \setminus \Sigma_n) \end{cases}$$

if and only if $w_\eta = \frac{u_\eta}{\phi_*}$ is a solution of

$$\begin{cases} -\mathrm{div}(\phi_*^2 \nabla w) = 0 & \text{in } \Omega \setminus \Sigma_n \\ w = \dfrac{\eta}{\phi_*} & \text{on } \partial(\Omega \setminus \Sigma_n). \end{cases}$$

Since the operator $L_{\phi_*} w := -\mathrm{div}(\phi_*^2 \nabla w)$ is uniformly elliptic and has smooth coefficients, we may deduce the existence of L_μ-harmonic measure ω_n^x on $\partial(\Omega \setminus \Sigma_n)$ and the Green kernel G_μ^n of $-L_\mu$ in $\Omega \setminus \Sigma_n$.

Let v_n be the unique solution of

$$\begin{cases} -L_\mu v = 0 & \text{in } \Omega \setminus \Sigma_n \\ v = Z\tilde{W} & \text{on } \partial(\Omega \setminus \Sigma_n). \end{cases} \tag{4.8}$$

Then by the representation formula we have

$$v_n(x) = \int_{\partial(\Omega\setminus\Sigma_n)} Z\tilde{W}\, \mathrm{d}\omega_n^x(y) = \int_{\partial\Omega \cap \mathrm{supp}\,\zeta} \zeta \tilde{W}\, \mathrm{d}\omega_n^x(y).$$

Proceeding as in the proof of [3, proposition 7.7], we may show that

$$v_n(x) \to \int_{\partial\Omega\cap\mathrm{supp}\,\zeta} \zeta(y)\, \mathrm{d}\omega_\Omega^x(y) =: v(x) \quad \text{as } n \to \infty.$$

On the other hand, the Poisson kernel P_μ^n of $-L_\mu$ in $\Omega \setminus \Sigma_n$ is well defined and given by

$$P_\mu^n(x,y) = -\frac{\partial G_\mu^n}{\partial \mathbf{n}^n}(x,y), \quad x \in \Omega \setminus \Sigma_n,\ y \in \partial(\Omega \setminus \Sigma_n),$$

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where $\mathbf{n}^n$ is the unit outer normal vector to $\partial(\Omega \setminus \Sigma_n)$. Hence,

$$v_n(x) = \int_{\partial(\Omega\setminus\Sigma_n)} P_\mu^n(x,y) Z(y) \tilde{W}(y) \, \mathrm{d}S_{\partial\Omega}(y)$$
$$= \int_{\partial\Omega\cap\operatorname{supp}\zeta} P_\mu^n(x,y)\zeta(y)\tilde{W}(y) \, \mathrm{d}S_{\partial\Omega}(y),$$

where $S_{\partial\Omega}$ is the $(N-1)$-dimensional surface measure on $\partial(\Omega \setminus \Sigma_n)$. Combining all above, we obtain

$$\int_{\partial\Omega\cap\operatorname{supp}\zeta} \zeta(y) \, \mathrm{d}\omega_n^x(y) = \int_{\partial\Omega\cap\operatorname{supp}\zeta} P_\mu^n(x,y)\tilde{W}(y)\zeta(y) \, \mathrm{d}S_{\partial\Omega}(y). \tag{4.9}$$

Put $\beta = \frac{1}{2}\min\{d_{\partial\Omega}(x), r_0\}$. Since $G_\mu^n(x,y) \nearrow G_\mu(x,y)$ for any $x \neq y$ and $x, y \in \Omega$, $\{G_\mu^n(x,\cdot)\}_n$ is uniformly bounded in $W^{2,\kappa}(\Omega_\beta \setminus \Sigma_{r_0})$ for any $\kappa > 1$. Thus, by the standard compact Sobolev embedding, there exists a subsequence, still denoted by index n, which converges to $G_\mu(x,\cdot)$ in $C^1(\overline{\Omega_\beta \setminus \Sigma_{r_0}})$ as $n \to \infty$. This implies that $P_\mu^n(x,\cdot) \to P_\mu(x,\cdot)$ uniformly on $\partial\Omega \setminus \Sigma_{r_0}$ as $n \to \infty$.

Therefore, by letting $n \to \infty$ in (4.9), we obtain

$$\int_{\partial\Omega} \zeta(y) \, \mathrm{d}\omega_\Omega^x(y) = \lim_{n\to\infty} \int_{\partial\Omega} \zeta(y)\tilde{W}(y) \, \mathrm{d}\omega_n^x(y)$$
$$= \lim_{n\to\infty} \int_{\partial\Omega} P_\mu^n(x,y)\zeta(y)\tilde{W}(y) \, \mathrm{d}S_{\partial\Omega}(y)$$
$$= \int_{\partial\Omega} P_\mu(x,y)\zeta(y)\tilde{W}(y) \, \mathrm{d}S_{\partial\Omega}(y). \tag{4.10}$$

By (4.10) and the fact that $\inf_{y\in\partial\Omega\setminus\Sigma_r} P_\mu(x_0,y) > 0 \quad \forall\, r > 0$, we deduce that

$$\omega_\Omega^x(E) = \mathbb{P}_\mu[\mathbb{1}_E \tilde{W}](x)$$

for any Borel set $E \subset \overline{E} \subset \partial\Omega \setminus \Sigma$. This implies in particular that $\omega_\Omega^{x_0}$ and $S_{\partial\Omega}$ are mutually absolutely continuous with respect to compact subsets of $\partial\Omega \setminus \Sigma$.

Now, assume $0 \leqslant h \in L^1(\partial\Omega; \mathrm{d}\omega_\Omega^{x_0})$ has compact support in $\partial\Omega \setminus \Sigma$ and $\operatorname{dist}(\operatorname{supp} h, \Sigma) = 4r > 0$. Then there exists a sequence of nonnegative functions $\{h_n\} \subset C(\partial\Omega)$ with compact support in $\partial\Omega \setminus \Sigma$ such that $\operatorname{dist}(\operatorname{supp} h_n, \Sigma) = 2r > 0$ for any $n \in \mathbb{N}$ and $h_n \to h$ in $L^1(\partial\Omega; \mathrm{d}\omega_\Omega^{x_0})$ as $n \to \infty$.

Applying (4.10) with ζ replaced by $|h_n - h_m|$ for $m, n \in \mathbb{N}$ and using the fact that $\inf_{y\in\Sigma_r\cap\partial\Omega}(P_\mu(x_0,y)\tilde{W}(y)) > 0$, we have

$$\int_{\partial\Omega} |h_m(y) - h_n(y)| \, \mathrm{d}\omega_\Omega^{x_0}(y) = \int_{\partial\Omega} P_\mu(x_0,y)|h_n(y) - h_n(y)|\tilde{W}(y) \, \mathrm{d}S_{\partial\Omega}(y)$$
$$\geqslant c \int_{\partial\Omega} |h_n(y) - h_n(y)| \, \mathrm{d}S_{\partial\Omega}(y).$$

This implies that $\{h_n\}$ is a Cauchy sequence in $L^1(\partial\Omega)$. Therefore, there exists $\tilde{h} \in L^1(\partial\Omega)$ such that $h_n \to \tilde{h}$ in $L^1(\partial\Omega)$. Since $\omega_\Omega^{x_0}$ and $S_{\partial\Omega}$ are mutually absolutely continuous with respect to compact subsets of $\partial\Omega \setminus \Sigma$ and h and $\tilde{h}$ have compact

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support in $\partial\Omega \setminus \Sigma$, we deduce that $h = \tilde{h}\ \omega_\Omega^{x_0}$-a.e. and $S_{\partial\Omega}$-a.e. in $\partial\Omega$. In particular, $h \in L^1(\partial\Omega)$.

Applying (4.10) with ζ replaced by h_n, for any $n \in \mathbb{N}$, we have

$$\int_{\partial\Omega} h_n(y)\,\mathrm{d}\omega_\Omega^{x_0}(y) = \int_{\partial\Omega} P_\mu(x_0,y)h_n(y)\tilde{W}(y)\,\mathrm{d}S_{\partial\Omega}(y). \tag{4.11}$$

By letting $n \to \infty$ in (4.11), we obtain (4.5).

Next, we assume $h \in L^1(\partial\Omega;\mathrm{d}\omega_\Omega^{x_0})$ and drop the assumption that $h \geqslant 0$, then we write $h = h_+ - h_-$ where $h_\pm \in L^1(\partial\Omega;\mathrm{d}\omega_\Omega^{x_0})$. By applying (4.5) for $h_\pm$, we deduce that (4.5) holds true for $h \in L^1(\partial\Omega;\mathrm{d}\omega_\Omega^{x_0})$. Moreover, we can show that $h_\pm \in L^1(\partial\Omega)$, which implies $h \in L^1(\partial\Omega)$. □

Proposition 4.3.

(i) *For any $h \in L^1(\partial\Omega;\mathrm{d}\omega_\Omega^{x_0})$ with compact support in $\partial\Omega \setminus \Sigma$, there holds*

$$-\int_\Omega \mathbb{K}_\mu[h\mathrm{d}\omega_\Omega^{x_0}]L_\mu\eta\,\mathrm{d}x = -\int_{\partial\Omega} \frac{\partial\eta}{\partial\mathbf{n}}(y)h(y)\tilde{W}(y)\,\mathrm{d}S_{\partial\Omega}(y), \quad \forall\eta \in \mathbf{X}_\mu(\Omega). \tag{4.12}$$

(ii) *For any $\nu \in \mathfrak{M}(\partial\Omega)$ with compact support in $\partial\Omega \setminus \Sigma$, there holds*

$$-\int_\Omega \mathbb{K}_\mu[\nu]L_\mu\eta\,\mathrm{d}x = -\int_{\partial\Omega} \frac{\partial\eta}{\partial\mathbf{n}}(y)\frac{1}{P_\mu(x_0,y)}\,\mathrm{d}\nu(y), \quad \forall\eta \in \mathbf{X}_\mu(\Omega), \tag{4.13}$$

where $P_\mu(x_0,y)$ is defined in (4.3) and $\mathbf{X}_\mu(\Omega)$ is defined by (2.8).

Proof.

(i) Let $\zeta \in C(\partial\Omega)$ with compact support in $\partial\Omega \setminus \Sigma$ such that $\mathrm{dist}\,(\mathrm{supp}\,\zeta, \Sigma) = r > 0$. We consider a function $Z \in C(\overline{\Omega})$ such that $Z(y) = \zeta(y)$ for any $y \in C(\partial\Omega)$ and $Z(y) = 0$ in $\Sigma_{\frac{r}{2}}$. Set $r_0 = \frac{1}{4}\min\{\beta_2, r\}$ where β_2 is the constant in (A.7). Let $\{\Sigma_n\}$ be as in (4.6), $\eta \in \mathbf{X}_\mu(\Omega)$ and v_n be the solution of (4.8). In view of the proof of proposition 4.2, $v_n \in C(\overline{\Omega \setminus \Sigma_n})$ and

$$v_n(x) = \int_{\partial\Omega} \zeta(y)\tilde{W}(y)\,\mathrm{d}\omega_n^x(y) = \int_{\partial\Omega} P_\mu^n(x,y)\tilde{W}(y)\zeta(y)\,\mathrm{d}S_{\partial\Omega}(y).$$

Put

$$v(x) = \int_{\partial\Omega} \zeta(y)\,\mathrm{d}\omega_\Omega^x(y) \quad \text{and} \quad w(x) = \int_{\partial\Omega} |\zeta(y)|\,\mathrm{d}\omega_\Omega^x(y).$$

Then $v_n(x) \to v(x)$ for any $x \in \Omega$ and $|v_n(x)| \leqslant w(x)$ in $\Omega \setminus \Sigma_n$. By [**30**, proposition 1.3.7],

$$-\int_{\Omega\setminus\Sigma_n} v_n L_\mu \tilde{Z}\,\mathrm{d}x = -\int_{\partial\Omega} \tilde{W}\zeta\frac{\partial\tilde{Z}}{\partial\mathbf{n}}\,\mathrm{d}S_{\partial\Omega}, \quad \forall\tilde{Z} \in C_0^2(\Omega \setminus \Sigma_n). \tag{4.14}$$

By approximation, the above equality is valid for any $\tilde{Z} \in C^{1,\gamma}(\overline{\Omega \setminus \Sigma_n})$, for some $\gamma \in (0,1)$ and $\Delta\tilde{Z} \in L^\infty(\Omega \setminus \Sigma_n)$. Hence, we may choose $\tilde{Z} = \eta_n$ in

(4.14), where η_n satisfies

$$\begin{cases} -L_\mu \eta_n = -L_\mu \eta & \text{in } \Omega \setminus \Sigma_n \\ \eta_n = 0 & \text{on } \partial(\Omega \setminus \Sigma_n) \end{cases}$$

to obtain

$$-\int_{\Omega\setminus\Sigma_n} v_n L_\mu \eta \,\mathrm{d}x = -\int_{\partial\Omega} \tilde{W}\zeta \frac{\partial \eta_n}{\partial \mathbf{n}} \,\mathrm{d}S_{\partial\Omega}. \tag{4.15}$$

We note that $\eta_n \to \eta$ locally uniformly in Ω and in $C^1(\overline{\Omega \setminus \Sigma_1})$. Therefore, by the dominated convergence theorem, letting $n \to \infty$ in (4.15), we obtain

$$-\int_{\Omega} v L_\mu \eta \,\mathrm{d}x = -\int_{\partial\Omega} \tilde{W}\zeta \frac{\partial \eta}{\partial \mathbf{n}} \,\mathrm{d}S_{\partial\Omega}. \tag{4.16}$$

Now, let $h \in L^1(\partial\Omega; \mathrm{d}\omega_\Omega^x)$ with compact support in $\partial\Omega \setminus \Sigma$ such that $\operatorname{dist}(\operatorname{supp} h, \Sigma) = 4r > 0$. By (4.5) we may construct a sequence $\{h_n\} \subset C(\partial\Omega)$ such that h_n has compact support in $\partial\Omega \setminus \Sigma$ with $\operatorname{dist}(\operatorname{supp} h_n, \Sigma) > r$ for any $n \in \mathbb{N}$. In addition, the same sequence can be constructed such that $h_n \to h$ in $L^1(\partial\Omega; \mathrm{d}\omega_\Omega^{x_0})$ and in $L^1(\partial\Omega)$.
Set

$$u_n(x) = \int_{\partial\Omega} K_\mu(x,y) h_n(y) \,\mathrm{d}\omega_\Omega^{x_0}(y) = \mathbb{K}_\mu[h_n \,\mathrm{d}\omega_\Omega^{x_0}](x), \quad x \in \Omega.$$

Since $K_\mu(\cdot, y) \in C^2(\Omega)$ for any $y \in \partial\Omega$, by the above equality, we deduce that $u_n \to u$ locally uniformly in Ω, where

$$u(x) = \int_{\partial\Omega} K_\mu(x,y) h(y) \,\mathrm{d}\omega_\Omega^{x_0}(y) = \mathbb{K}_\mu[h \,\mathrm{d}\omega_\Omega^{x_0}](x), \quad x \in \Omega.$$

By (4.16) with $v = u_n$ and $\zeta = h_n$, there holds

$$-\int_{\Omega} u_n L_\mu \eta \,\mathrm{d}x = -\int_{\partial\Omega} \tilde{W} h_n \frac{\partial \eta}{\partial \mathbf{n}} \,\mathrm{d}S_{\partial\Omega}. \tag{4.17}$$

Now, by [**3**, theorem 9.2], there exists a positive constant $C = C(N, \Omega, \Sigma, \mu, \kappa)$ such that $\|u_n\|_{L^\kappa(\Omega;\phi_{\mu,\Sigma})} \leqslant C \int_\Omega |h_n| \,\mathrm{d}\omega_\Omega^{x_0}(y)$ for all $n \in \mathbb{N}$ and for any $1 < \kappa < \min\left\{\frac{N+1}{N-1}, \frac{N+\alpha_-+1}{N+\alpha_--1}\right\}$. This in turn implies that $\{u_n\}$ is equi-integrable in $L^1(\Omega; \phi_{\mu,\Sigma})$. Therefore, by Vitali's convergence theorem, $u_n \to u$ in $L^1(\Omega; \phi_{\mu,\Sigma})$. Letting $n \to \infty$ in (4.17), we obtain (4.12).

(ii) Assume $\operatorname{dist}(\operatorname{supp} \nu, \Sigma) = 4r > 0$ and let $\{h_n\}$ be a sequence of functions in $C(\partial\Omega)$ with compact support in $\partial\Omega \setminus \Sigma$ such that $\operatorname{dist}(\operatorname{supp} h_n, \Sigma) \geqslant r$ and $h_n \rightharpoonup \nu$, i.e.

$$\int_{\partial\Omega} \zeta h_n \,\mathrm{d}S_{\partial\Omega} \to \int_{\partial\Omega} \zeta \mathrm{d}\nu \quad \forall \zeta \in C(\partial\Omega). \tag{4.18}$$

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In addition, we assume that $\|h_n\|_{L^1(\partial\Omega)} \leqslant C\|\nu\|_{\mathfrak{M}(\partial\Omega)}$ for every $n \geqslant 1$, for some positive constant C independent of n.
Set

$$u_n(x) = \int_{\partial\Omega} K_\mu(x,y)\frac{h_n(y)}{\tilde{W}(y)P_\mu(x_0,y)}\,\mathrm{d}\omega_\Omega^{x_0}(y).$$

By (4.5) and (4.18), we have

$$u_n(x) = \int_{\partial\Omega} K_\mu(x,y)h_n(y)\,\mathrm{d}S_{\partial\Omega}(y) \to \int_{\partial\Omega} K_\mu(x,y)\,\mathrm{d}\nu(y) =: u(x).$$

It means $u_n \to u$ a.e. in Ω.
Finally, equality (4.13) can be obtained by proceeding as in the proof of (i) and hence we omit its proof. □

5. Semilinear equations with a power absorption nonlinearity

5.1. Keller–Osserman estimates

In this section, we prove Keller–Osserman type estimates for nonnegative solutions of equation (E_+).

LEMMA 5.1. *Assume $p > 1$. Let $u \in C^2(\Omega)$ be a nonnegative solution of equation (E_+). Then there exists a positive constant $C = C(\Omega, \Sigma, \mu, p)$ such that*

$$0 \leqslant u(x) \leqslant Cd_{\partial\Omega}(x)^{-\frac{2}{p-1}}, \quad \forall x \in \Omega. \tag{5.1}$$

Proof. Let β_0 be as in § A.1 and $\eta_{\beta_0} \in C_c^\infty(\mathbb{R}^N)$ such that

$$0 \leqslant \eta_{\beta_0} \leqslant 1, \quad \eta_{\beta_0} = 1 \text{ in } \overline{\Omega}_{\frac{\beta_0}{4}} \quad \text{and} \quad \operatorname{supp}\eta_{\beta_0} \subset \Omega_{\frac{\beta_0}{2}},$$

where Ω_ϵ is defined in (2.2). For $\varepsilon \in \left(0, \frac{\beta_0}{16}\right)$, we define

$$V_\varepsilon := 1 - \eta_{\beta_0} + \eta_{\beta_0}(d_{\partial\Omega} - \varepsilon)^{-\frac{2}{p-1}} \quad \text{in } \Omega \setminus \overline{\Omega}_\varepsilon.$$

Then $V_\varepsilon \geqslant 0$ in $\Omega \setminus \overline{\Omega}_\epsilon$. It can be checked that there exists $C = C(\Omega, \beta_0, \mu, p) > 1$ such that the function $W_\varepsilon := CV_\varepsilon$ satisfies

$$-L_\mu W_\varepsilon + W_\varepsilon^p = C(-L_\mu V_\varepsilon + C^{p-1}V_\varepsilon^p) \geqslant 0 \quad \text{in } \Omega \setminus \overline{\Omega}_\varepsilon. \tag{5.2}$$

Combining (E_+) and (5.2) yields

$$-L_\mu(u - W_\varepsilon) + u^p - W_\varepsilon^p \leqslant 0 \quad \text{in } \Omega \setminus \overline{\Omega}_\varepsilon. \tag{5.3}$$

We see that $(u - W_\varepsilon)^+ \in H_0^1(\Omega \setminus \overline{\Omega}_\varepsilon)$ and $(u - W_\varepsilon)^+$ has compact support in $\Omega \setminus \overline{\Omega}_\varepsilon$. By using $(u - W_\varepsilon)^+$ as a test function for (5.3), we deduce that

$$\begin{aligned} 0 &\geqslant \int_{\Omega\setminus\Omega_\varepsilon} |\nabla(u - W_\varepsilon)^+|^2 \mathrm{d}x - \mu \int_{\Omega\setminus\Omega_\varepsilon} \frac{[(u-W_\varepsilon)^+]^2}{d_\Sigma^2} \mathrm{d}x \\ &\quad + \int_{\Omega\setminus\Omega_\varepsilon} (u^p - W_\varepsilon^p)(u - W_\varepsilon)^+ \, \mathrm{d}x \\ &\geqslant \int_{\Omega\setminus\Omega_\varepsilon} |\nabla(u - W_\varepsilon)^+|^2 \, \mathrm{d}x - \mu \int_{\Omega\setminus\Omega_\varepsilon} \frac{[(u-W_\varepsilon)^+]^2}{d_\Sigma^2} \, \mathrm{d}x \\ &\geqslant \lambda_{\mu,\Sigma} \int_{\Omega\setminus\Omega_\varepsilon} |(u - W_\varepsilon)^+|^2 \, \mathrm{d}x. \end{aligned}$$

This and the assumption $\lambda_{\mu,\Sigma} > 0$ imply $(u - W_\varepsilon)^+ = 0$, whence $u \leqslant W_\varepsilon$ in $\Omega \setminus \overline{\Omega}_\varepsilon$. Letting $\varepsilon \to 0$, we obtain the desired result. □

The following theorem is the main tool in the study of the boundary removable singularities for nonnegative solutions of equation (E_+). We assume additionally that Ω is a C^3 domain which is needed to apply proposition A.2.

THEOREM 5.2. *Let $p > 1$, $F \subset \Sigma$ be a compact subset of Σ and $d_F(x) = \mathrm{dist}\,(x, F)$. We additionally assume that Ω is a C^3 bounded domain. If $u \in C^2(\Omega)$ is a nonnegative solution of (E_+) satisfying*

$$\lim_{x\in\Omega,\ x\to\xi} \frac{u(x)}{\tilde{W}(x)} = 0 \quad \forall \xi \in \partial\Omega \setminus F, \quad \textit{locally uniformly in } \partial\Omega \setminus F, \tag{5.4}$$

then there exists a positive constant $C = C(N, \Omega, \Sigma, \mu, p)$ such that

$$u(x) \leqslant C d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_- -1} \quad \forall x \in \Omega, \tag{5.5}$$

$$|\nabla u(x)| \leqslant C d_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_- -1} \quad \forall x \in \Omega. \tag{5.6}$$

Proof. The proof is in the spirit of [**30**, proposition 3.4.3]. Let β_5 be the positive constant defined in proposition A.2. Let $\xi \in (\Sigma_{\beta_5} \cap \partial\Omega) \setminus F$ and put $d_{F,\xi} = \frac{1}{16} d_F(\xi) < 1$. Denote

$$\Omega^\xi := \frac{1}{d_{F,\xi}}\Omega = \{y \in \mathbb{R}^N : \ d_{F,\xi}\, y \in \Omega\} \quad \text{and} \quad \Sigma^\xi := \frac{1}{d_{F,\xi}}\Sigma = \{y \in \mathbb{R}^N : \ d_{F,\xi}\, y \in \Sigma\}.$$

If u is a nonnegative solution of (E_+) in Ω then the function

$$u^\xi(y) := d_{F,\xi}^{\frac{2}{p-1}} u(d_{F,\xi} y), \quad y \in \Omega^\xi,$$

is a nonnegative solution of

$$-\Delta u^\xi - \frac{\mu}{|\mathrm{dist}(y, \Sigma^\xi)|^2} u^\xi + \left(u^\xi\right)^p = 0 \tag{5.7}$$

in Ω^ξ.

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Now we note that u^ξ is a nonnegative L_μ subharmonic function and satisfies [by (5.4)]

$$\lim_{y\in\Omega^\xi,\, y\to P}\frac{u^\xi(y)}{\tilde W^\xi(y)}=0\quad \forall P\in B\left(\frac{1}{d_{F,\xi}}\xi,2\right)\cap\partial\Omega^\xi,$$

where

$$\tilde W^\xi(y)=1-\eta_{\frac{\beta_3}{d_{F,\xi}}}+\eta_{\frac{\beta_3}{d_{F,\xi}}}W^\xi(y)\quad \text{in } \Omega^\xi\setminus\Sigma^\xi,$$

$\eta_{\frac{\beta_3}{d_{F,\xi}}}$ is the scaled version of the function η_{β_3} defined before (2.4), and

$$W^\xi(y)=\begin{cases}(d_{\partial\Omega^\xi}(y)+d_{\Sigma^\xi}(y))^2 d_{\Sigma^\xi}(y)^{-\alpha_+} & \text{if } \mu<H^2,\\ (d_{\partial\Omega^\xi}(y)+d_{\Sigma^\xi}(y))^2 d_{\Sigma^\xi}(y)^{-H}|\ln d_{\Sigma^\xi}(y)| & \text{if } \mu=H^2,\end{cases}\qquad x\in\Omega^\xi\setminus\Sigma^\xi.$$

Set $R_0=\min\{\beta_5,1\}$. In view of the proof of [**3**, lemma 6.2 and estimate (6.7)], there exists a positive constant c depending on Ω,Σ,μ and

$$\int_{B\left(\frac{1}{d_F(\xi)}\xi,2R_0\right)\cap\Omega^\xi} u^\xi(y)\, d_{\partial\Omega^\xi}(y)\, d_{\Sigma^\xi}(y)^{-\alpha_-}\,\mathrm{d}y \tag{5.8}$$

such that

$$u^\xi(y)\leqslant c\, d_{\partial\Omega^\xi}(y)\, d_{\Sigma^\xi}(y)^{-\alpha_-}\quad \forall y\in B\left(\frac{1}{d_F(\xi)}\xi,R_0\right)\cap\Omega^\xi. \tag{5.9}$$

Let $r_0=\frac{R_0}{16}$ and let $w_{r_0,\xi}$ be the supersolution of (5.7) in $\mathscr{B}(\frac{1}{d_{F,\xi}}\xi,r_0)\cap\Omega^\xi$ constructed in proposition A.2 with $R=r_0$ and $z=\frac{1}{d_{F,\xi}}\xi$.

Taking into account of (5.9) and using a similar argument as in the proof of lemma 5.1, we can show that

$$u^\xi(y)\leqslant w_{r_0,\xi}(y)\quad \forall y\in\mathscr{B}\left(\frac{1}{d_{F,\xi}}\xi,r_0\right)\cap\Omega^\xi.$$

By (5.8), (5.9) and the above inequality, we may deduce that

$$u^\xi(y)\leqslant c\, d_{\partial\Omega^\xi}(y)\, d_{\Sigma^\xi}(y)^{-\alpha_-}\quad \forall y\in B\left(\frac{1}{d_F(\xi)}\xi,\frac{r_0}{16}\right)\cap\Omega^\xi, \tag{5.10}$$

where c depends only on Ω,Σ,μ,p, the C^3 characteristic of Ω^ξ and the C^2 characteristic of Σ^ξ. As $d_{F,\xi}\leqslant 1$ the C^3 characteristic of Ω (respectively the C^2 characteristic of Σ) is also a C^3 characteristic of Ω^ξ (respectively a C^2 characteristic of Σ^ξ), therefore this constant c can be taken to be independent of ξ. Thus, for any $\xi\in(\Sigma_{\beta_5}\cap\partial\Omega)\setminus F$ such that $d_F(x)\leqslant 16$, there holds

$$u(x)\leqslant c\, d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_-} d_F(\xi)^{-\frac{2}{p-1}+\alpha_--1}\quad \forall x\in B\left(\xi,r_1 d_F(\xi)\right)\cap\Omega, \tag{5.11}$$

where $r_1=\frac{r_0}{16^2}$.

Now, we consider three cases.

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Case 1: $x \in \Sigma_{\frac{r_1}{32}} \cap \Omega$ and $d_F(x) < 1$. If $d_{\partial\Omega}(x) \leqslant \frac{r_1}{8+r_1} d_F(x)$ then there exits a unique point in $\xi \in \partial\Omega \setminus F$ such that $|x-\xi| = d_{\partial\Omega}(x)$. Hence,

$$d_F(\xi) \leqslant d_{\partial\Omega}(x) + d_F(x) \leqslant 2\frac{4+r_1}{8+r_1} d_F(x) < 16, \tag{5.12}$$

$d_F(x) \leqslant \frac{8+r_1}{8} d_F(\xi)$ and $d_{\partial\Omega}(x) \leqslant \frac{r_1}{8} d_F(\xi)$. This, combined with (5.11), (5.12) and the fact that $d_F(x) \approx d_F(\xi)$, yields

$$u(x) \leqslant C d_\Sigma(x)^{-\alpha_-} d_F(\xi)^{-\frac{2}{p-1}+\alpha_- -1} \leqslant C d_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_- -1}.$$

If $d_{\partial\Omega}(x) > \frac{r_1}{8+r_1} d_F(x) \geqslant \frac{r_1}{8+r_1} d_\Sigma(x)$ then by (5.1) and the fact that $d_{\partial\Omega}(x) \approx d_F(x) \approx d_\Sigma(x)$, we obtain

$$u(x) \leqslant C d_{\partial\Omega}(x)^{-\frac{2}{p-1}} \leqslant C d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_- -1}.$$

Thus, (5.5) holds for every $x \in \Sigma_{\frac{r_1}{4}}$ such that $d_F(x) < 1$.

Case 2: $x \in \Sigma_{\frac{r_1}{32}} \cap \Omega$ and $d_F(x) \geqslant 1$. Let ξ be the unique point in $\partial\Omega \setminus F$ such that $|x-\xi| = d_{\partial\Omega}(x)$. Since u is an L_μ-subharmonic function in $B(\xi, r_1) \cap \Omega$, in view of the proof of (5.10), we obtain

$$\begin{aligned} u(x) &\leqslant C d_{\partial\Omega}(x) d_\Sigma(x)^{-\alpha_-} \leqslant C d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_- -1} \\ &\forall x \in B\left(\xi, \frac{\beta_3}{2}\right) \cap \Omega, \end{aligned}$$

where C depend only on Ω, Σ, μ, p.

Case 3: $x \in \Omega \setminus \Sigma_{\frac{r_1}{32}}$. The proof is similar to the one of [**21**, proposition A.3] and we omit it. (ii) Let $x_0 \in \Omega$. Put $\ell = d_{\partial\Omega}(x_0)$ and

$$\Omega^\ell := \frac{1}{\ell}\Omega = \{y \in \mathbb{R}^N : \ \ell y \in \Omega\} \quad \text{and} \quad \Sigma^\ell := \frac{1}{\ell}\Sigma = \{y \in \mathbb{R}^N : \ \ell y \in \Sigma\}.$$

If $x \in B\left(x_0, \frac{\ell}{2}\right)$ then $y = \ell^{-1}x$ belongs to $B\left(y_0, \frac{1}{2}\right)$, where $y_0 = \ell^{-1}x_0$. Also we have that $\frac{1}{2} \leqslant d_{\Omega^\ell}(y) \leqslant \frac{3}{2}$ and $\frac{1}{2} \leqslant d_{\Sigma^\ell}(y)$ for each $y \in B\left(y_0, \frac{1}{2}\right)$. Set $v(y) = u(\ell y)$ for $y \in B(y_0, \frac{1}{2})$ then v satisfies

$$-\Delta v - \frac{\mu}{d^2_{\Sigma^\ell}} v + \ell^2 |v|^p = 0 \quad \text{in } B\left(y_0, \frac{1}{2}\right).$$

By the standard elliptic estimates and (5.1) we have

$$\sup_{y \in B\left(y_0, \frac{1}{4}\right)} |\nabla v(y)| \leqslant C \sup_{y \in B\left(y_0, \frac{1}{3}\right)} |v(y)| \leqslant C v(y_0),$$

This, together with the equality $\nabla v(y) = \ell \nabla u(x)$, estimate (5.5) implies

$$\begin{aligned} |\nabla u(x_0)| &\leqslant C\ell^{-1} d_{\partial\Omega}(x_0) d_\Sigma^{-\alpha_-}(x_0)\, d_F(x_0)^{-\frac{2}{p-1}+\alpha_- -1} \\ &\leqslant C d_\Sigma(x_0)^{-\alpha_-} d_F(x_0)^{-\frac{2}{p-1}+\alpha_- -1}. \end{aligned}$$

Therefore, estimate (5.6) follows since x_0 is an arbitrary point. The proof is complete. $\square$

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5.2. Removable boundary singularities

This subsection is devoted to the study of removable boundary singularities for nonnegative solutions of equation (E_+) in the supercritical range. More precisely we will prove theorems 2.1 and 2.2.

Proof of theorem 2.1. We will only consider the case $\mu < H^2$ and $p = \frac{\alpha_+ + 1}{\alpha_+ - 1}$, since the proof in the other cases is very similar. Let u be a nonnegative solution of (E_+) satisfying (2.5). By (5.5) with $F = \Sigma$, there holds

$$u(x) \leqslant C d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_+} \quad \forall x \in \Omega, \tag{5.13}$$

for some constant C independent of u.

Let $\{\Omega_n\}$ be a C^2 exhaustion of Ω and we write

$$\mathbb{G}_\mu^{\Omega_n}\left[u^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}\right](x) = \int_{\Omega_n} G_\mu^{\Omega_n}(x, y) u(y)^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}\, \mathrm{d}y.$$

By the representation formula in Ω_n, we have that

$$u(x_0) + \mathbb{G}_\mu^{\Omega_n}\left[u^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}\right](x_0) = \int_{\partial\Omega_n} u(y)\, \mathrm{d}\omega_{\Omega_n}^{x_0}(y). \tag{5.14}$$

By (5.13), the definition of $\tilde{W}$ in (2.4), estimate (A.9) and proposition 4.1, we deduce

$$\limsup_{n\to\infty} \int_{\partial\Omega_n} u(y)\, \mathrm{d}\omega_{\Omega_n}^{x_0}(y) \leqslant C \limsup_{n\to\infty} \int_{\partial\Omega_n} \tilde{W}(y)\, \mathrm{d}\omega_{\Omega_n}^{x_0}(y) = C\omega_\Omega^{x_0}(\partial\Omega). \tag{5.15}$$

Since $G_\mu^{\Omega_n}(x, y) \uparrow G_\mu(x, y)$ for $x, y \in \Omega, x \neq y$, (5.14) and (5.15) yield

$$\mathbb{G}_\mu[u^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}](x_0) \leqslant C\omega_\Omega^{x_0}(\partial\Omega).$$

Hence, there exists another positive constant C independent of u such that

$$\int_\Omega u(y)^{\frac{\alpha_+ + 1}{\alpha_+ - 1}} \phi_{\mu,\Sigma}(y)\, \mathrm{d}y \leqslant C. \tag{5.16}$$

Consequently, the function $v = u + \mathbb{G}_\mu[u^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}]$ is a nonnegative L_μ-harmonic in Ω. This, together with theorem 3.3, implies the existence of a measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that

$$u + \mathbb{G}_\mu\left[u^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}\right] = \mathbb{K}_\mu[\nu] \quad \text{in } \Omega. \tag{5.17}$$

By proposition 4.1 and (5.13), we may deduce that ν has compact support in Σ.

Next we will show that $\nu \equiv 0$. Suppose by contradiction that $\nu \not\equiv 0$. Let $1 < M \in \mathbb{N}$ and $v_{M,n}$ be the positive solution of

$$\begin{cases} -L_\mu^{\Omega_n} v_{M,n} + v_{M,n}^{\frac{\alpha_+ + 1}{\alpha_+ - 1}} = 0 & \text{in } \Omega_n \\ v_{M,n} = Mu & \text{on } \partial\Omega_n. \end{cases}$$

Since Mu is a supersolution of the above problem, we have that $0 \leqslant v_{M,n} \leqslant Mu$ in Ω_n for any $n \in \mathbb{N}$. As a result, there exists $v_M \in C^2(\Omega)$ such that $v_{M,n} \to v_M$ locally uniformly in Ω and $L_\mu v_M + v_M^{\frac{\alpha_+ + 1}{\alpha_+ - 1}} = 0$ in Ω. As $v_M \leqslant Mu$, it follows that v_M satisfies (2.5) and hence thanks to theorem 5.2, estimate (5.5) holds for v_M with $F = \Sigma$, namely

$$v_M(x) \leqslant C d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_+} \quad \forall x \in \Omega, \tag{5.18}$$

for some constant C independent of v_M. By using an argument similar to the one leading to (5.16), we derive

$$\int_\Omega v_M(y)^{\frac{\alpha_+ + 1}{\alpha_+ - 1}} \phi_{\mu,\Sigma}(y)\, \mathrm{d}y \leqslant C_0, \tag{5.19}$$

for some constant C_0 independent of v_M. Also, by the representation formula we have

$$v_{M,n}(x) + \mathbb{G}_\mu^{\Omega_n}\left[v_{M,n}^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}\right](x) = M \int_{\partial\Omega_n} u(y)\, \mathrm{d}\omega_{\Omega_n}^x(y), \quad \forall x \in \Omega. \tag{5.20}$$

From (5.17), we have

$$\begin{aligned} \int_{\partial\Omega_n} u(y)\, \mathrm{d}\omega_{\Omega_n}^x(y) &= \int_{\partial\Omega_n} \left(\mathbb{K}_\mu[\nu](y) - \mathbb{G}_\mu\left[u^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}\right](y)\right) \mathrm{d}\omega_{\Omega_n}^x(y) \\ &= \mathbb{K}_\mu[\nu](x) - \int_{\partial\Omega_n} \mathbb{G}_\mu\left[u^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}\right](y)\, \mathrm{d}\omega_{\Omega_n}^x(y). \end{aligned}$$

By proposition 2.7 (ii), we find

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \mathbb{G}_\mu\left[u^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}\right](y)\, \mathrm{d}\omega_{\Omega_n}^x(y) = 0.$$

Hence, letting $n \to \infty$ in (5.20), we obtain

$$v_M + \mathbb{G}_\mu\left[v_M^{\frac{\alpha_+ + 1}{\alpha_+ - 1}}\right] = M\mathbb{K}_\mu[\nu] \quad \text{in } \Omega, \quad \forall M > 0.$$

We see that the above display contradicts with (5.18) and (5.19). The proof is complete. □

Next we turn to

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Proof of theorem 2.2. Without loss of generality, we may assume that $z = 0$. Let $\zeta : \mathbb{R} \to [0, \infty)$ be a smooth function such that $0 \leqslant \zeta \leqslant 1$, $\zeta(t) = 0$ for $|t| \leqslant 1$ and $\zeta(t) = 1$ for $|t| > 2$. For $\varepsilon > 0$, we set $\zeta_\varepsilon(x) = \zeta(\frac{|x|}{\varepsilon})$. Since $u \in C^2(\Omega)$, there holds

$$L_\mu(\zeta_\varepsilon u) = u\Delta\zeta_\varepsilon + \zeta_\varepsilon u^p + 2\nabla\zeta_\varepsilon \cdot \nabla u \quad \text{in} \quad \Omega.$$

By (5.5) and (5.6) with $F = \{0\} \subset \Sigma$, (1.4) and the estimate $\int_{\Sigma_\beta} d_\Sigma(x)^{-\alpha}\,\mathrm{d}x \lesssim \beta^{N-\alpha}$ for $\alpha < N - k$, we have

$$\int_\Omega \zeta_\varepsilon u^p \phi_{\mu,\Sigma}\,\mathrm{d}x \lesssim \varepsilon^{-\frac{2p}{p-1}+(\alpha_--1)p} \int_{\Omega\cap\{|x|>\varepsilon\}} d_\Sigma(x)^{(p+1)(1-\alpha_-)}\,\mathrm{d}x \lesssim \varepsilon^{-\frac{2p}{p-1}+(\alpha_--1)p}, \tag{5.21}$$

$$\begin{aligned}\int_\Omega u|\Delta\zeta_\varepsilon|\phi_{\mu,\Sigma}\,\mathrm{d}x &\leqslant \varepsilon^{-\frac{2}{p-1}+\alpha_--3} \int_{\Omega\cap\{\varepsilon<|x|<2\varepsilon\}} d_\Sigma(x)^{2-2\alpha_-}\,\mathrm{d}x \\ &\lesssim \varepsilon^{N-\frac{2}{p-1}-\alpha_--1} \lesssim 1,\end{aligned} \tag{5.22}$$

$$\begin{aligned}\int_\Omega |\nabla\zeta_\varepsilon||\nabla u|\phi_{\mu,\Sigma}\,\mathrm{d}x &\lesssim \varepsilon^{-\frac{2}{p-1}+\alpha_--2} \int_{\Omega\cap\{\varepsilon<|x|<2\varepsilon\}} d_\Sigma(x)^{-2\alpha_-+1}\,\mathrm{d}x \\ &\lesssim \varepsilon^{N-\frac{2}{p-1}-\alpha_--1} \lesssim 1.\end{aligned} \tag{5.23}$$

The estimates in (5.21) hold because of the assumption $p > 1$ if $\alpha_+ = 1$ or $p < \frac{\alpha_++1}{\alpha_+-1}$ if $\alpha_+ > 1$. For the last estimate in (5.22) and (5.23), we have used the assumption $p \geqslant \frac{N-\alpha_-+1}{N-\alpha_--1}$.

Estimates (5.21)–(5.23) imply that $L_\mu(\zeta_\varepsilon u) \in L^1(\Omega; \phi_{\mu,\Sigma})$. By [**3**, lemma 8.5], we have

$$-\int_\Omega \zeta_\varepsilon u L_\mu \eta\,\mathrm{d}x = -\int_\Omega (u\Delta\zeta_\varepsilon + \zeta_\varepsilon u^p + 2\nabla\zeta_\varepsilon \cdot \nabla u)\,\eta\,\mathrm{d}x, \quad \forall \eta \in \mathbf{X}_\mu(\Omega).$$

Taking $\eta = \phi_{\mu,\Sigma}$, we obtain

$$\lambda_{\mu,\Sigma}\int_\Omega \zeta_\varepsilon u\phi_{\mu,\Sigma}\,\mathrm{d}x + \int_\Omega \zeta_\varepsilon u^p\phi_{\mu,\Sigma}\,\mathrm{d}x = -\int_\Omega (u\Delta\zeta_\varepsilon + 2\nabla\zeta_\varepsilon \cdot \nabla u)\,\phi_{\mu,\Sigma}\,\mathrm{d}x.$$

By (5.22)–(5.23), we have

$$\lambda_{\mu,\Sigma}\int_\Omega \zeta_\varepsilon u\phi_{\mu,\Sigma}\,\mathrm{d}x + \int_\Omega \zeta_\varepsilon u^p\phi_{\mu,\Sigma}\,\mathrm{d}x \leqslant C\varepsilon^{N-\frac{2}{p-1}-\alpha_--1}.$$

By letting $\varepsilon \to 0$ and Fatou's lemma, we deduce that

$$\lambda_{\mu,\Sigma}\int_\Omega u\phi_{\mu,\Sigma}\,\mathrm{d}x + \int_\Omega u^p\phi_{\mu,\Sigma}\,\mathrm{d}x \lesssim \begin{cases} 0 & \text{if } p > \dfrac{N+\alpha_-+1}{N-\alpha_--1}, \\ 1 & \text{if } p = \dfrac{N+\alpha_-+1}{N-\alpha_--1}. \end{cases}$$

This implies that $u \equiv 0$ if $p > \frac{N-\alpha_-+1}{N-\alpha_--1}$ or $u \in L^p(\Omega; \phi_{\mu,\Sigma})$ if $p = \frac{N-\alpha_-+1}{N-\alpha_--1}$.

The rest of the proof can proceed as in the proof of theorem 2.1 and we omit it. □

5.3. Existence of solutions in the supercritical range

In this subsection, we discuss the existence of solutions for the following problem

$$\begin{cases} -L_\mu u + |u|^{p-1} u = 0 \text{in } \Omega, \\ \qquad \operatorname{tr}_{\mu,\Sigma}(u) = \nu, \end{cases} \tag{5.24}$$

where $p > 1$ and $\nu \in \mathfrak{M}(\partial\Omega)$. We will focus on the supercritical case $p \geqslant \min\{\frac{N+1}{N-1}, \frac{N-\alpha_-+1}{N-\alpha_--1}\}$. In particular, we will give various sufficient conditions for the existence of solutions to (5.24).

To this purpose, we use Besov space (see e.g. [**1**, **35**]). For $\sigma > 0$, $1 \leqslant \kappa < \infty$, we denote by $W^{\sigma,\kappa}(\mathbb{R}^d)$ the Sobolev space over $\mathbb{R}^d$. If σ is not an integer the Besov space $B^{\sigma,\kappa}(\mathbb{R}^d)$ coincides with $W^{\sigma,\kappa}(\mathbb{R}^d)$. When σ is an integer we denote $\Delta_{x,y} f := f(x+y) + f(x-y) - 2f(x)$ and

$$B^{1,\kappa}(\mathbb{R}^d) := \left\{ f \in L^\kappa(\mathbb{R}^d) : \frac{\Delta_{x,y} f}{|y|^{1+\frac{d}{\kappa}}} \in L^\kappa(\mathbb{R}^d \times \mathbb{R}^d) \right\},$$

with norm

$$\|f\|_{B^{1,\kappa}} := \left(\|f\|_{L^\kappa}^\kappa + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Delta_{x,y} f|^\kappa}{|y|^{\kappa+d}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{\kappa}}.$$

Then

$$B^{m,\kappa}(\mathbb{R}^d) := \left\{ f \in W^{m-1,\kappa}(\mathbb{R}^d) : D_x^\alpha f \in B^{1,\kappa}(\mathbb{R}^d) \ \forall \alpha \in \mathbb{N}^d \text{ such that } |\alpha| = m-1 \right\},$$

with norm

$$\|f\|_{B^{m,\kappa}} := \left(\|f\|_{W^{m-1,\kappa}}^\kappa + \sum_{|\alpha|=m-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|D_x^\alpha \Delta_{x,y} f|^\kappa}{|y|^{\kappa+d}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{\kappa}}.$$

These spaces are fundamental because they are stable under the real interpolation method developed by Lions and Petree.

It is well known that if $1 < \kappa < \infty$ and $\alpha > 0$, $L_{\alpha,\kappa}(\mathbb{R}^d) = W^{\alpha,\kappa}(\mathbb{R}^d)$ if $\alpha \in \mathbb{N}$. If $\alpha \notin \mathbb{N}$ then the positive cone of their dual coincide, i.e. $(L_{-\alpha,\kappa'}(\mathbb{R}^d))^+ = (B^{-\alpha,\kappa'}(\mathbb{R}^d))^+$, always with equivalent norms.

LEMMA 5.3. *Let $k \geqslant 1$, p be as in (2.14), and $\nu \in \mathfrak{M}^+(\mathbb{R}^k)$ with compact support in $B^k\left(0, \frac{R}{2}\right)$ for some $R > 0$. Set*

$$\vartheta := \frac{\alpha_+ + 1 - p(\alpha_+ - 1)}{p}. \tag{5.25}$$

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For $x \in \mathbb{R}^{k+1}$, we write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^k$. Then there exists a constant $C = C(R, N, k, \mu, p) > 1$ such that

$$\begin{aligned}
&C^{-1} \|\nu\|^p_{B^{-\vartheta,p}(\mathbb{R}^k)} \\
&\leqslant \int_{B^k(0,R)} \int_0^R x_1^{N-k-1-(p+1)(\alpha_- -1)} \\
&\quad \times \left(\int_{B^k(0,R)} (x_1 + |x' - y'|)^{-(N-2\alpha_-)} \, \mathrm{d}\nu(y') \right)^p \mathrm{d}x_1 \, \mathrm{d}x' \\
&\leqslant C \|\nu\|^p_{B^{-\vartheta,p}(\mathbb{R}^k)} .
\end{aligned}$$

Proof. The proof is very similar to that of [**19**, lemma 8.1], and hence we omit it. □

THEOREM 5.4. *Let $k \geqslant 1$, p be as in (2.14), and $\nu \in \mathfrak{M}^+(\partial\Omega)$ with compact support in Σ. Then there exists a constant $C = C(\Omega, \Sigma, \mu) > 1$ such that*

$$C^{-1} \|\nu\|_{B^{-\vartheta,p}(\Sigma)} \leqslant \|\mathbb{K}_\mu[\nu]\|_{L^p(\Omega;\phi_{\mu,\Sigma})} \leqslant C \|\nu\|_{B^{-\vartheta,p}(\Sigma)} ,$$

where ϑ is given in (5.25).

Proof. By (A.7), there exists $\xi^j \in \Sigma$, $j = 1, 2, \ldots, m_2$ (where $m_2 \in \mathbb{N}$ depends on N, Σ), and $\beta_2 \in (0, \frac{\beta_0}{4})$ such that $\Omega \cap \Sigma_{\beta_2} \subset \cup_{j=1}^{m_2} \mathcal{V}_\Sigma(\xi^j, \frac{\beta_0}{4}) \cap \Omega$.

Assume $\nu \in \mathfrak{M}^+(\partial\Omega)$ with compact support in $\Sigma \cap \mathcal{V}_\Sigma(\xi^j, \frac{\beta_0}{4})$ for some $j \in \{1, \ldots, m_2\}$.

On one hand, from (1.4), (3.3) and since $p < \frac{\alpha_+ + 1}{\alpha_+ - 1}$ and $\alpha_+ \geqslant \alpha_-$, we have

$$\begin{aligned}
&\int_{\Omega \cap \mathcal{V}_\Sigma\left(\xi^j, \frac{\beta_0}{2}\right)} \mathbb{K}_\mu[\nu]^p \phi_{\mu,\Sigma} \, \mathrm{d}x \\
&\quad \gtrsim \nu(\partial\Omega)^p \int_{\Omega \cap \mathcal{V}_\Sigma\left(\xi^j, \frac{\beta_0}{2}\right)} d_{\partial\Omega}(x)^{p+1} d_\Sigma(x)^{-(p+1)\alpha_-} \, \mathrm{d}x \gtrsim \nu(\partial\Omega)^p.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{\Omega \setminus \mathcal{V}_\Sigma(\xi^j, \frac{\beta_0}{2})} \mathbb{K}_\mu[\nu]^p \phi_{\mu,\Sigma} \, \mathrm{d}x \\
&\quad \lesssim \nu(\partial\Omega)^p \int_{\Omega \setminus \mathcal{V}_\Sigma(\xi^j, \frac{\beta_0}{2})} d_{\partial\Omega}(x)^{p+1} d_\Sigma(x)^{-(p+1)\alpha_-} \, \mathrm{d}x \lesssim \nu(\partial\Omega)^p.
\end{aligned}$$

Combining the above estimates, we obtain

$$\begin{aligned}
\int_\Omega \phi_{\mu,\Sigma} \mathbb{K}_\mu[\nu]^p \, \mathrm{d}x &= \int_{\Omega \setminus \mathcal{V}_\Sigma(\xi^j, \frac{\beta_0}{2})} \phi_{\mu,\Sigma} \mathbb{K}_\mu[\nu]^p \, \mathrm{d}x + \int_{\Omega \cap \mathcal{V}_\Sigma\left(\xi^j, \frac{\beta_0}{2}\right)} \phi_{\mu,\Sigma} \mathbb{K}_\mu[\nu]^p \, \mathrm{d}x \\
&\approx \int_{\Omega \cap \mathcal{V}_\Sigma\left(\xi^j, \frac{\beta_0}{2}\right)} \phi_{\mu,\Sigma} \mathbb{K}_\mu[\nu]^p \, \mathrm{d}x. \qquad (5.26)
\end{aligned}$$

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For any $x \in \mathbb{R}^N$, we write $x = (x', x''', x_N)$ where $x' = (x_1, \ldots, x_k)$, $x''' = (x_{k+1}, \ldots, x_{N-1})$ and define the C^2 function

$$\Phi(x) := (x', x_{k+1} - \Gamma^{\xi^j}_{k+1,\Sigma}(x'), \ldots, x_{N-1} - \Gamma^{\xi^j}_{N-1,\Sigma}(x'), x_N - \Gamma^{\xi^j}_{N,\partial\Omega}(x_1, \ldots, x_{N-1})).$$

Taking into account the local representation of Σ and $\partial\Omega$ in § A.1, we may deduce that $\Phi : \mathcal{V}_\Sigma(\xi^j, \beta_0) \to B^k(0, \beta_0) \times B^{N-1-k}(0, \beta_0) \times (-\beta_0, \beta_0)$ is C^2 diffeomorphism and $\Phi(x) = (x', 0_{\mathbb{R}^{N-k}})$ for $x = (x', x''', x_N) \in \Sigma$. In view of the proof of [**1**, lemma 5.2.2], there exists a measure $\overline{\nu} \in \mathfrak{M}^+(\mathbb{R}^k)$ with compact support in $B^k\left(0, \frac{\beta_0}{4}\right)$ such that for any Borel $E \subset B^k\left(0, \frac{\beta_0}{4}\right)$, there holds $\overline{\nu}(E) = \nu(\Phi^{-1}(E \times \{0_{\mathbb{R}^{N-k}}\}))$.

Set $\psi = (\psi', \psi''', \psi_N) = \Phi(x)$ then

$$\psi' = x',\ \psi''' = (x_{k+1} - \Gamma^{\xi^j}_{k+1,\Sigma}(x'), \ldots, x_{N-1} - \Gamma^{\xi^j}_{N-1,\Sigma}(x')) \text{ and } \psi_N = x_N - \Gamma^{\xi^j}_{N,\partial\Omega}(x_1, \ldots, x_{N-1}).$$

By (1.4), (A.6) and (3.3), we have

$$\begin{aligned} &\phi_{\mu,\Sigma}(x) \approx \psi_N(\psi_N + |\psi'''|)^{-\alpha_-}, \\ &K_\mu(x,y) \approx \psi_N(\psi_N + |\psi'''|)^{-\alpha_-}(\psi_N + |\psi'''| + |\psi' - y'|)^{-(N-2\alpha_-)}, \\ &\quad \forall x \in \mathcal{V}(\xi^j, \beta_0) \cap \Omega,\ \forall y = (y', y''', y_N) \in \mathcal{V}(\xi^j, \beta_0) \cap \Sigma. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\Omega \cap \mathcal{V}(\xi^j, \beta_0/2)} \phi_{\mu,\Sigma} \mathbb{K}^p_\mu[\nu] \, \mathrm{d}x \\ &\approx \int_{B^k\left(0,\frac{\beta_0}{2}\right)} \int_0^{\frac{\beta_0}{2}} \int_{B^{N-k-1}\left(0,\frac{\beta_0}{2}\right)} \psi_N^{p+1}(\psi_N + |\psi'''|)^{-(p+1)\alpha_-} \\ &\quad \left(\int_{B^k\left(0,\frac{\beta_0}{4}\right)} (\psi_N + |\psi'''| + |\psi' - y'|)^{-(N-2\alpha_-)} \, \mathrm{d}\overline{\nu}(y') \right)^p \mathrm{d}\psi''' \, \mathrm{d}\psi_N \, \mathrm{d}\psi' \qquad (5.27) \\ &\approx \int_{B^k\left(0,\frac{\beta_0}{2}\right)} \int_0^{\frac{\beta_0}{2}} r^{N-k-1-(p+1)(\alpha_- - 1)} \\ &\quad \left(\int_{B^k\left(0,\frac{\beta_0}{2}\right)} (r + |\psi' - y'|)^{-(N-2\alpha_- - 2)} \, \mathrm{d}\overline{\nu}(y') \right)^p \mathrm{d}r \, \mathrm{d}\psi'. \end{aligned}$$

Since $\nu \mapsto \nu \circ \Phi^{-1}$ is a C^2 diffeomorphism between $\mathfrak{M}^+(\Sigma \cap \mathcal{V}_\Sigma(\xi^j, \beta_0)) \cap B^{-\vartheta,p}(\Sigma \cap \mathcal{V}_\Sigma(\xi^j, \beta_0))$ and $\mathfrak{M}^+(B^k(0, \beta_0)) \cap B^{-\vartheta,p}(B^k(0, \beta_0))$, using (5.26), (5.27) and lemma 5.3, we derive that

$$C^{-1}\|\nu\|_{B^{-\vartheta,p}(\Sigma)} \leqslant \|\mathbb{K}_\mu[\nu]\|_{L^p(\Omega;\phi_{\mu,\Sigma})} \leqslant C\|\nu\|_{B^{-\vartheta,p}(\Sigma)}.$$

If $\nu \in \mathfrak{M}^+(\partial\Omega)$ has compact support in Σ, we may write $\nu = \sum_{j=1}^{m_2} \nu_j$, where $\nu_j \in \mathfrak{M}^+(\partial\Omega)$ with compact support in $\mathcal{V}(\xi^j, \frac{\beta_0}{4}) \cap \Sigma$. By (5.26), we can show that

$$\|\mathbb{K}_\mu[\nu]\|_{L^p(\Omega;\phi_{\mu,\Sigma})} \approx \sum_{j=1}^{m_2} \|\mathbb{K}_\mu[\nu_j]\|_{L^p(\Omega;\phi_{\mu,\Sigma})} \approx C\sum_{j=1}^{m_2}\|\nu_j\|_{B^{-\vartheta,p}(\Sigma)} \approx Cm_2\|\nu\|_{B^{-\vartheta,p}(\Sigma)}.$$

The proof is complete. □

Using theorem 5.4 and proposition 3.5, we are ready to prove theorem 2.5.

Proof of theorem 2.5. If ν is a positive measure which vanishes on Borel sets $E \subset \Sigma$ with $\mathrm{Cap}^{\mathbb{R}^k}_{\vartheta,p'}$-capacity zero then there exists an increasing sequence $\{\nu_n\}$ of positive measures in $B^{-\vartheta,p}(\Sigma)$ which converges weakly to ν (see [**10**, **16**]). By theorem 5.4, we have $\mathbb{K}_\mu[\nu_n] \in L^p(\Omega;\phi_{\mu,\Sigma})$, hence we may apply proposition 3.5 with $g(t) = |t|^{p-1}t$ to deduce that there exists a unique nonnegative weak solution u_n of (5.24) with $\mathrm{tr}_{\mu,\Sigma}(u_n) = \nu_n$.

Since $\{\nu_n\}$ is an increasing sequence of positive measures, by theorem 3.4, $\{u_n\}$ is increasing and its limit is denoted by u. Moreover,

$$-\int_\Omega u_n L_\mu\zeta\,\mathrm{d}x + \int_\Omega u_n^p\zeta\,\mathrm{d}x = -\int_\Omega \mathbb{K}_\mu[\nu_n]L_\mu\zeta\,\mathrm{d}x \quad \forall\zeta\in\mathbf{X}_\mu(\Omega). \tag{5.28}$$

By taking $\zeta = \phi_{\mu,\Sigma}$ in (5.28), we obtain

$$\int_\Omega (\lambda_{\mu,\Sigma}u_n + u_n^p)\,\phi_{\mu,\Sigma}\,\mathrm{d}x = \lambda_{\mu,\Sigma}\int_\Omega \mathbb{K}_\mu[\nu_n]\phi_{\mu,\Sigma}\,\mathrm{d}x,$$

which implies that $\{u_n\}$ and $\{u_n^p\}$ are uniformly bounded in $L^1(\Omega;\phi_{\mu,\Sigma})$. Therefore, $u_n \to u$ in $L^1(\Omega;\phi_{\mu,\Sigma})$ and in $L^p(\Omega;\phi_{\mu,\Sigma})$. By letting $n\to\infty$ in (5.28), we deduce

$$\int_\Omega -uL_\mu\zeta\,\mathrm{d}x + \int_\Omega u^p\zeta\,\mathrm{d}x = -\int_\Omega \mathbb{K}_\mu[\nu]L_\mu\zeta\,\mathrm{d}x \quad \forall\zeta\in\mathbf{X}_\mu(\Omega).$$

This means u is the unique weak solution of (5.24) with $\mathrm{tr}_{\mu,\Sigma}(u) = \nu$. □

Proof of theorem 2.6.

1. Suppose u is a weak solution of (5.24) with $\mathrm{tr}_{\mu,\Sigma}(u) = \nu$. Let $\beta > 0$. Since

$$\phi_{\mu,\Sigma}(x) \approx C(\beta)\,d_{\partial\Omega}(x) \text{ and } K_\mu(x,y) \approx C(\beta)\,d_{\partial\Omega}(x)|x-y|^{-N}, \\ \forall (x,y)\in(\Omega\setminus\Sigma_\beta)\times\partial\Omega, \tag{5.29}$$

proceeding as in the proof of [**32**, theorem 3.1], we may prove that ν is absolutely continuous with respect to the Bessel capacity $\mathrm{Cap}^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}$.

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2. We assume that $\nu \in \mathfrak{M}^+(\partial\Omega) \cap B^{-\frac{2}{p},p}(\partial\Omega)$ has compact support $F \in \partial\Omega \setminus \Sigma$. Then by (5.29), we may apply [**32**, Theorem A] to deduce that $\mathbb{K}_\mu[\nu] \in L^p(\Omega \setminus \Sigma_\beta; \phi_{\mu,\Sigma})$ for any $\beta > 0$. Denote $g_n(t) = \max\{\min\{|t|^{p-1}t, n\}, -n\}$. By applying proposition 3.5 with $g = g_n$, we deduce that there exists a unique weak solution $v_n \in L^1(\Omega; \phi_{\mu,\Sigma})$ of

$$\begin{cases} -L_\mu v_n + g_n(v_n) = 0 & \text{in } \Omega, \\ \mathrm{tr}_{\mu,\Sigma}(v_n) = \nu, \end{cases} \tag{5.30}$$

such that $0 \leqslant v_n \leqslant \mathbb{K}_\mu[\nu]$ in Ω. Furthermore, by (3.5), $\{v_n\}$ is non-increasing. Set $v = \lim_{n\to\infty} v_n$, then $0 \leqslant v \leqslant \mathbb{K}_\mu[\nu]$ in Ω.
Since v_n is a weak solution of (5.30), we have

$$-\int_\Omega v_n L_\mu \zeta \,\mathrm{d}x + \int_\Omega g_n(v_n)\zeta \,\mathrm{d}x = -\int_\Omega \mathbb{K}_\mu[\nu_n] L_\mu \zeta \,\mathrm{d}x \quad \forall \zeta \in \mathbf{X}_\mu(\Omega).$$

By taking $\phi_{\mu,\Sigma}$ as a test function, we obtain

$$\int_\Omega (\lambda_{\mu,\Sigma} v_n + g_n(v_n))\, \phi_{\mu,\Sigma} \,\mathrm{d}x = \lambda_{\mu,\Sigma} \int_\Omega \mathbb{K}_\mu[\nu] \phi_{\mu,\Sigma} \,\mathrm{d}x, \tag{5.31}$$

which, together with by Fatou's lemma, implies that $v, v^p \in L^1(\Omega; \phi_{\mu,\Sigma})$ and

$$\int_\Omega (\lambda_{\mu,\Sigma} v + v^p)\, \phi_{\mu,\Sigma} \,\mathrm{d}x \leqslant \lambda_{\mu,\Sigma} \int_\Omega \mathbb{K}_\mu[\nu] \phi_{\mu,\Sigma} \,\mathrm{d}x.$$

Hence, $v + \mathbb{G}_\mu[v^p]$ is a nonnegative L_μ-harmonic function. By the representation theorem 3.3, there exists a unique $\overline{\nu} \in \mathfrak{M}^+(\partial\Omega)$ such that $v + \mathbb{G}_\mu[v^p] = \mathbb{K}_\mu[\overline{\nu}]$. Since $v \leqslant \mathbb{K}_\mu[\nu]$, by proposition 2.7 (i), $\overline{\nu} = \mathrm{tr}_{\mu,\Sigma}(v) \leqslant \mathrm{tr}_{\mu,\Sigma}(\mathbb{K}_\mu[\nu]) = \nu$ and hence $\overline{\nu}$ has compact support in F.
Let $\beta > 0$ be small enough such that $F \cap \overline{\Sigma}_{4\beta} = \emptyset$. We consider a cut-off function $\psi_\beta \in C^\infty(\mathbb{R}^N)$ such that $0 \leqslant \psi_\beta \leqslant 1$ in $\mathbb{R}^N$, $\psi_\beta = 1$ in $\Omega \setminus \Sigma_{\frac{\beta}{2}}$ and $\psi_\beta = 0$ in $\overline{\Sigma}_{\frac{\beta}{4}}$. Let ϕ_0 be the eigenfunction associated to $-\Delta$ in Ω such that $\sup_{x\in\Omega} \phi_0 = 1$. Let $\eta \in C^\infty(\partial\Omega)$ such that $\eta = 0$ on $\partial\Omega \cap \Sigma_{2\beta}$. We consider the lifting $R[\eta]$ in [**32**, (1.11)]. Then $R \in C^2(\overline{\Omega})$ has compact support in $\overline{\Omega}_{\beta_0}$ for some $\beta_0 > 0$ small enough. In addition, $|\nabla R[\eta] \cdot \nabla \phi_0| \lesssim \phi_0$ in Ω, and $R[\eta] = \eta$ for any $x \in \partial\Omega$.
Then the function $\psi_{\beta,\eta} = \psi_\beta R[\eta]\phi_0 \in C^{1,\gamma}(\overline{\Omega}) \cap \mathbf{X}_\mu(\Omega)$ for any $\gamma \in (0,1)$, $\psi_{\beta,\eta} = 0$ on $\partial\Omega$ and has compact support in $\overline{\Omega} \setminus \Sigma_{\frac{\beta}{4}}$. Hence, by (4.13) and the fact that $\frac{\partial \psi_{\beta,\zeta}}{\partial \mathbf{n}} = \frac{\partial \phi_0}{\partial \mathbf{n}} \eta$ on $\partial\Omega$, we obtain

$$\int_\Omega (-v L_\mu \psi_{\beta,\eta} + v^p \psi_{\beta,\eta}) \,\mathrm{d}x = -\int_{\partial\Omega} \frac{\partial \phi_0}{\partial \mathbf{n}} \frac{\eta}{P_\mu(x_0, y)} \,\mathrm{d}\overline{\nu}(y). \tag{5.32}$$

Also,

$$\int_\Omega (-v_n L_\mu \psi_{\beta,\zeta} + g_n(v_n) \psi_{\beta,\zeta}) \,\mathrm{d}x = -\int_{\partial\Omega} \frac{\partial \phi_0}{\partial \mathbf{n}} \frac{\eta}{P_\mu(x_0, y)} \,\mathrm{d}\nu(y). \tag{5.33}$$

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Since $v \leqslant v_n \leqslant \mathbb{K}_\mu[\nu]$ and $\mathbb{K}_\mu[\nu] \in L^p(\Omega \setminus \Sigma_{\frac{\beta}{16}}; \phi_{\mu,\Sigma})$, by letting $n \to \infty$ in (5.33), we obtain by the dominated convergence theorem that

$$\int_\Omega (-vL_\mu \psi_{\beta,\zeta} + v^p \psi_{\beta,\zeta})\,\mathrm{d}x = -\int_{\partial\Omega} \frac{\partial \phi_0}{\partial \mathbf{n}} \frac{\eta}{P_\mu(x_0,y)}\,\mathrm{d}\nu(y). \tag{5.34}$$

From (5.32) and (5.34), we deduce that

$$-\int_{\partial\Omega} \frac{\partial \phi_0}{\partial \mathbf{n}} \frac{\eta}{P_\mu(x_0,y)}\,\mathrm{d}\nu(y) = -\int_{\partial\Omega} \frac{\partial \phi_0}{\partial \mathbf{n}} \frac{\eta}{P_\mu(x_0,y)}\,\mathrm{d}\overline{\nu}(y),$$

which implies that $\nu = \overline{\nu}$, since $-\frac{\partial \phi_0}{\partial \mathbf{n}} \approx 1$ in $\partial\Omega$, $P_\mu(x_0,y) \approx 1$ in $\partial\Omega \setminus \Sigma_{\frac{\beta}{4}}$ and ν, $\overline{\nu}$ have compact support in $\partial\Omega \setminus \overline{\Sigma}_{4\beta}$.

3. If $\nu \in \mathfrak{M}^+(\partial\Omega)$ vanishes on Borel sets $E \subset \partial\Omega$ with zero $\mathrm{Cap}^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}$-capacity and has compact support in $\partial\Omega \setminus \Sigma$ then there exists a nondecreasing sequence $\{\nu_n\}$ of positive measures in $B^{-\frac{2}{p},p}(\partial\Omega)$ which converges to ν (see [**10**, **16**]). Let u_n be the unique weak solution of (5.24) with $\mathrm{tr}_{\mu,\Sigma}(u_n) = \nu_n$. Since $\{\nu_n\}$ is nondecreasing, by (3.5), $\{u_n\}$ is nondecreasing. Moreover, $0 \leqslant u_n \leqslant \mathbb{K}_\mu[\nu_n] \leqslant \mathbb{K}_\mu[\nu]$. Denote $u = \lim_{n\to\infty} u_n$. By an argument similar to the one leading to (5.31), we obtain

$$\int_\Omega (\lambda_{\mu,\Sigma} u_n + u_n^p)\,\phi_{\mu,\Sigma}\,\mathrm{d}x = \lambda_{\mu,\Sigma} \int_\Omega \mathbb{K}_\mu[\nu_n]\phi_{\mu,\Sigma}\,\mathrm{d}x,$$

which yields that $u, u^p \in L^1(\Omega; \phi_{\mu,\Sigma})$. By the dominated convergence theorem, we derive

$$\int_\Omega (-uL_\mu\zeta + u^p\zeta)\,\mathrm{d}x = -\int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta\,\mathrm{d}x \quad \forall \zeta \in \mathbf{X}_\mu(\Omega),$$

and thus u is the unique weak solution of (5.24).

4. If $\mathbb{1}_F \nu$ is absolutely continuous with respect to $\mathrm{Cap}^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}$ for any compact set $F \subset \partial\Omega \setminus \Sigma$, we set $\nu_n = \mathbb{1}_{\partial\Omega \setminus \Sigma_{\frac{1}{n}}} \nu$ and u_n the weak solution of (5.24) with $\mathrm{tr}_{\mu,\Sigma}(u_n) = \nu_n$. By using an argument similar to that in case 3, we obtain the desired result. □

6. Semilinear equations with a power source nonlinearity

In this section, we study the following problem

$$\begin{cases} -L_\mu u = |u|^{p-1}u & \text{in } \Omega, \\ \mathrm{tr}_{\mu,\Sigma}(u) & = \sigma\nu, \end{cases} \tag{BVP$^\sigma_-$}$$

where $p > 1$, σ is a positive parameter and $\nu \in \mathfrak{M}^+(\partial\Omega)$.

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We remark that a positive function u is a weak solution of (BVP$^\sigma_-$) if and only if

$$u = \mathbb{G}_\mu[u^p] + \sigma \mathbb{K}_\mu[\nu] \quad \text{a.e. in } \Omega. \tag{6.1}$$

In the following proposition, we give a necessary and sufficient condition for the existence of solutions to problem (BVP$^\sigma_-$).

PROPOSITION 6.1. *Assume* $\mu \leqslant H^2$, $p > 1$ *and* $\nu \in \mathfrak{M}^+(\partial\Omega)$. *Then problem* (BVP$^\sigma_-$) *admits a weak solution if and only if there exists a positive constant* $C > 0$ *such that*

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^p] \leqslant C\, \mathbb{K}_\mu[\nu] \quad a.e.\ in\ \Omega.$$

Proof. The proof is similar to that of [**20**, proposition 6.2] with some minor modifications, and hence we omit it. □

6.1. Preparative results

For $\alpha \leqslant N$, set

$$\mathcal{N}_\alpha(x,y) := \frac{\max\{|x-y|, d_\Sigma(x), d_\Sigma(y)\}^\alpha}{|x-y|^{N-2}\max\{|x-y|, d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}^2}, \quad (x,y) \in \overline{\Omega}\times\overline{\Omega}, x \neq y,$$

$$\mathbb{N}_\alpha[\omega](x) := \int_{\overline{\Omega}} \mathcal{N}_\alpha(x,y)\,\mathrm{d}\omega(y), \quad \omega \in \mathfrak{M}^+(\overline{\Omega}). \tag{6.2}$$

Let $\alpha < N$, $b > 0$, $\theta > -N + k$ and $s > 1$. We define the capacity $\mathrm{Cap}^{b,\theta}_{\mathbb{N}_\alpha,s}$ by

$$\mathrm{Cap}^{b,\theta}_{\mathbb{N}_\alpha,s}(E) := \inf\left\{\int_{\overline{\Omega}} d^b_{\partial\Omega} d^\theta_\Sigma \phi^s\,\mathrm{d}x : \ \phi \geqslant 0,\ \mathbb{N}_\alpha[d^b_{\partial\Omega} d^\theta_\Sigma \phi] \geqslant \mathbb{1}_E\right\}$$

for any Borel set $E \subset \overline{\Omega}$.

Here $\mathbb{1}_E$ denotes the indicator function of E. By [**1**, theorem 2.5.1], we have

$$(\mathrm{Cap}^{b,\theta}_{\mathbb{N}_\alpha,s}(E))^{\frac{1}{s}} = \sup\{\omega(E) : \omega \in \mathfrak{M}^+(E), \|\mathbb{N}_\alpha[\omega]\|_{L^{s'}(\Omega; d^b_{\partial\Omega} d^\theta_\Sigma)} \leqslant 1\}.$$

If $\mu < \frac{N^2}{4}$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$, then, by (3.1) and (3.3), we can show that

$$G_\mu(x,y) \approx d_{\partial\Omega}(x)\, d_{\partial\Omega}(y)(d_\Sigma(x)\, d_\Sigma(y))^{-\alpha_-}\mathcal{N}_{2\alpha_-}(x,y) \quad \forall x,y \in \Omega, x \neq y \tag{6.3}$$

and

$$K_\mu(x,y) \approx d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_-}\mathcal{N}_{2\alpha_-}(x,y) \quad \forall (x,y) \in \Omega \times \partial\Omega. \tag{6.4}$$

Therefore, if the integral equation

$$v = \mathbb{N}_{2\alpha_-}[(d_{\partial\Omega} d_\Sigma^{-\alpha_-})^{p+1} v^p] + \ell \mathbb{N}_{2\alpha_-}[\nu] \tag{6.5}$$

has a solution v for some $\ell > 0$ then $\tilde{v} = d_{\partial\Omega} d_\Sigma^{-\alpha_-} v$ satisfies

$$\tilde{v} \approx \mathbb{G}_\mu[\tilde{v}^p] + \ell \mathbb{K}_\mu[\nu]. \tag{6.6}$$

This, together with [**5**, proposition 2.7], implies that equation (6.1) has a positive solution u for some $\sigma > 0$, whence problem (BVP$^\sigma_-$) has a weak positive solution u for some $\sigma > 0$.

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In order to show that (6.5) possesses a solution, we will apply the results in [**23**] which we recall here for the sake of completeness.

Let $\mathbf{Z}$ be a metric space and $\omega \in \mathfrak{M}^+(\mathbf{Z})$. Let $J : \mathbf{Z} \times \mathbf{Z} \to (0, \infty]$ be a Borel positive kernel such that J is symmetric and $1/J$ satisfies a quasi-metric inequality, i.e. there is a constant $C > 1$ such that for all $x, y, z \in \mathbf{Z}$,

$$\frac{1}{J(x,y)} \leqslant C\left(\frac{1}{J(x,z)} + \frac{1}{J(z,y)}\right). \tag{6.7}$$

Under these conditions, one can define the quasi-metric $\mathbf{d}$ by

$$\mathbf{d}(x,y) := \frac{1}{J(x,y)}$$

and denote by $\mathfrak{B}(x,r) := \{y \in \mathbf{Z} : \ \mathbf{d}(x,y) < r\}$ the open $\mathbf{d}$-ball of radius $r > 0$ and centre x. Note that this set can be empty.

For $\omega \in \mathfrak{M}^+(\mathbf{Z})$ and a positive function ϕ, we define the potentials $\mathbb{J}[\omega]$ and $\mathbb{J}[\phi,\omega]$ by

$$\mathbb{J}[\omega](x) := \int_{\mathbf{Z}} J(x,y)\,\mathrm{d}\omega(y) \quad \text{and} \quad \mathbb{J}[\phi,\omega](x) := \int_{\mathbf{Z}} J(x,y)\phi(y)\,\mathrm{d}\omega(y).$$

For $t > 1$ the capacity $\mathrm{Cap}^{\omega}_{\mathbb{J},t}$ in $\mathbf{Z}$ is defined for any Borel set $E \subset \mathbf{Z}$ by

$$\mathrm{Cap}^{\omega}_{\mathbb{J},t}(E) := \inf\left\{\int_{\mathbf{Z}} \phi(x)^t\,\mathrm{d}\omega(x) : \ \phi \geqslant 0, \ \mathbb{J}[\phi,\omega] \geqslant \mathbb{1}_E\right\}.$$

Proposition 6.2. ([**23**]) *Let $p > 1$ and $\lambda, \omega \in \mathfrak{M}^+(\mathbf{Z})$ such that*

$$\int_0^{2r} \frac{\omega\left(\mathfrak{B}(x,s)\right)}{s^2}\,\mathrm{d}s \leqslant C\int_0^r \frac{\omega\left(\mathfrak{B}(x,s)\right)}{s^2}\,\mathrm{d}s, \tag{6.8}$$

$$\sup_{y\in\mathfrak{B}(x,r)} \int_0^r \frac{\omega\left(\mathfrak{B}(y,r)\right)}{s^2}\mathrm{d}s \leqslant C\int_0^r \frac{\omega\left(\mathfrak{B}(x,s)\right)}{s^2}\mathrm{d}s, \tag{6.9}$$

for any $r > 0$, $x \in \mathbf{Z}$, where $C > 0$ is a constant. Then the following statements are equivalent.

1. *The equation $v = \mathbb{J}[|v|^p,\omega] + \ell\mathbb{J}[\lambda]$ has a positive solution for $\ell > 0$ small.*
2. *For any Borel set $E \subset \mathbf{Z}$, there holds $\int_E \mathbb{J}[\mathbb{1}_E\lambda]^p\mathrm{d}\omega \leqslant C\,\lambda(E)$.*
3. *The following inequality holds $\mathbb{J}[\mathbb{J}[\lambda]^p,\omega] \leqslant C\mathbb{J}[\lambda] < \infty$ $\omega - a.e.$*
4. *For any Borel set $E \subset \mathbf{Z}$ there holds $\lambda(E) \leqslant C\,\mathrm{Cap}^{\omega}_{\mathbb{J},p'}(E)$.*

We will point out below that $\mathbb{N}_\alpha$ defined in (6.2) with $\mathrm{d}\omega = d_{\partial\Omega}(x)^b\,d_\Sigma(x)^\theta\mathbb{1}_\Omega(x)$ $\mathrm{d}x$ satisfies all assumptions of $\mathbb{J}$ in proposition 6.2, for some appropriate $b, \theta \in \mathbb{R}$. Let us first prove the quasi-metric inequality.

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LEMMA 6.3. *Let $\alpha \leqslant N$. There exists a positive constant $C = C(\Omega, \Sigma, \alpha)$ such that*

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant C\left(\frac{1}{\mathcal{N}_\alpha(x,z)} + \frac{1}{\mathcal{N}_\alpha(z,y)}\right), \quad \forall x,y,z \in \overline{\Omega}. \tag{6.10}$$

Proof. **Case 1:** $0 < \alpha \leqslant N$**.** We first assume that $|x-y| < 2|x-z|$. Then by the triangle inequality, we have $d_\Sigma(z) \leqslant |x-z| + d_\Sigma(x) \leqslant 2\max\{|x-z|, d_\Sigma(x)\}$ hence

$$\max\{|x-z|, d_\Sigma(x), d_\Sigma(z)\} \leqslant 2\max\{|x-z|, d_\Sigma(x)\}.$$

If $|x-z| \geqslant d_\Sigma(x)$ then $|x-z| \geqslant d_{\partial\Omega}(x)$ which implies that $|x-z| \geqslant \frac{d_{\partial\Omega}(x)+|x-y|}{4}$. Now,

$$\begin{aligned}
&\frac{|x-z|^{N-2}\max\{|x-z|, d_{\partial\Omega}(x), d_{\partial\Omega}(z)\}^2}{\max\{|x-z|, d_\Sigma(x), d_\Sigma(z)\}^\alpha} \\
&\quad \geqslant 2^{-\alpha}|x-z|^{N-\alpha} \geqslant 2^{-2N+\alpha}(|x-y| + d_{\partial\Omega}(x))^{N-\alpha} \\
&\quad = 2^{-2N+\alpha}\frac{(|x-y| + d_{\partial\Omega}(x))^N}{(|x-y| + d_{\partial\Omega}(x))^\alpha} \gtrsim \frac{|x-y|^{N-2}\max\{|x-y|, d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}^2}{\max\{|x-y|, d_\Sigma(x), d_\Sigma(y)\}^\alpha},
\end{aligned} \tag{6.11}$$

since $d_{\partial\Omega}(x) \leqslant d_\Sigma(x)$.

If $|x-z| \leqslant d_\Sigma(x)$ then

$$\begin{aligned}
&\frac{|x-z|^{N-2}\max\{|x-z|, d_{\partial\Omega}(x), d_{\partial\Omega}(z)\}^2}{\max\{|x-z|, d_\Sigma(x), d_\Sigma(z)\}^\alpha} \\
&\quad \geqslant 2^{-\alpha-2}\, d_\Sigma(x)^{-\alpha}|x-z|^{N-2}\max\{|x-y|, d_{\partial\Omega}(x)\}^2 \\
&\quad \gtrsim \frac{|x-y|^{N-2}\max\{|x-y|, d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}^2}{\max\{|x-y|, d_\Sigma(x), d_\Sigma(y)\}^\alpha},
\end{aligned} \tag{6.12}$$

since $d_{\partial\Omega}(y) \leqslant |x-y| + d_{\partial\Omega}(x) \leqslant 2\max\{|x-y|, d_\Sigma(x)\}$. Combining (6.11)–(6.12), we obtain (6.10).

Next we consider the case $2|x-z| \leqslant |x-y|$. Then $\frac{1}{2}|x-y| \leqslant |y-z|$, thus by symmetry we obtain (6.10).

Case 2: $\alpha \leqslant 0$**.** Let $b \in [0,2]$, since $d_\Sigma(x) \leqslant |x-y| + d_\Sigma(y)$, it follows that

$$\max\{|x-y|, d_\Sigma(x), d_\Sigma(y)\} \leqslant |x-y| + \min\{d_\Sigma(x), d_\Sigma(y)\}.$$

Using the above estimate, we obtain

$$\begin{aligned}
&|x-y|^{N-b}\max\{|x-y|, d_\Sigma(x), d_\Sigma(y)\}^{-\alpha} \\
&\lesssim |x-y|^{N-b-\alpha} + \min\{d_\Sigma(x), d_\Sigma(y)\}^{-\alpha}|x-y|^{N-b} \\
&\lesssim |x-z|^{N-b-\alpha} + |y-z|^{N-b-\alpha} + \min\{d_\Sigma(x), d_\Sigma(y)\}^{-\alpha}(|x-z|^{N-b} + |y-z|^{N-b}) \\
&\lesssim \frac{|x-z|^{N-b}}{\max\{|x-z|, d_\Sigma(x), d_\Sigma(z)\}^\alpha} + \frac{|z-y|^{N-b}}{\max\{|z-y|, d_\Sigma(z), d_\Sigma(y)\}^\alpha}.
\end{aligned} \tag{6.13}$$

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Since $d_{\partial\Omega}(x) \leqslant |x-y| + d_{\partial\Omega}(y)$, we can easily show that $\max\{|x-y|, d_{\partial\Omega}(x), d_{\partial\Omega}(y)\} \leqslant |x-y| + \min\{d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}$. Hence,

$$\frac{1}{\mathcal{N}_\alpha(x,y)} = \frac{\max\{|x-y|, d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}^2 |x-y|^{N-2}}{\max\{|x-y|, d_\Sigma(x), d_\Sigma(y)\}^\alpha}$$
$$\leqslant \frac{2|x-y|^N}{\max\{|x-y|, d_\Sigma(x), d_\Sigma(y)\}^\alpha} + \frac{2\min\{d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}^2 |x-y|^{N-2}}{\max\{|x-y|, d_\Sigma(x), d_\Sigma(y)\}^\alpha}.$$

The desired result follows by the above inequality and (6.13). □

Next we give sufficient conditions for (6.8) and (6.9) to hold.

LEMMA 6.4. *Let $b>0$, $\theta+b>k-N$ and $\mathrm{d}\omega = d_{\partial\Omega}(x)^b\, d_\Sigma(x)^\theta \mathbb{1}_\Omega(x)\,\mathrm{d}x$. Then*

$$\omega(B(x,s)) \approx \max\{d_{\partial\Omega}(x), s\}^b \max\{d_\Sigma(x), s\}^\theta s^N, \quad \text{for all } x\in\Omega \text{ and } 0<s\leqslant 4\mathrm{diam}(\Omega). \tag{6.14}$$

Proof. Let $\beta_0, \beta_1, \beta_2$ be as in § A.1 and $s<\frac{\beta_2}{32}$. We first assume that $x\in\Sigma_{\frac{\beta_2}{4}}$.

Case 1: $d_{\partial\Omega}(x)\geqslant 2s$. Let $\Gamma=\partial\Omega$ or Σ. Then $\frac{1}{2}d_\Gamma(x)\leqslant d_\Gamma(y)\leqslant \frac{3}{2}d_\Gamma(x)$ for any $y\in B(x,s)$, therefore (6.14) follows easily in this case.

Case 2: $d_{\partial\Omega}(x)\leqslant 2s$ and $d_\Sigma(x)\geqslant 2s$. By estimate (2.9) in [5, Lemma 2.3], we have that

$$\int_{B(x,s)\cap\Omega} d_{\partial\Omega}(y)^b\,\mathrm{d}y \approx \max\{d_{\partial\Omega}(x), s\}^b s^N. \tag{6.15}$$

Therefore,

$$\int_{B(x,s)\cap\Omega} d_{\partial\Omega}(y)^b\, d_\Sigma(y)^\theta\,\mathrm{d}y \approx d_\Sigma(x)^\theta \int_{B(x,s)\cap\Omega} d_{\partial\Omega}(y)^b \mathrm{d}y \approx \max\{d_{\partial\Omega}(x), s\}^b \max\{d_\Sigma(x), s\}^\theta s^N.$$

Case 3: $d_{\partial\Omega}(x)\leqslant 2s$ and $d_\Sigma(x)\leqslant 2s$. By (A.2), there exists $\xi\in\Sigma$ such that $B(x,s)\cap\Omega\subset V_\Sigma(\xi,\beta_0)$. If $y\in B(x,s)$, then $|y'-x'|<s$ and $d_\Sigma(y)\leqslant d_\Sigma(x)+|x-y|\leqslant 3s$. Thus, by (A.3), $\delta_\Sigma^\xi(y)\leqslant C_1 s$ for any $y\in B(x,s)\cap\Omega$, where C_1 depends on $\|\Sigma\|_{C^2}, N$ and k. Therefore,

$$\int_{B(x,s)\cap\Omega} d_{\partial\Omega}(y)^b\, d_\Sigma(y)^\theta \mathrm{d}y \lesssim \int_{\{|x'-y'|<s\}}\int_{\{\delta_\Sigma^\xi(y)\leqslant C_1 s\}} (\delta_\Sigma^\xi(y))^{\theta+b}\,\mathrm{d}y''\mathrm{d}y' \approx s^{N+\theta+b}$$
$$\approx \max\{d_{\partial\Omega}(x), s\}^b \max\{d_\Sigma(x), s\}^\theta s^N.$$

Here the similar constants depend on $N, k, \|\Sigma\|_{C^2}$ and β_0.

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Case 4: $d_{\partial\Omega}(x) \leqslant 2s$ and $d_\Sigma(x) \leqslant 2s$ and $\theta < 0$. We have that $d_\Sigma(y)^\theta \geqslant 3^\theta s^\theta$ for any $y \in B(x,s)$. Hence,

$$\int_{B(x,s)\cap\Omega} d_{\partial\Omega}(y)^b\, d_\Sigma(y)^\theta\, \mathrm{d}y \gtrsim s^\theta \int_{B(x,s)\cap\Omega} d_{\partial\Omega}(y)^b\, \mathrm{d}y$$
$$\approx \max\{d_{\partial\Omega}(x), s\}^b \max\{d_\Sigma(x), s\}^\theta s^N.$$

Case 5: $d_{\partial\Omega}(x) \leqslant 2s$ and $d_\Sigma(x) \leqslant 2s$ and $\theta \geqslant 0$. By (A.7), there exists $\xi \in \Sigma$ such that $B(x,s)\cap\Omega \subset \mathcal{V}_\Sigma(\xi, \beta_0)$. Let C_Σ, $C_{\partial\Omega}$ be as in (A.3), A be as in (A.5) and $C_2 = \max\{C_\Sigma \|\Sigma\|_{C^2}, C_{\partial\Omega}\|\partial\Omega\|_{C^2}\}(A+1)$.

We first assume that $d_{\partial\Omega}(x) \leqslant \frac{s}{12NC_2}$ $d_\Sigma(x) \leqslant \frac{s}{12NC_2}$. Set

$$\mathcal{A} := \{\psi = (\psi', \psi'') \in \Omega : |x' - \psi'| < r_0,\ |\delta(\psi)| < r_0,\ |\delta_{2,\Sigma}(\psi)| < r_0\},$$

where $r_0 = \frac{s}{12N(A+1)}$. By (A.3), we have $\delta^\xi_\Sigma(x) \leqslant \frac{s}{12N(A+1)}$ and $\delta^\xi_{\partial\Omega}(x) \leqslant \frac{s}{12N(A+1)}$. In addition for any $y \in \mathcal{A}$, we have

$$|x'' - y''| \leqslant \delta^\xi_\Sigma(x) + \delta^\xi_\Sigma(y) + \left(\sum_{i=k+1}^N |\Gamma^\xi_{i,\Sigma}(x') - \Gamma^\xi_{i,\Sigma}(y')|^2\right)^{\frac{1}{2}}$$
$$\leqslant \frac{s}{3} + (N-k)\|\Sigma\|_{C^2}|x' - y'| + A(\delta^\xi_{2,\Sigma}(y) + \delta^\xi(y)) < s,$$

where in the last inequality we used (A.5). This implies that $\mathcal{A} \subset B(x,s)$. Consequently,

$$\int_{B(x,s)\cap\Omega} d_{\partial\Omega}(y)^b\, d_\Sigma(y)^\theta \mathrm{d}y$$
$$\approx \int_{B(x,s)\cap\Omega} \delta^\xi(y)^b (\delta^\xi_\Sigma(y))^\theta\, \mathrm{d}y \gtrsim \int_{\mathcal{A}} \delta^\xi(y)^b(\delta^\xi(y) + \delta^\xi_{2,\Sigma}(y))^\theta\, \mathrm{d}y$$
$$\approx s^{N+\theta+b} \approx C \approx \max\{d_{\partial\Omega}(x), s\}^b \max\{d_\Sigma(x), s\}^\theta s^N. \tag{6.16}$$

If $d_{\partial\Omega}(x) \leqslant \frac{s}{12NC_2}$ and $d_\Sigma(x) \geqslant \frac{s}{12NC_2}$ then

$$\int_{B(x,s)} d_{\partial\Omega}(y)^b\, d_\Sigma(y)^\theta\, \mathrm{d}y \geqslant \int_{B(x, \frac{s}{24NC_2})} d_{\partial\Omega}(y)^b\, d_\Sigma(y)^\theta\, \mathrm{d}y$$

and hence (6.16) follows by case 2.

If $d_{\partial\Omega}(x) \geqslant \frac{s}{12NC_2}$ then

$$\int_{B(x,s)} d_{\partial\Omega}(y)^b\, d_\Sigma(y)^\theta\, \mathrm{d}y \geqslant \int_{B\left(x, \frac{s}{24NC_2}\right)} d_{\partial\Omega}(y)^b\, d_\Sigma(y)^\theta\, \mathrm{d}y$$

and hence (6.16) follows by case 1.

Next we consider $x \in \Omega \setminus \Sigma_{\frac{\beta_2}{4}}$ and $s < \frac{\beta_2}{32}$. Then $d_\Sigma(y) \approx 1$ for any $y \in \Omega \cap B(x,s)$. This, together with (6.15), implies the desired result.

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If $\frac{\beta_2}{16} \leqslant s \leqslant 4\mathrm{diam}(\Omega)$ then $\omega(B(x,s)) \approx 1$, hence estimate (6.14) follows straightforward. The proof is complete. □

LEMMA 6.5. *Let* $\alpha < N$, $b > 0$, $\theta > \max\{k-N-b, -b-\alpha\}$ *and* $\mathrm{d}\omega = d_{\partial\Omega}(x)^b\, d_\Sigma(x)^\theta\, \mathbb{1}_\Omega(x)\,\mathrm{d}x$. *Then* (6.8) *holds.*

Proof. We note that if $s \geqslant (4\mathrm{diam}\,(\Omega))^{N-\alpha}$ then $\omega(\mathfrak{B}(x,s)) = \omega(\overline{\Omega}) < \infty$, where $\mathfrak{B}(x,s)$ is defined after (6.7), namely $\mathfrak{B}(x,s) = \{y \in \Omega \setminus \Sigma : \mathbf{d}(x,y) < s\}$ and $\mathbf{d}(x,y) = \frac{1}{\mathcal{N}_\alpha(x,y)}$. Let $M = (4\mathrm{diam}\,\Omega)^{N-\alpha}$. We first note that it is enough to show that

$$\omega\left(\mathfrak{B}(x,t)\right) \approx \begin{cases} d_{\partial\Omega}(x)^{b-\dfrac{2N}{N-2}}\, d_\Sigma(x)^{\theta+\dfrac{\alpha N}{N-2}}\, s^{\dfrac{N}{N-2}} & \text{if } s \leqslant d_{\partial\Omega}(x)^N\, d_\Sigma(x)^{-\alpha}, \\ s^{\dfrac{b+N}{N}}\, d_\Sigma(x)^{\theta+\alpha\dfrac{b+N}{N}} & \text{if } d_{\partial\Omega}(x)^N\, d_\Sigma(x)^{-\alpha} \\ & \quad < s \leqslant d_\Sigma(x)^{N-\alpha}, \\ s^{\dfrac{b+\theta+N}{N-\alpha}} & \text{if } d_\Sigma(x)^{N-\alpha} \leqslant s \leqslant M, \\ M^{\dfrac{b+\theta+N}{N-\alpha}} & \text{if } M \leqslant s. \end{cases} \tag{6.17}$$

Indeed, by the above display, we can easily deduce that

$$\int_0^s \frac{\omega\left(\mathfrak{B}(x,t)\right)}{t^2}\,\mathrm{d}t$$
$$\approx \begin{cases} d_{\partial\Omega}(x)^{b-\dfrac{2N}{N-2}}\, d_\Sigma(x)^{\theta+\dfrac{\alpha N}{N-2}}\, s^{\dfrac{2}{N-2}} & \text{if } s \leqslant d_{\partial\Omega}(x)^N\, d_\Sigma(x)^{-\alpha}, \\ s^{\dfrac{b}{N}}\, d_\Sigma(x)^{\theta+\alpha\dfrac{b+N}{N}} & \text{if } d_{\partial\Omega}(x)^N\, d_\Sigma(x)^{-\alpha} \\ & \quad < s \leqslant d_\Sigma(x)^{N-\alpha}, \\ s^{\dfrac{b+\theta+\alpha}{N-\alpha}} & \text{if } d_\Sigma(x)^{N-\alpha} \leqslant s \leqslant M, \\ M^{\dfrac{b+\theta+\alpha}{N-\alpha}} & \text{if } M \leqslant s, \end{cases} \tag{6.18}$$

since $b > 0$ and $b + \theta + \alpha > 0$. This in turn implies (6.8).

In order to show (6.17), we will consider three cases.

Case 1: $s \leqslant d_{\partial\Omega}(x)^N\, d_\Sigma(x)^{-\alpha}$.

(a) Let $y \in \mathfrak{B}(x,s)$ be such that $d_{\partial\Omega}(x) \leqslant |x-y|$ and $d_\Sigma(x) \leqslant |x-y|$. Then

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^{N-\alpha},$$

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thus if $|x-y|^{N-\alpha} \leqslant s \leqslant d_{\partial\Omega}(x)^N\, d_\Sigma(x)^{-\alpha} \leqslant d_{\partial\Omega}(x)^{N-\alpha}$ then $d_{\partial\Omega}(x) \approx d_\Sigma(x) \approx |x-y|$. Hence, there exist constants C_1, C_2 depending only on α, N such that

$$\begin{aligned}
&\Big\{y \in \Omega : |x-y| \leqslant C_1(d_\Sigma(x)^\alpha d_{\partial\Omega}(x)^{-2}s)^{\frac{1}{N-2}},\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) \leqslant |x-y|\Big\} \\
&\quad\subset \left\{y \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant s,\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) \leqslant |x-y|\right\} \\
&\quad\subset \Big\{y \in \Omega : |x-y| \leqslant C_2(d_\Sigma(x)^\alpha d_{\partial\Omega}(x)^{-2}s)^{\frac{1}{N-2}},\, d_{\partial\Omega}(x) \\
&\quad\leqslant |x-y|,\, d_\Sigma(x) \leqslant |x-y|\Big\}. \qquad (6.19)
\end{aligned}$$

(b) Let $y \in \mathfrak{B}(x,s)$ be such that $d_{\partial\Omega}(x) \leqslant |x-y|$ and $d_\Sigma(x) > |x-y|$. Then

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^N\, d_\Sigma(x)^{-\alpha},$$

thus if $|x-y|^N d_\Sigma(x)^{-\alpha} \leqslant s$, then $|x-y|^N \leqslant s d_\Sigma(x)^\alpha \leqslant d_{\partial\Omega}(x)^N$. Thus, $d_{\partial\Omega}(x) \approx |x-y|$. Hence, there exist constants C_1, C_2 depending only on α, N such that

$$\begin{aligned}
&\Big\{y \in \Omega : |x-y| \leqslant C_1(d_\Sigma(x)^\alpha d_{\partial\Omega}(x)^{-2}s)^{\frac{1}{N-2}},\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) > |x-y|\Big\} \\
&\quad\subset \left\{y \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant s,\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) > |x-y|\right\} \\
&\quad\subset \Big\{y \in \Omega : |x-y| \leqslant C_2(d_\Sigma(x)^\alpha d_{\partial\Omega}(x)^{-2}s)^{\frac{1}{N-2}},\, d_{\partial\Omega}(x) \\
&\quad\leqslant |x-y|,\, d_\Sigma(x) > |x-y|\Big\}.
\end{aligned}$$

(c) Let $y \in \mathfrak{B}(x,s)$ be such that $d_{\partial\Omega}(x) > |x-y|$. Then, $d_\Sigma(x) \geqslant d_{\partial\Omega}(x) > |x-y|$ and

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^{N-2} d_{\partial\Omega}(x)^2 d_\Sigma(x)^{-\alpha}.$$

Hence, there exist constants C_1, C_2 depending only on α, N such that

$$\begin{aligned}
&\Big\{x \in \Omega : |x-y| \leqslant C_1(d_\Sigma(x)^\alpha d_{\partial\Omega}(x)^{-2}s)^{\frac{1}{N-2}},\, d_{\partial\Omega}(x) > |x-y|\Big\} \\
&\quad\subset \left\{x \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant s,\, d_{\partial\Omega}(x) > |x-y|\right\} \\
&\quad\subset \Big\{x \in \Omega : |x-y| \leqslant C_2(d_\Sigma(x)^\alpha d_{\partial\Omega}(x)^{-2}s)^{\frac{1}{N-2}},\, d_{\partial\Omega}(x) > |x-y|\Big\}. \qquad (6.20)
\end{aligned}$$

Combining (6.19)–(6.20) and lemma 6.4, we deduce

$$\omega(\mathfrak{B}(x,s)) \approx \omega(B(x,s_1)) \approx d_{\partial\Omega}(x)^{b-\frac{2N}{N-2}}\, d_\Sigma(x)^{\theta+\frac{\alpha N}{N-2}} s^{\frac{N}{N-2}},$$

where $s_1 = (d_\Sigma(x)^\alpha d_{\partial\Omega}(x)^{-2}s)^{\frac{1}{N-2}}$.

Case 2: $d_{\partial\Omega}(x)^N\, d_\Sigma(x)^{-\alpha} < s \leqslant d_\Sigma(x)^{N-\alpha}$.

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(a) Let $y \in \mathfrak{B}(x,s)$ be such that $d_{\partial\Omega}(x) \leqslant |x-y|$ and $d_\Sigma(x) \leqslant |x-y|$. Then

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^{N-\alpha}.$$

Thus, if $|x-y|^{N-\alpha} \leqslant s \leqslant d_\Sigma(x)^{N-\alpha}$ then $d_\Sigma(x) \approx |x-y|$. Hence, there exist constants C_1, C_2 which depending only on α, N such that

$$\begin{aligned}
&\left\{y \in \Omega : |x-y| \leqslant C_1(d_\Sigma(x)^\alpha s)^{\frac{1}{N}},\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) \leqslant |x-y|\right\}\\
&\quad \subset \left\{y \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant s,\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) \leqslant |x-y|\right\}\\
&\quad \subset \left\{y \in \Omega : |x-y| \leqslant C_2(d_\Sigma(x)^\alpha s)^{\frac{1}{N}},\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) \leqslant |x-y|\right\}.
\end{aligned} \tag{6.21}$$

(b) Let $y \in \mathfrak{B}(x,s)$ be such that $d_{\partial\Omega}(x) \leqslant |x-y|$ and $d_\Sigma(x) > |x-y|$. Then

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^N\, d_\Sigma(x)^{-\alpha}.$$

Hence, there exist constants C_1, C_2 depending only on α, N such that

$$\begin{aligned}
&\left\{y \in \Omega : |x-y| \leqslant C_1(d_\Sigma(x)^\alpha s)^{\frac{1}{N}},\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) > |x-y|\right\}\\
&\quad \subset \left\{y \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant s,\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) > |x-y|\right\}\\
&\quad \subset \left\{y \in \Omega : |x-y| \leqslant C_2(d_\Sigma(x)^\alpha s)^{\frac{1}{N}},\, d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) > |x-y|\right\}.
\end{aligned}$$

(c) Let $y \in \mathfrak{B}(x,s)$ be such that $d_{\partial\Omega}(x) > |x-y|$. Then $d_\Sigma(x) > |x-y|$. In addition,

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^{N-2} d_{\partial\Omega}(x)^2\, d_\Sigma(x)^{-\alpha},$$

$$|x-y|^{N-2} d_{\partial\Omega}(x)^2\, d_\Sigma(x)^{-\alpha} \geqslant |x-y|^N\, d_\Sigma(x)^{-\alpha},$$

and

$$|x-y| \leqslant (d_\Sigma(x)^\alpha s)^{\frac{1}{N}} = (d_\Sigma(x)^\alpha s)^{\frac{1}{N-2}}\, (d_\Sigma(x)^\alpha s)^{-\frac{2}{N(N-2)}} \leqslant \left(d_\Sigma(x)^\alpha d_{\partial\Omega}(x)^{-2} s\right)^{\frac{1}{N-2}},$$

since $d_{\partial\Omega}(x)^N\, d_\Sigma(x)^{-\alpha} < s$. Hence, there exist constants C_1, C_2 depending only on α, N such that

$$\begin{aligned}
&\left\{y \in \Omega : |x-y| \leqslant C_1(d_\Sigma(x)^\alpha s)^{\frac{1}{N}},\, d_{\partial\Omega}(x) > |x-y|\right\}\\
&\quad \subset \left\{y \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant s,\, d_{\partial\Omega}(x) > |x-y|\right\}\\
&\quad \subset \left\{y \in \Omega : |x-y| \leqslant C_2(d_\Sigma(x)^\alpha s)^{\frac{1}{N}},\, d_{\partial\Omega}(x) > |x-y|\right\}.
\end{aligned} \tag{6.22}$$

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Combining (6.21)–(6.22) and lemma 6.4, we derive

$$\omega(\mathfrak{B}(x,s)) \approx \omega(B(x,s_2)) \approx s^{\frac{b+N}{N}}\, d_\Sigma(x)^{\theta+\alpha\frac{b+N}{N}},$$

where $s_2 = (d_\Sigma(x)^\alpha s)^{\frac{1}{N}}$.

Case 3: $d_\Sigma(x)^{N-\alpha} < s \leqslant (4\mathrm{diam}\,(\Omega))^{N-\alpha}$.

(a) Let $y \in \mathfrak{B}(x,s)$ be such that $d_{\partial\Omega}(x) \leqslant |x-y|$ and $d_\Sigma(x) \leqslant |x-y|$. Then

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^{N-\alpha}.$$

Hence, there exist constants C_1, C_2 which depend only on α, N such that

$$\begin{aligned}
&\left\{y \in \Omega : |x-y| \leqslant C_1 s^{\frac{1}{N-\alpha}},\ d_{\partial\Omega}(x) \leqslant |x-y|,\ d_\Sigma(x) \leqslant |x-y|\right\}\\
&\quad\subset \left\{y \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant s,\ d_{\partial\Omega}(x) \leqslant |x-y|,\ d_\Sigma(x) \leqslant |x-y|\right\}\\
&\quad\subset \left\{y \in \Omega : |x-y| \leqslant C_2 s^{\frac{1}{N-\alpha}},\ d_{\partial\Omega}(x) \leqslant |x-y|,\ d_\Sigma(x) \leqslant |x-y|\right\}.
\end{aligned} \tag{6.23}$$

(b) Let $y \in \mathfrak{B}(x,s)$ be such that $d_{\partial\Omega}(x) \leqslant |x-y|$ and $d_\Sigma(x) > |x-y|$. Then

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^N\, d_\Sigma(x)^{-\alpha}.$$

On one hand, if $\alpha > 0$, we have

$$|x-y|^N\, d_\Sigma(x)^{-\alpha} \leqslant |x-y|^{N-\alpha}$$

and since $d_\Sigma(x)^{N-\alpha} < s$, we have

$$|x-y| \leqslant (d_\Sigma(x)^\alpha s)^{\frac{1}{N}} = s^{\frac{1}{N-\alpha}}\, s^{-\frac{\alpha}{N(N-\alpha)}}\, d_\Sigma(x)^{\frac{\alpha}{N}} \leqslant s^{\frac{1}{N-\alpha}}.$$

On the other hand, if $\alpha \leqslant 0$ then

$$|x-y|^N\, d_\Sigma(x)^{-\alpha} \geqslant |x-y|^{N-\alpha}$$

and since $d_\Sigma(x)^{N-\alpha} < s$, we obtain

$$|x-y| \leqslant s^{\frac{1}{N-\alpha}} = s^{\frac{1}{N}}\, s^{\frac{\alpha}{N(N-\alpha)}} \leqslant (d_\Sigma(x)^\alpha s)^{\frac{1}{N}}.$$

Hence, there exist constants C_1, C_2 which depend only on α, N such that

$$\begin{aligned}
&\left\{y \in \Omega : |x-y| \leqslant C_1 s^{\frac{1}{N-\alpha}},\ d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) > |x-y|\right\}\\
&\quad\subset \left\{y \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant s,\ d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) > |x-y|\right\}\\
&\quad\subset \left\{y \in \Omega : |x-y| \leqslant C_2 s^{\frac{1}{N-\alpha}},\ d_{\partial\Omega}(x) \leqslant |x-y|,\, d_\Sigma(x) > |x-y|\right\}.
\end{aligned} \tag{6.24}$$

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(c) Let $y \in \mathfrak{B}(x,s)$ be such that $d_{\partial\Omega}(x) > |x-y|$. Then $d_\Sigma(x) > |x-y|$ and

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^{N-2} d_{\partial\Omega}(x)^2 d_\Sigma(x)^{-\alpha}.$$

In view of the proof of (6.22), we may deduce the existence of positive constants C_1, C_2 depending only on α, N such that

$$\begin{aligned}&\left\{y \in \Omega : |x-y| \leqslant C_1 (d_\Sigma(x)^\alpha s)^{\frac{1}{N}},\, d_{\partial\Omega}(x) > |x-y|\right\}\\ &\quad \subset \left\{y \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \geqslant s,\, d_{\partial\Omega}(x) > |x-y|\right\}\\ &\quad \subset \left\{y \in \Omega : |x-y| \leqslant C_2 (d_\Sigma(x)^\alpha s)^{\frac{1}{N}},\, d_{\partial\Omega}(x) > |x-y|\right\}.\end{aligned}$$

This and (6.24) imply the existence of two positive constants $\tilde{C}_1, \tilde{C}_2$ depending only on α, N such that

$$\begin{aligned}&\left\{y \in \Omega : |x-y| \leqslant \tilde{C}_1 s^{\frac{1}{N-\alpha}},\, d_{\partial\Omega}(x) > |x-y|\right\}\\ &\quad \subset \left\{y \in \Omega : \frac{1}{\mathcal{N}_\alpha(x,y)} \leqslant s,\; d_{\partial\Omega}(x) > |x-y|\right\}\\ &\quad \subset \left\{y \in \Omega : |x-y| \leqslant \tilde{C}_2 s^{\frac{1}{N-\alpha}},\; d_{\partial\Omega}(x) > |x-y|\right\}.\end{aligned} \tag{6.25}$$

Combining (6.23)–(6.25) and lemma 6.4, we obtain

$$\omega(\mathfrak{B}(x,s)) \approx \omega(B(x,s_3)) \approx s^{\frac{b+\theta+N}{N-\alpha}},$$

where $s_3 = s^{\frac{1}{N-\alpha}}$.

The proof is complete. □

LEMMA 6.6. *Let* $\alpha < N,\ b > 0,\ \theta > \max\{k-N-b, -b-\alpha\}$ *and* $\mathrm{d}\omega = d_{\partial\Omega}(x)^b\, d_\Sigma(x)^\theta\ \mathbb{1}_\Omega(x)\,\mathrm{d}x$. *Then* (6.9) *holds.*

Proof. Let $y \in \mathfrak{B}(x,s)$. We will consider three cases.

Case 1: $d_{\partial\Omega}(x) \leqslant 2|x-y|$ and $d_\Sigma(x) \leqslant 2|x-y|$. We can easily show that $d_{\partial\Omega}(y) \leqslant 3|x-y|$, $d_\Sigma(y) \leqslant 3|x-y|$ and

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^{N-\alpha}.$$

Therefore, $|x-y| \lesssim s^{\frac{1}{N-\alpha}}$, which implies that $d_\Sigma(x) + d_\Sigma(y) \lesssim s^{\frac{1}{N-\alpha}}$. By (6.18), we can easily show that

$$\int_0^s \frac{\omega\left(\mathfrak{B}(x,t)\right)}{t^2}\,\mathrm{d}t \approx \int_0^s \frac{\omega\left(\mathfrak{B}(y,t)\right)}{t^2}\,\mathrm{d}t \approx s^{\frac{b+\theta+\alpha}{N-\alpha}}.$$

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Case 2: $d_{\partial\Omega}(x) \leqslant 2|x-y|$ and $d_\Sigma(x) \geqslant 2|x-y|$. In this case, we have that $d_{\partial\Omega}(y) \leqslant 3|x-y|$, $\frac{1}{2}\, d_\Sigma(x) \leqslant d_\Sigma(y) \leqslant \frac{3}{2}\, d_\Sigma(x)$ and

$$\frac{1}{\mathcal{N}_\alpha(x,y)} \approx |x-y|^N\, d_\Sigma^{-\alpha}(x) \approx |x-y|^N\, d_\Sigma(y)^{-\alpha}.$$

This implies

$$d_{\partial\Omega}(x)^N\, d_\Sigma(x)^{-\alpha} \lesssim s \quad \text{and} \quad d_{\partial\Omega}(y)^N\, d_\Sigma(y)^{-\alpha} \lesssim s.$$

By (6.18), we obtain

$$\int_0^s \frac{\omega\left(\mathfrak{B}(x,t)\right)}{t^2}\,\mathrm{d}t \approx \int_0^s \frac{\omega\left(\mathfrak{B}(y,t)\right)}{t^2}\,\mathrm{d}t \approx \begin{cases} s^{\dfrac{b+\theta+\alpha}{N-\alpha}} & \text{if } d_\Sigma(x)^{N-\alpha} \leqslant s, \\ s^{\dfrac{b}{N}}\, d_\Sigma(x)^{\theta+\alpha\dfrac{b+N}{N}} & \text{if } s \leqslant d_\Sigma(x)^{N-\alpha}. \end{cases}$$

Case 3: $d_{\partial\Omega}(x) \geqslant 2|x-y|$. We first note that $d_\Sigma(x) \geqslant 2|x-y|$,

$$\frac{1}{2} d_{\partial\Omega}(x) \leqslant d_{\partial\Omega}(y) \leqslant \frac{3}{2} d_{\partial\Omega}(x) \quad \text{and} \quad \frac{1}{2}\, d_\Sigma(x) \leqslant d_\Sigma(y) \leqslant \frac{3}{2}\, d_\Sigma(x).$$

From (6.18), we infer that

$$\int_0^s \frac{\omega\left(\mathfrak{B}(x,t)\right)}{t^2}\,\mathrm{d}t \approx \int_0^s \frac{\omega\left(\mathfrak{B}(y,t)\right)}{t^2}\,\mathrm{d}t.$$

Combining cases 1–3, we derive (6.9). □

By applying proposition 6.2 with $J(x,y) = \mathcal{N}_{2\alpha_-}(x,y)$, $\mathrm{d}\omega = (d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_-})^{p+1}\mathrm{d}x$ and $\mathrm{d}\lambda = \mathrm{d}\nu$, we obtain the following result for any $\nu \in \mathfrak{M}^+(\partial\Omega)$.

Theorem 6.7. *Let p satisfy (2.11). Then the following statements are equivalent.*

1. *The equation*

$$v = \mathbb{N}_{2\alpha_-}[|v|^p \left(d_{\partial\Omega} d_\Sigma^{-\alpha_-}\right)^{p+1}] + \ell\mathbb{N}_{2\alpha_-}[\nu]$$

has a positive solution for $\ell > 0$ small.

2. *For any Borel set $E \subset \overline{\Omega}$, there holds*

$$\int_E \mathbb{N}_{2\alpha_-}[\mathbb{1}_E\nu]^p \left(d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_-}\right)^{p+1} \mathrm{d}x \leqslant C\,\nu(E).$$

3. *The following inequality holds*

$$\mathbb{N}_{2\alpha_-}[\mathbb{N}_{2\alpha_-}[\nu]^p \left(d_{\partial\Omega} d_\Sigma^{-\alpha_-}\right)^{p+1}] \leqslant C\mathbb{N}_{2\alpha_-}[\nu] < \infty \quad a.e.$$

4. *For any Borel set $E \subset \overline{\Omega}$ there holds*

$$\nu(E) \leqslant C\,\mathrm{Cap}^{p+1,-\alpha_-(p+1)}_{\mathbb{N}_{2\alpha_-},p'}(E).$$

Here we implicitly extend ν to whole $\overline{\Omega}$ by setting $\nu(\Omega) = 0$.

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6.2. Existence and nonexistence results in the case $\mu < \frac{N^2}{4}$

We first show that theorem 2.7 is a direct consequence of theorem 6.7.

Proof of theorem 2.7. We will use theorem 6.7 and show that statements 1–4 of the present theorem are equivalent to statements 1–4 of theorem 6.7 respectively. By (6.3)–(6.6) and [**5**, proposition 2.7], we can easily show that equation (6.5) has a solution v for some $\ell > 0$ if and only if equation

$$u = \mathbb{G}_\mu[u^p] + \sigma \mathbb{K}_\mu[\nu] \tag{6.26}$$

has a positive solution u for some $\sigma > 0$. This and the fact that u is a weak solution of (P_-^σ) if and only if u is represented by (6.26) imply that statement 1 of theorem 6.7 is equivalent to statement 1 of the present theorem. In addition, in light of (6.3) and (6.4), we can deduce that statements 2–3 of theorem 6.7 are equivalent to statements 2–3 of the present theorem respectively.

Therefore, it remains to prove that, under condition (2.14), statement 4 of this theorem is equivalent to statement 4 of theorem 6.7. It is enough to show that for any compact subset $E \subset \Sigma$, there holds

$$\mathrm{Cap}^{\Sigma}_{\vartheta,p'}(E) \approx \mathrm{Cap}^{p+1,-\alpha_-(p+1)}_{\mathbb{N}_{2\alpha_-},p'}(E),$$

where ϑ is defined in (2.15). Under condition (2.14), in view of (5.26)–(5.27), we may employ a similar argument as in the proof of [**20**, Estimate (6.36)] to reach the desired result. □

Remark 6.8. By [**3**, theorems B.1 and B.2], the following statements are valid.

(i) If $1 < p < \min\{\frac{N+1}{N-1}, \frac{N-\alpha_-+1}{N-\alpha_--1}\}$ then

$$\int_\Omega \mathbb{K}_\mu[|\nu|]^p \phi_{\mu,\Sigma}\, dx \leqslant C(\Omega,\Sigma,\mu,p)\, |\nu|(\Omega)^p, \quad \forall \nu \in \mathfrak{M}(\partial\Omega). \tag{6.27}$$

(ii) If $1 < p < \frac{N+1}{N-1}$ and $\nu \in \mathfrak{M}(\partial\Omega)$ has compact support in $\partial\Omega \setminus \Sigma$ then (6.27) holds true.

(iii) If $1 < p < \frac{N-\alpha_-+1}{N-\alpha_--1}$ and $\nu \in \mathfrak{M}(\partial\Omega)$ has compact support in Σ then (6.27) holds true.

Hence, if one of the above cases occurs, we see that statement 2 of theorem 6.7 holds true, which implies the existence of solution of (BVP_-^σ) for some $\sigma > 0$.

Remark 6.9. Assume $\mu < \frac{N^2}{4}$ and $p \geqslant \frac{N-\alpha_-+1}{N-\alpha_--1}$. Then for any $z \in \Sigma$ and any $\sigma > 0$, problem (6.26) with $\nu = \delta_z$ does not admit any positive weak solution. Indeed, suppose by contradiction that for some $z \in \Sigma$ and $\sigma > 0$, there exists a positive solution $u \in L^p(\Omega; \phi_{\mu,\Sigma})$ of equation (6.26). Without loss of generality, we can assume that $z = 0 \in \Sigma$ and $\sigma = 1$. From (6.26), $u(x) \geqslant \mathbb{K}_\mu[\delta_0](x) = K_\mu(x,0)$ for a.e. $x \in \Omega$. Let

$\mathcal{C}$ be a cone of vertex 0 such that $\mathcal{C} \subset \Omega$ and there exist $r > 0$, $0 < \ell < 1$ satisfying for any $x \in \mathcal{C}$, $|x| < r$ and $d_\Sigma(x) \geqslant d_{\partial\Omega}(x) > \ell|x|$. Then, by (3.3) and (1.4),

$$\int_\Omega u(x)^p \phi_{\mu,\Sigma}(x)\,\mathrm{d}x \geqslant \int_{\mathcal{C}} K_\mu(x,0)^p \phi_{\mu,\Sigma}(x)\,\mathrm{d}x \geqslant \int_{\mathcal{C}} |x|^{1-\alpha_- -(N-\alpha_- -1)p}\,\mathrm{d}x \approx \int_0^r t^{N-\alpha_- -(N-\alpha_- -1)p}\,\mathrm{d}t.$$

Since $p \geqslant \frac{N-\alpha_- +1}{N-\alpha_- -1}$, the last integral is divergent, hence $u \notin L^p(\Omega;\phi_{\mu,\Sigma})$, which leads to a contradiction.

Remark 6.10. Assume $\mu < \frac{N^2}{4}$ and $p \geqslant \frac{N+1}{N-1}$. Proceeding as in remark 6.10, we may show that any $z \in \partial\Omega \setminus \Sigma$ and any $\sigma > 0$, problem (6.26) with $\nu = \delta_z$ does not admit any positive weak solution.

By using the above capacities and theorem 6.7, we are able to prove theorem 2.8.

Proof of theorem 2.8 when $\boldsymbol{\mu} < \frac{\boldsymbol{N}^2}{4}$. The fact that statements 1–3 are equivalent follows by using a similar argument as in the proof of theorem 2.7. Hence, it remains to show that statement 4 is equivalent to statements 1–3. Since

$$K_\mu(x,z) \approx C\,\mathrm{dist}\,(z,\Sigma)\, d_{\partial\Omega}(x)\, d_\Sigma(x)^{-\alpha_-} |x-z|^{-N}, \quad \forall x \in \Omega,\ z \in \partial\Omega \setminus \Sigma,$$

we may proceed as in the proof of [**20**, estimate (6.40)] to obtain the desired result. □

When $p \geqslant \frac{\alpha_+ +1}{\alpha_+ -1}$, the nonexistence occurs, as shown in the following remark.

Remark 6.11. We additionally assume that Ω is C^3. If $p \geqslant \frac{\alpha_+ +1}{\alpha_+ -1}$ then, for any measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ with compact support in Σ and any $\sigma > 0$, there is no solution of problem (6.26). Indeed, it can be proved by contradiction. Suppose that we can find $\sigma > 0$ and a measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ with compact support in Σ such that there exists a solution $0 \leqslant u \in L^p(\Omega;\phi_{\mu,\Sigma})$ of (6.26). It follows that $\mathbb{K}_\mu[\nu] \in L^p(\Omega;\phi_{\mu,\Sigma})$. Therefore, by proposition 3.5, there is a unique nontrivial nonnegative solution v of

$$\begin{cases} -L_\mu v + |v|^{p-1}v = 0 & \text{in } \Omega, \\ \mathrm{tr}_{\mu,\Sigma}(v) = \nu. \end{cases}$$

Moreover, $v \leqslant \mathbb{K}_\mu[\nu]$ in Ω. This, together with proposition 3.2 and the fact that ν has compact support in Σ, implies $v(x) \leqslant \mathbb{K}_\mu[\nu](x) \lesssim d_{\partial\Omega}(x)\nu(\Sigma)$ for x near $\partial\Omega \setminus \Sigma$. Therefore, by theorem 2.1, we have that $v \equiv 0$, which leads to a contradiction.

Remark 6.12. If $\alpha_- > 1$ and $p \geqslant \frac{\alpha_- +1}{\alpha_- -1}$ then for any measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ with compact support in $\partial\Omega$ and any $\sigma > 0$, there is no solution of (6.26). Indeed, it can be proved by contradiction. Suppose that we can find a measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ with compact support in $\partial\Omega \setminus \Sigma$ and $\sigma > 0$ such that there exists a solution $0 \leqslant u \in L^p(\Omega;\phi_{\mu,\Sigma})$ of (6.26). Then by theorem 2.8, estimate (2.13) holds for some constant $C > 0$.

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For simplicity, we assume that $0 \in \Sigma$. Let $\{x_n\} \subset \Omega$ be such that $\operatorname{dist}(x_n, \operatorname{supp}\nu) > \varepsilon > 0$ for any $n \in \mathbb{N}$ and $x_n \to 0$ as $n \to \infty$. Then there exists a positive constant $C_1 = C_1(\varepsilon, \Omega, \Sigma, \mu)$ such that

$$C \geqslant \frac{\int_\Omega G_\mu(x_n, y)\mathbb{K}_\mu[\nu](y)^p \mathrm{d}y}{\mathbb{K}_\mu[\nu](x_n)}$$
$$\gtrsim C_1 d_{\partial\Omega}(x_n)^{-1}\, d_\Sigma(x_n)^{\alpha_-}\nu(\partial\Omega)^{p-1}\int_\Omega (d_{\partial\Omega}(y)\, d_\Sigma(y)^{-\alpha_-})^p G_\mu(x_n, y)\,\mathrm{d}y.$$

Set

$$F(x_n, y) := d_{\partial\Omega}(x_n)^{-1}\, d_\Sigma(x_n)^{\alpha_-}(d_{\partial\Omega}(y)\, d_\Sigma(y)^{-\alpha_-})^p G_\mu(x_n, y).$$

Then

$$\liminf_{n\to\infty} F(x_n, y) \gtrsim d_{\partial\Omega}(y)^{p+1}|y|^{2\alpha_- - N - \alpha_-(p+1)}.$$

Let $\mathcal{C}$ be a cone of vertex 0 such that $\mathcal{C} \subset \Omega$ and there exist $r > 0$, $0 < \ell < 1$ satisfying for any $x \in \mathcal{C}$, $|x| < r$ and $d_\Sigma(x) \geqslant d_{\partial\Omega}(x) > \ell|x|$. Combining all above we have that

$$C \gtrsim \int_{\mathcal{C}} d_{\partial\Omega}(y)^{p+1}|y|^{2\alpha_- - N - \alpha_-(p+1)}\mathrm{d}y \gtrsim \int_{\mathcal{C}} |y|^{p+1+2\alpha_- - N - \alpha_-(p+1)}\,\mathrm{d}y$$
$$\approx \int_0^r s^{p(1-\alpha_-)+\alpha_-}\,\mathrm{d}s = +\infty,$$

since $p \geqslant \frac{\alpha_-+1}{\alpha_--1}$. This is clearly a contradiction.

6.3. Existence results in the case $\Sigma = \{0\}$ and $\mu = \frac{N^2}{4}$

Let $0 < \varepsilon < N$. For any $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ such that $x \neq y$, we set

$$\mathcal{N}_{1,\varepsilon}(x, y) := \frac{\max\{|x-y|, |x|, |y|\}^N}{|x-y|^{N-2}\max\{|x-y|, d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}^2} + \max\{|x-y|, d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}^{-\varepsilon}$$

and

$$\mathcal{N}_{N-\varepsilon}(x, y) := \frac{\max\{|x-y|, |x|, |y|\}^{N-\varepsilon}}{|x-y|^{N-2}\max\{|x-y|, d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}^2}.$$

Put

$$G_{H^2,\varepsilon}(x, y) := |x-y|^{2-N}\left(1 \wedge \frac{d_{\partial\Omega}(x)\, d_{\partial\Omega}(y)}{|x-y|^2}\right)\left(1 \wedge \frac{|x||y|}{|x-y|^2}\right)^{-\frac{N}{2}}$$
$$+ \frac{d_{\partial\Omega}(x)\, d_{\partial\Omega}(y)}{(|x||y|)^{\frac{N}{2}}}\max\{|x-y|, d_{\partial\Omega}(x), d_{\partial\Omega}(y)\}^{-\varepsilon},$$
$$x, y \in \Omega \setminus \{0\},\ x \neq y$$

and

$$\tilde{G}_{H^2,\varepsilon}(x,y) := d_{\partial\Omega}(x)\, d_{\partial\Omega}(y)(|x||y|)^{-\frac{N}{2}}\mathcal{N}_{N-\varepsilon}(x,y), \quad \forall x,y \in \Omega\setminus\{0\},\, x\neq y. \tag{6.28}$$

Note that

$$\left|\ln\left(\min\left\{|x-y|^{-2},(d_{\partial\Omega}(x)\,d_{\partial\Omega}(y))^{-1}\right\}\right)\right| \leqslant C(\Omega,\varepsilon)\max\{|x-y|,d_{\partial\Omega}(x),d_{\partial\Omega}(y)\}^{-\varepsilon},$$

which, together with (3.2), implies

$$G_{H^2}(x,y) \lesssim G_{H^2,\varepsilon}(x,y), \quad \forall x,y\in\Omega,\, x\neq y. \tag{6.29}$$

Next, from the estimates

$$G_{H^2,\varepsilon}(x,y) \approx d_{\partial\Omega}(x)\,d_{\partial\Omega}(y)(|x||y|)^{-\frac{N}{2}}\mathcal{N}_{1,\varepsilon}(x,y), \quad x,y\in\Omega,\, x\neq y,$$

$$\mathcal{N}_{1,\varepsilon}(x,y) \leqslant C(\varepsilon,\Omega)\mathcal{N}_{N-\varepsilon}(x,y), \quad x,y\in\Omega,\, x\neq y,$$

we obtain

$$G_{H^2,\varepsilon}(x,y) \lesssim \tilde{G}_{H^2,\varepsilon}(x,y), \quad \forall x,y\in\Omega,\, x\neq y. \tag{6.30}$$

Set

$$\tilde{\mathbb{G}}_{H^2,\varepsilon}[|u|^p](x) := \int_\Omega \tilde{G}_{H^2,\varepsilon}(x,y)|u|^p\,\mathrm{d}y,$$

$$\mathbb{N}_{N-\varepsilon}[\tau](x) := \int_\Omega \mathcal{N}_{N-\varepsilon}(x,y)\,\mathrm{d}\tau(y).$$

Proceeding as in the proof of theorem 6.7, we obtain the following result

THEOREM 6.13. *Let $0<\varepsilon<2$, $1<p<\frac{N+2-2\varepsilon}{N-2}$ and $\nu\in\mathfrak{M}^+(\partial\Omega)$. Then the following statements are equivalent.*

1. *The equation*

$$u = \tilde{\mathbb{G}}_{H^2,\varepsilon}[u^p] + \sigma d_{\partial\Omega}|\cdot|^{-\frac{N}{2}}\mathbb{N}_{N-\varepsilon}[\nu] \tag{6.31}$$

has a positive solution for $\sigma>0$ small.

2. *For any Borel set $E\subset\overline{\Omega}$, there holds*

$$\int_E \mathbb{N}_{N-\varepsilon}[\mathbb{1}_E\nu]^p(x)\phi_{H^2,\Sigma}(x)^{p+1}\,\mathrm{d}x \leqslant C\nu(E).$$

3. *The following inequality holds*

$$\mathbb{N}_{N-\varepsilon}[\mathbb{N}_{N-\varepsilon}[\nu]^p\left(d_{\partial\Omega}d_\Sigma^{-\frac{N}{2}}\right)^{p+1}] \leqslant C\,\mathbb{N}_{N-\varepsilon}[\nu]<\infty \quad a.e.$$

4. *For any Borel set $E\subset\overline{\Omega}$ there holds*

$$\nu(E) \leqslant C\,\mathrm{Cap}^{p+1,-\frac{N}{2}(p+1)}_{\mathbb{N}_{N-\varepsilon},p'}(E).$$

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THEOREM 6.14. *We assume that at least one of the statements 1–4 of theorem 6.13 is valid. Then problem* (P^σ_-) *with* $\mu = H^2$ *admits a positive weak solution for* $\sigma > 0$ *small.*

Proof. By theorem 6.13, there exists a solution u to equation (6.31) for $\sigma > 0$ small. By (6.29) and (6.30), we have $u \gtrsim \mathbb{G}_{H^2}[u^p] + \sigma\mathbb{K}_{H^2}[\nu]$. By [**5**, proposition 2.7], we deduce that equation

$$u = \mathbb{G}_{H^2}[u^p] + \sigma\mathbb{K}_{H^2}[\nu] \tag{6.32}$$

has a solution for $\sigma > 0$ small. This means that it admits a positive weak solution for $\sigma > 0$. □

REMARK 6.15. Let $\Sigma = \{0\} \subset \partial\Omega$, $\mu = \frac{N^2}{4}$, $\nu = \delta_0$ and $1 < p < \frac{N+2}{N-2}$. Then there exists $\varepsilon > 0$ small enough such that $1 < p < \frac{N+2}{N-2+2\varepsilon} \leqslant \frac{N+2-2\varepsilon}{N-2}$. In addition, we have

$$\int_\Omega \mathbb{N}_{N-\varepsilon}[\delta_0]^p(x)\phi_{H^2,\Sigma}(x)^{p+1}\,\mathrm{d}x \lesssim \int_\Omega |x|^{(p+1)(1-\frac{N}{2})-p\varepsilon}\,\mathrm{d}x < \infty.$$

Hence, statement 2 of theorem 6.13 is satisfied. This and theorem 6.14 imply that equation (6.32) has a solution for $\sigma > 0$ small.

Proof of theorem 2.8 when $\boldsymbol{\Sigma = \{0\}}$ *and* $\boldsymbol{\mu = \frac{N^2}{4}}$. Let $\varepsilon > 0$ be small enough such that $1 < p < \frac{N+2-2\varepsilon}{N-2}$. Let $K = \mathrm{supp}\,(\nu) \Subset \partial\Omega \setminus \{0\}$ and $\tilde\beta = \frac{1}{2}\mathrm{dist}\,(K, \{0\}) > 0$. By (3.4), we can easily show that

$$\mathbb{K}_{H^2,\varepsilon}[\nu] := d_{\partial\Omega} d_\Sigma^{-\frac{N}{2}}\mathbb{N}_{N-\varepsilon}[\nu] \approx \mathbb{K}_{\frac{N^2}{4}}[\nu]. \tag{6.33}$$

Hence, by proposition 6.1, (6.33), theorems 6.13 and 6.14, it is enough to show that

$$\tilde{\mathbb{G}}_{H^2,\varepsilon}[\mathbb{K}_{H^2,\varepsilon}[\nu]^p] \approx \mathbb{G}_{H^2}[\mathbb{K}_{H^2}[\nu]^p] \quad \text{in } \Omega.$$

By (6.29) and (6.30), it is sufficient to show that

$$\tilde{\mathbb{G}}_{H^2,\varepsilon}[\mathbb{K}_{H^2}[\nu]^p] \lesssim \mathbb{G}_{H^2}[\mathbb{K}_{H^2}[\nu]^p] \quad \text{in } \Omega. \tag{6.34}$$

Indeed, on one hand, since $1 < p < \frac{N+2-2\varepsilon}{N-2}$, for any $x \in \Omega$ there holds

$$\begin{aligned}
&\int_{B(0,\frac{\tilde\beta}{4})\cap\Omega} \tilde G_{H^2,\varepsilon}(x,y)\mathbb{K}_{H^2}[\nu](y)^p\,\mathrm{d}y \approx \nu(K)^p \\
&\quad\int_{B\left(0,\frac{\tilde\beta}{4}\right)\cap\Omega} \tilde G_{H^2,\varepsilon}(x,y)\, d_{\partial\Omega}(y)^p|y|^{-\frac{pN}{2}}\,\mathrm{d}y \\
&\lesssim \nu(K)^p d_{\partial\Omega}(x)|x|^{-\frac{N}{2}} \int_{B\left(0,\frac{\tilde\beta}{4}\right)\cap\Omega} |x-y|^{-\varepsilon}\left(d_{\partial\Omega}(y)|y|^{-\frac{N}{2}}\right)^{p+1}\,\mathrm{d}y \\
&\quad + \nu(K)^p d_{\partial\Omega}(x)|x|^{-\frac{N}{2}} \int_{B\left(0,\frac{\tilde\beta}{4}\right)\cap\Omega} |x-y|^{-N+1} d_{\partial\Omega}(y)^p|y|^{N-\varepsilon-\frac{(p+1)N}{2}}\,\mathrm{d}y \\
&\lesssim \nu(K)^p d_{\partial\Omega}(x)|x|^{-\frac{N}{2}}.
\end{aligned} \tag{6.35}$$

The implicit constants in the above inequalities depend only on $\Omega, K, \tilde\beta, p, \varepsilon$.

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On the other hand, we have

$$\int_{B\left(0,\frac{\tilde\beta}{4}\right)\cap\Omega} G_{H^2}(x,y)\mathbb{K}_{H^2}[\nu](y)^p\,\mathrm{d}y \gtrsim \nu(K)^p d_{\partial\Omega}(x)|x|^{-\frac{N}{2}}\int_{B(0,\frac{\tilde\beta}{4})\cap\Omega} d_{\partial\Omega}(y)^{p+1}\,\mathrm{d}y$$
$$\gtrsim \nu(K)^p d_{\partial\Omega}(x)|x|^{-\frac{N}{2}}, \tag{6.36}$$

where the implicit constants in the above inequalities depend only on $\Omega, K, \tilde\beta, p$.

Hence, by (6.35) and (6.36), we have that

$$\int_{B(0,\frac{\tilde\beta}{4})\cap\Omega} \tilde G_{H^2,\varepsilon}(x,y)\mathbb{K}_{H^2}[\nu](y)^p\,\mathrm{d}y \lesssim \int_{B(0,\frac{\tilde\beta}{4})\cap\Omega} G_{H^2}(x,y)\mathbb{K}_{H^2}[\nu](y)^p\mathrm{d}y \quad \forall x\in\Omega. \tag{6.37}$$

Next, by (3.2) and (6.28), for any $x\in\Omega$ and $y\in\Omega\setminus B(0,\frac{\tilde\beta}{4})$, we have

$$\tilde G_{H^2,\varepsilon}(x,y) \approx d_{\partial\Omega}(x)\,d_{\partial\Omega}(y)(|x||y|)^{-\frac{N-2}{2}}\mathcal{N}_N(x,y) \lesssim G_{H^2}(x,y).$$

This and (6.33) yield

$$\int_{\Omega\setminus B(0,\frac{\tilde\beta}{4})} \tilde G_{H^2,\varepsilon}(x,y)\mathbb{K}_{H^2}[\nu](y)^p\,\mathrm{d}y \lesssim \int_{\Omega\setminus B(0,\frac{\tilde\beta}{4})} G_{H^2}(x,y)\mathbb{K}_{H^2}[\nu](y)^p\,\mathrm{d}y \quad \forall x\in\Omega. \tag{6.38}$$

Combining (6.37) and (6.38), we deduce (6.34). The proof is complete. □

REMARK 6.16. If $p<\frac{N+1}{N-1}$, by using a similar argument to the one in remark 6.15, we obtain that statement 2 of theorem 6.13 holds true. Consequently, under the assumptions of theorem 2.8, equation (6.26) has a positive solution for $\sigma>0$ small.

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Appendix A. Barriers

Appendix A.1. Local representation of Σ and ∂Ω

In this subsection, we present the local representation of Σ and $\partial\Omega$.

If $k=0$ we always assume that $\Sigma=\{0\}$. If $k\in\mathbb{N}$ such that $1\leqslant k\leqslant N-1$, we set $x=(x_1,\ldots,x_k,x_{k+1},\ldots,x_N)\in\mathbb{R}^N$ and $x=(x',x'')$ where $x'=(x_1,..,x_k)\in\mathbb{R}^k$ and $x''=(x_{k+1},\ldots,x_N)\in\mathbb{R}^{N-k}$. For $\beta>0$, we denote by $B(x,\beta)$ the ball in $\mathbb{R}^N$ with centre $x\in\mathbb{R}^N$ and radius β, and by $B^k(x',\beta)$ the ball in $\mathbb{R}^k$ with centre at $x'\in\mathbb{R}^k$ and radius β. For any $\xi\in\Sigma$, we set

$$V_\Sigma(\xi,\beta) := \{x=(x',x''): |x'-\xi'|<\beta,\ |x_i-\Gamma_i^\xi(x')|<\beta,\ \forall i=k+1,\ldots,N\},$$

for some functions $\Gamma_i^\xi:\mathbb{R}^k\to\mathbb{R}$, $i=k+1,\ldots,N$.

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Since Σ is a C^2 compact submanifold in $\mathbb{R}^N$ without boundary, there exists $\beta_0 > 0$ such that the followings hold.

(i) For any $x \in \Sigma_{6\beta_0}$, there is a unique $\xi \in \Sigma$ satisfying $|x - \xi| = d_\Sigma(x)$.

(ii) $d_\Sigma \in C^2(\Sigma_{4\beta_0})$, $|\nabla d_\Sigma| = 1$ in $\Sigma_{4\beta_0}$ and there exists $\eta \in L^\infty(\Sigma_{4\beta_0})$ such that

$$\Delta d_\Sigma(x) = \frac{N-k-1}{d_\Sigma(x)} + \eta(x) \quad \text{in } \Sigma_{4\beta_0}.$$

(See [**34**, lemma 2.2] and [**13**, lemma 6.2].)

(iii) For any $\xi \in \Sigma$, there exist C^2 functions $\Gamma^\xi_{i,\Sigma} \in C^2(\mathbb{R}^k;\mathbb{R})$, $i = k+1,\dots,N$, such that for any $\beta \in (0, 6\beta_0)$ and $V_\Sigma(\xi,\beta) \subset \Omega$ (upon relabelling and reorienting the coordinate axes if necessary), there holds

$$V_\Sigma(\xi,\beta) \cap \Sigma = \{x = (x', x'') : |x' - \xi'| < \beta,\ x_i = \Gamma^\xi_{i,\Sigma}(x'),\ \forall i = k+1,\dots,N\}. \tag{A.1}$$

(iv) There exist $m_1 \in \mathbb{N}$ and points $\xi^j \in \Sigma$, $j = 1,\dots,m_1$, and $\beta_1 \in (0,\beta_0)$ such that

$$\Sigma_{2\beta_1} \subset \cup_{j=1}^{m_1} V_\Sigma(\xi^j, \beta_0). \tag{A.2}$$

Now for $\xi \in \Sigma$, set

$$\delta^\xi_\Sigma(x) := \left(\sum_{i=k+1}^N |x_i - \Gamma^\xi_{i,\Sigma}(x')|^2 \right)^{\frac{1}{2}}, \quad x = (x', x'') \in V_\Sigma(\xi, 4\beta_0).$$

Then we see that there exists a constant C_Σ depending only on N, Σ such that

$$d_\Sigma(x) \leqslant \delta^\xi_\Sigma(x) \leqslant C_\Sigma \|\Sigma\|_{C^2} d_\Sigma(x), \quad \forall x \in V_\Sigma(\xi, 2\beta_0), \tag{A.3}$$

where

$$\|\Sigma\|_{C^2} := \sup\{||\Gamma^{\xi^j}_{i,\Sigma}||_{C^2(B^k_{5\beta_0}((\xi^j)'))} : i = k+1,\dots,N,\ j = 1,\dots,m_1\} < \infty,$$

with $\xi^j = ((\xi^j)', (\xi^j)'') \in \Sigma$, $j = 1,\dots,m_1$, being the points in (A.2).

Moreover, β_1 can be chosen small enough such that for any $x \in \Sigma_{\beta_1}$,

$$B(x, \beta_1) \subset V_\Sigma(\xi, \beta_0),$$

where $\xi \in \Sigma$ satisfies $|x - \xi| = d_\Sigma(x)$.

In the following, when $\Sigma = \partial\Omega$ or $\Sigma \subset \partial\Omega$ is a C^2 submanifold, we will keep the same notations β_0 and β_1 for which (i)–(iv) hold.

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When $\Sigma = \partial\Omega$, we assume that

$$V_{\partial\Omega}(\xi,\beta)\cap\Omega = \left\{x=(x_1,\dots,x_N) : \sum_{i=1}^{N-1}|x_i-\xi_i|^2<\beta^2,\ 0 < x_N - \Gamma^{\xi}_{N,\partial\Omega}(x_1,\dots,x_{N-1}) < \beta\right\}.$$

We also find that (A.1) with $\Sigma = \partial\Omega$ becomes

$$V_{\partial\Omega}(\xi,\beta)\cap\partial\Omega = \left\{x=(x_1,\dots,x_N) : \sum_{i=1}^{N-1}|x_i-\xi_i|^2<\beta^2,\ x_N = \Gamma^{\xi}_{N,\partial\Omega}(x_1,\dots,x_{N-1})\right\}.$$

Thus, when $\Sigma\subset\partial\Omega$ is a C^2 compact submanifold in $\mathbb{R}^N$ without boundary, of dimension $0\leqslant k\leqslant N-1$, for any $x\in\Sigma$, we have that

$$\Gamma^{\xi}_{N,\Sigma}(x') = \Gamma^{\xi}_{N,\partial\Omega}(x',\Gamma^{\xi}_{k+1,\Sigma}(x'),\dots,\Gamma^{\xi}_{N-1,\Sigma}(x')). \tag{A.4}$$

Let $\xi\in\Sigma$. For any $x\in V_{\Sigma}(\xi,\beta_0)\cap\Omega$, we define

$$\delta^{\xi}(x) := x_N - \Gamma^{\xi}_{N,\partial\Omega}(x_1,\dots,x_{N-1}),$$

and

$$\delta^{\xi}_{2,\Sigma}(x) := \left(\sum_{i=k+1}^{N-1}|x_i-\Gamma^{\xi}_{i,\Sigma}(x')|^2\right)^{\frac{1}{2}}.$$

Then by (A.4), there exists a constant $A>1$ which depends only on N, k, $\|\Sigma\|_{C^2}$, $\|\partial\Omega\|_{C^2}$ and β_0 such that

$$A^{-1}(\delta^{\xi}_{2,\Sigma}(x)+\delta^{\xi}(x))\leqslant\delta^{\xi}_{\Sigma}(x)\leqslant A(\delta^{\xi}_{2,\Sigma}(x)+\delta^{\xi}(x)),\quad \forall x\in V_{\Sigma}(\xi,\beta_0)\cap\Omega. \tag{A.5}$$

Thus, by (A.3) and (A.5), for any $\gamma\in\mathbb{R}$, we can show that there exists a constant $C>1$ which depends on $N,k,\|\Sigma\|_{C^2},\|\partial\Omega\|_{C^2},\beta_0,\gamma$ such that

$$C^{-1}\delta(x)^2(\delta_{2,\Sigma}(x)+\delta(x))^{\gamma}\leqslant d_{\partial\Omega}(x)^2\,d_{\Sigma}(x)^{\gamma}\leqslant C\delta(x)^2(\delta_{2,\Sigma}(x)+\delta(x))^{\gamma}. \tag{A.6}$$

We set

$$\mathcal{V}_{\Sigma}(\xi,\beta_0) := \{(x',x''): |x'-\xi'|<\beta_0,\ |\delta(x)|<\beta_0,\ |\delta_{2,\Sigma}|<\beta_0\}.$$

We may assume that

$$\mathcal{V}_{\Sigma}(\xi,\beta_0)\cap\Omega = \{(x',x''): |x'-\xi'|<\beta_0, 0<\delta(x)<\beta_0,\ |\delta_{2,\Sigma}|<\beta_0\},$$
$$\mathcal{V}_{\Sigma}(\xi,\beta_0)\cap\partial\Omega = \{(x',x''): |x'-\xi'|<\beta_0, \delta(x)=0,\ |\delta_{2,\Sigma}|<\beta_0\},$$
$$\mathcal{V}_{\Sigma}(\xi,\beta_0)\cap\Sigma = \{(x',x''): |x'-\xi'|<\beta_0, \delta(x)=0,\ |\delta_{2,\Sigma}|=0\}.$$

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We also assume that there exist $m_2 \in \mathbb{N}$ and points $\xi^j \in \Sigma$, $j = 1, \ldots, m_2$, and $\beta_2 \in (0, \beta_1)$ such that

$$\Sigma_{2\beta_2} \cap \Omega \subset \cup_{j=1}^{m_2} \mathcal{V}_\Sigma(\xi^j, \beta_0) \cap \Omega. \tag{A.7}$$

We recall that the distance $\tilde{d}_\Sigma$ is defined in (2.3) as

$$\tilde{d}_\Sigma(x) = \sqrt{|\mathrm{dist}^{\partial\Omega}(\xi_x, \Sigma)|^2 + |x - \xi_x|^2},$$

where $\mathrm{dist}^{\partial\Omega}$ denotes the geodesic distance on $\partial\Omega$.

Proposition A.1 [14, lemma 2.1]. *There exists $\beta_3 = \beta_3(\Sigma, \Omega)$ small enough such that, for any $x \in \Omega \cap \Sigma_{\beta_3}$ the following expansions hold*

$$\begin{aligned}
&\tilde{d}_\Sigma^2(x) = d_\Sigma^2(x)(1 + f_1(x)),\\
&\nabla d_{\partial\Omega}(x) \cdot \nabla \tilde{d}_\Sigma(x) = \frac{d_{\partial\Omega}(x)}{\tilde{d}_\Sigma(x)},\\
&|\nabla \tilde{d}_\Sigma(x)|^2 = 1 + f_2(x),\\
&\tilde{d}_\Sigma(x)\Delta \tilde{d}_\Sigma(x) = N - k - 1 + f_3(x),
\end{aligned}$$

where f_i, $i = 1, 2, 3$, satisfy

$$\sum_{i=1}^{3} |f_i(x)| \leqslant C_1(\beta_3, N)\tilde{d}_\Sigma(x), \quad \forall x \in \Omega \cap \Sigma_{\beta_3}. \tag{A.8}$$

We may choose β_3 small enough such that

$$\frac{1}{2} d_\Sigma(x) \leqslant \tilde{d}_\Sigma(x) \leqslant 2 d_\Sigma(x) \quad \text{in } \Omega \cap \Sigma_{\beta_3}. \tag{A.9}$$

Appendix A.2. Barriers

In this subsection, we assume that Ω is a C^3 open bounded domain. Then there exists $\beta_4 > 0$ depending on C^3 characteristic of Ω such that for any $x \in \Omega_{\beta_4}$ the followings hold.

(i) There exists a unique $\sigma(x) \in \partial\Omega$ such that

$$d_{\partial\Omega}(x) = |x - \sigma(x)|, \quad \sigma(x) = x - d_{\partial\Omega}(x)\nabla d_{\partial\Omega}(x)$$
$$\text{and} \quad \nabla d_{\partial\Omega}(x) = \frac{x - \sigma(x)}{|x - \sigma(x)|}.$$

(ii) $\sigma(x) \in C^2(\overline{\Omega}_{\beta_4})$ and $d_{\partial\Omega} \in C^3(\overline{\Omega}_{\beta_4})$.

(iii) For any $i = 1, \ldots, N$ there holds

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$$|\nabla\sigma_i(x)\cdot\nabla d_{\partial\Omega}(x)| \leqslant \left\|D^2 d_{\partial\Omega}\right\|_{L^\infty(\overline{\Omega}_{\beta_4})} d_{\partial\Omega}(x).$$

For any $(x,z)\in\overline{\Omega}_{\beta_4}\times\partial\Omega$, set

$$d_z(x) := \sqrt{d^2_{\partial\Omega}(x)+|\sigma(x)-z|^2}.$$

Then

$$\frac{1}{2}|x-z| \leqslant d_z(x) \leqslant \sqrt{5}|x-z|.$$

Finally, for any $0<R\leqslant\beta_4$, we set

$$\mathscr{B}(z,R) := \{x\in\overline{\Omega}_{\beta_4} :\ d_z(x)<R\}$$

Proposition A.2. *Let $\beta_5=\frac{1}{16}\min\{\beta_3,\beta_4\}$, $R_0\in(0,\beta_5]$ and $0<R\leqslant R_0$. For any $z\in\overline{\Sigma}_{R_0}\cap\partial\Omega$, there is a supersolution $w:=w_{R,z}$ of (E_+) in $\mathscr{B}(z,R)$ such that*

$$w\in C(\Omega\cap B(z,R)),\qquad \lim_{x\in\Omega\cap\mathscr{B}(z,R),\ x\to\xi}\frac{w(x)}{\tilde{W}(x)}=0 \quad \text{for any}\ \ \xi\in\partial\Omega\cap\mathscr{B}(z,R),$$

$$w(x)\to\infty \ \text{as}\ \operatorname{dist}(x,F)\to 0,\ \text{for any compact subset}\ F\subset\Omega\cap\partial\mathscr{B}(z,R).$$

More precisely, for $\gamma\in(\alpha_-,\alpha_+)$, w can be constructed as

$$w(x)=\begin{cases}\Lambda(R^2-d_z(x)^2)^{-b}\,\mathrm{e}^{Md_{\partial\Omega}(x)}d_{\partial\Omega}(x)\tilde{d}_\Sigma(x)^{-\gamma} & \text{if}\ \ \mu<H^2,\\ \Lambda(R^2-d_z(x)^2)^{-b}\,\mathrm{e}^{Md_{\partial\Omega}(x)}d_{\partial\Omega}(x)\tilde{d}_\Sigma(x)^{-H}\sqrt{\left|\ln\dfrac{\tilde{d}_\Sigma(x)}{16R_0}\right|} & \text{if}\ \ \mu=H^2,\end{cases}$$

with $M<0$ depending only on the C^2 characteristic of $\partial\Omega$, $b>\frac{2(p+1)-2(p-1)\min\{\gamma,0\}}{p-1}$ and $\Lambda>0$ large enough depending only on γ,N,b,p,M,R_0, the C^2 characteristic of Σ and the C^3 characteristic of $\partial\Omega$.

Proof. Without loss of generality, we assume $z=0\in\overline{\Sigma}_{R_0}\cap\partial\Omega$.

Case 1: $\mu<H^2$. Set

$$w(x) := \Lambda(R^2-d_0^2(x))^{-b}d_{\partial\Omega}(x)\,\mathrm{e}^{Md_{\partial\Omega}(x)}\tilde{d}_\Sigma(x)^{-\gamma}\quad\text{for}\ x\in\Omega\cap\mathscr{B}(0,R),$$

where $\gamma>0,b$ and $\Lambda>0$ will be determined later on.

Then, by straightforward computation and using proposition A.1, we obtain

$$-L_\mu w+w^p=\Lambda(R^2-d_0(x)^2)^{-b-2}d_{\partial\Omega}(x)\,\mathrm{e}^{Md_{\partial\Omega}(x)}\tilde{d}_\Sigma^{-\gamma-2}(x)(I_1+I_2+I_3+I_4),$$

where

$$I_1 := -\tilde{d}_\Sigma^2\left(4b(b+1)|\nabla d_0|^2d_0^2+2b(R^2-d_0^2)(|\nabla d_0|^2+d_0\Delta d_0)\right),$$

$$I_2 := -(R^2-d_0^2)^2\left(\gamma^2-\gamma(N-k)+\mu+\gamma(\gamma+1)f_2-\gamma f_3+\mu f_1-2\gamma Md_{\partial\Omega}\right),$$

$$I_3 := -(R^2-d_0^2)^2\tilde{d}_\Sigma^2 d_{\partial\Omega}^{-1}\left(\Delta d_{\partial\Omega}(1+Md_{\partial\Omega})+2M+M^2d_{\partial\Omega}\right),$$

$$I_4 := -4b(R^2-d_0^2)\frac{d_0}{d_{\partial\Omega}}(\tilde{d}_\Sigma^2\nabla d_0\nabla d_{\partial\Omega}(1+Md_{\partial\Omega})-\gamma\tilde{d}_\Sigma d_{\partial\Omega}\nabla d_0\cdot\nabla\tilde{d}_\Sigma),$$

$$I_5 := \Lambda^{p-1}(R^2-d_0^2)^{-(p-1)b+2}\,\mathrm{e}^{M(p-1)\,d_{\partial\Omega}}d_{\partial\Omega}^{p-1}\tilde{d}_\Sigma^{-(p-1)\gamma+2}.$$

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By (i)–(iii), we have

$$|I_1| \leqslant C_1(R_0, b, \Omega, N)\tilde{d}_\Sigma^2. \tag{A.10}$$

Also,

$$|I_4| \leqslant C_2(R_0, \Omega, N, M, \gamma, b)(R^2 - d_0^2)\tilde{d}_\Sigma. \tag{A.11}$$

Next we choose $\gamma \in (\alpha_-, \alpha_+)$, then $\gamma^2 - \gamma(N-k) + \mu < 0$. In addition, there exist $0 < \geqslant_0 < 1$, $\delta_0 > 0$ and $M < 0$ such that if $\tilde{d}_\Sigma \leqslant \delta_0$ then

$$\Delta d_{\partial\Omega}(1 + M d_{\partial\Omega}) + 2M + M^2 d_{\partial\Omega} < - \geqslant_0$$

and by (A.8),

$$\gamma^2 - \gamma(N-k) + \mu + \gamma(\gamma+1)f_2 - \gamma f_3 + \mu f_1 - 2\gamma M d_{\partial\Omega} < -\varepsilon_0.$$

It follows that if $\tilde{d}_\Sigma \leqslant \delta_0$ then

$$I_2 \geqslant \epsilon_0(R^2 - d_0^2)^2. \tag{A.12}$$

We set

$$\mathcal{A}_1 := \left\{x \in \Omega \cap \mathscr{B}(0,R) : \tilde{d}_\Sigma(x) \leqslant c_1(R^2 - d_0(x)^2)\right\} \quad \text{where } c_1 = \frac{\epsilon_0}{4\max\{\sqrt{C_1}, C_2\}},$$

$$\mathcal{A}_2 := \left\{x \in \Omega \cap \mathscr{B}(0,R) : d_\Sigma(x) \leqslant \delta_0\right\}, \quad \mathcal{A}_3 := \{x \in \Omega \cap \mathscr{B}(0,R) : \tilde{d}_\Sigma(x) \geqslant \delta_0\}.$$

In $\mathcal{A}_1 \cap \mathcal{A}_2$, by (A.10), (A.11) and (A.12), we have

$$I_1 + I_2 + I_3 + I_4 \geqslant \frac{\varepsilon_0(R^2 - d_0^2)^2}{2}. \tag{A.13}$$

In $\mathcal{A}_1^c \cap \mathcal{A}_2$, we have $\tilde{d}_\Sigma \geqslant c_1(R^2 - d_0^2)$. If $d_{\partial\Omega}(x) \leqslant c_2(R^2 - d_0(x)^2)^2$, where

$$c_2 = \min\left\{\frac{\varepsilon_0}{3C_1}, \frac{\varepsilon_0^2 c_1^2}{9C_2^2}\right\},$$

then we can show that

$$I_3 \geqslant c_2^{-1}\frac{\varepsilon_0}{2}\tilde{d}_\Sigma^2 + c_2^{-\frac{1}{2}}c_1\frac{\varepsilon_0}{2}\tilde{d}_\Sigma(R^2 - d_0^2),$$

This, together with (A.10) and (A.11), implies (A.13). If $d_{\partial\Omega}(x) \geqslant c_2(R^2 - d_0(x)^2)^2$, then by proposition A.1, $\tilde{d}_\Sigma(x) \geqslant c_2 c_3(\beta_3, \Sigma)(R^2 - d_0(x)^2)^2$. Therefore,

$$I_5 \geqslant c_4(R_0, M, p, \gamma, c_1, c_2, c_3)\Lambda^{p-1}(R^2 - d_0^2)^{-(p-1)b + 2 + 2(p-1) - 2(p-1)\min\{\gamma, 0\} + 2}\, d_\Sigma.$$

If we choose $b > \frac{2(p+1) - 2(p-1)\min\{\gamma,0\}}{p-1} =: b_0$, then there exists Λ large enough depending on c_4, R_0, b, p, γ such that

$$I_5 \geqslant I_1 + I_4. \tag{A.14}$$

This and (A.14) yield

$$I_1 + I_2 + I_3 + I_4 + I_5 \geqslant 0. \tag{A.15}$$

Similarly we may show that (A.15) is valid in $\mathcal{A}_3$ for some positive constant Λ depending on $M, R_0, b, p, \gamma, \Omega, \Sigma$.

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Combining the above estimates, we deduce that for $\gamma \in (\alpha_-, \alpha_+)$, $b > b_0$ and $\Lambda > 0$ large enough, there holds

$$-L_\mu w + w^p \geqslant 0 \quad \text{in} \quad \Omega \cap \mathscr{B}(0, R).$$

Case 2: $\mu = H^2$**.** First we note that $\frac{\tilde{d}_\Sigma}{16R_0} \leqslant \frac{1}{2}$ in $\Omega \cap \mathscr{B}(0, R)$. Set

$$w(x) := \Lambda(R^2 - d_0(x)^2)^{-b} d_{\partial\Omega}(x)\, \mathrm{e}^{M d_{\partial\Omega}(x)} \tilde{d}_\Sigma(x)^{-H} \left(-\ln \frac{\tilde{d}_\Sigma(x)}{16R_0}\right)^{\frac{1}{2}}$$
$$\text{for } x \in \Omega \cap \mathscr{B}(0, R),$$

where $\gamma > 0, b$ and $\Lambda > 0$ will be determined later on. Then, by straightforward calculations we have

$$-L_\mu w + w^p = \Lambda(R^2 - d_0^2)^{-b-2} \tilde{d}_\Sigma^{-H-2} \left(-\ln \frac{\tilde{d}_\Sigma}{16R_0}\right)^{-\frac{3}{2}} (\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4),$$

where

$$\tilde{I}_1 := -\tilde{d}_\Sigma^2 \left(\ln \frac{\tilde{d}_\Sigma}{16R_0}\right)^2 \left(4b(b+1)|\nabla d_0|^2 d_0^2 + 2b(R^2 - d_0^2)(|\nabla d_0|^2 + d_0 \Delta d_0)\right),$$

$$\tilde{I}_2 := -(R^2 - d_0^2)^2 \left(\frac{1}{2}\left(-\ln \frac{\tilde{d}_\Sigma}{16R_0}\right)(f_3 - (N-k-1)f_2) - \frac{1}{4}\right.$$
$$+ \left(\ln \frac{\tilde{d}_\Sigma}{16R_0}\right)^2 \left(H(H+1)f_2 - Hf_3 + H^2 f_1 - 2HMd_{\partial\Omega}\right)$$
$$\left. + M(-d_{\partial\Omega} \ln \frac{\tilde{d}_\Sigma}{16R_0})\right),$$

$$\tilde{I}_3 := -(R^2 - d_0^2)^2 \left(\ln \frac{\tilde{d}_\Sigma}{16R_0}\right)^2 \tilde{d}_\Sigma^2 d_{\partial\Omega}^{-1} \left(\Delta d_{\partial\Omega}(1 + Md_{\partial\Omega}) + 2M + M^2 d_{\partial\Omega}\right),$$

$$\tilde{I}_4 := -4b(R^2 - d_0^2)\frac{d_0}{d_{\partial\Omega}} \left(-\ln \frac{\tilde{d}_\Sigma}{16R_0}\right) \left(\left(-\ln \frac{\tilde{d}_\Sigma}{16R_0}\right) \tilde{d}_\Sigma^2 \nabla d_0 \nabla d_{\partial\Omega}(1 + Md_{\partial\Omega})\right.$$
$$\left. - H\left(-\ln \frac{\tilde{d}_\Sigma}{16R_0}\right) \tilde{d}_\Sigma d_{\partial\Omega} \nabla d_0 \cdot \nabla \tilde{d}_\Sigma - \frac{1}{2}\tilde{d}_\Sigma \nabla d_0 \cdot \nabla \tilde{d}_\Sigma\right),$$

$$\tilde{I}_5 := \Lambda^{p-1}(R^2 - d_0^2)^{-(p-1)b+2} \left(\ln \frac{\tilde{d}_\Sigma}{16R_0}\right)^{\frac{p-1}{2}+2} \mathrm{e}^{M(p-1)\, d_{\partial\Omega}} d_{\partial\Omega}^{p-1} \tilde{d}_\Sigma^{-(p-1)H+2}.$$

By (i)–(iii) and the fact that $-\ln \frac{\tilde{d}_\Sigma}{16R_0} \geqslant \ln 2$, we have

$$|\tilde{I}_1| \leqslant \tilde{C}_1(R_0, b, \Omega, N)\tilde{d}_\Sigma^2 \left(\ln \frac{\tilde{d}_\Sigma}{16R_0}\right)^2. \tag{A.16}$$

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Also,

$$|\tilde{I}_4| \leqslant \tilde{C}_2(R_0,\Omega,N,M,k,b)(R^2-d_0^2)\tilde{d}_\Sigma \left|\ln \frac{\tilde{d}_\Sigma}{16R_0}\right|. \tag{A.17}$$

Next we choose $\delta_0>0$ and $M<0$ such that if $\tilde{d}_\Sigma \leqslant \delta_0$ then

$$\tilde{I}_3 > \varepsilon_0(R^2-d_0^2)^2\left(\ln \frac{\tilde{d}_\Sigma}{16R_0}\right)^2 \tilde{d}_\Sigma^2 d_{\partial\Omega}^{-1}$$

and

$$\tilde{I}_2 > \varepsilon_0(R^2-d_0^2)^2. \tag{A.18}$$

We set

$$\tilde{\mathscr{A}}_1 := \left\{ x\in\Omega\cap\mathscr{B}(0,R) : d_\Sigma(x) \leqslant \tilde{c}_1 \frac{(R^2-d_0(x)^2)}{|\ln \frac{\tilde{d}_\Sigma(x)}{16R_0}|}\right\}$$

where $\tilde{c}_1 = \dfrac{\epsilon_0}{4\max\{\sqrt{\tilde{C}_2},\tilde{C}_3\}}$,

$$\tilde{\mathscr{A}}_2 := \left\{x\in\Omega\cap\mathscr{B}(0,R) : d_\Sigma(x)\leqslant\delta_0\right\}, \quad \tilde{\mathscr{A}}_3 := \{x\in\Omega\cap\mathscr{B}(0,R) : d_\Sigma(x)\geqslant\delta_0\}.$$

In $\tilde{\mathscr{A}}_1\cap\tilde{\mathscr{A}}_2$, by (A.16), (A.17) and (A.18), we have

$$\tilde{I}_1+\tilde{I}_2+\tilde{I}_3+\tilde{I}_4 \geqslant \frac{\varepsilon_0(R^2-d_0^2)^2}{2}. \tag{A.19}$$

In $\tilde{\mathscr{A}}_1^c\cap\tilde{\mathscr{A}}_2$, we have $\tilde{d}_\Sigma \geqslant \tilde{c}_1\frac{(R^2-d_0^2)}{|\ln\frac{\tilde{d}_\Sigma}{16R_0}|}$. If $d_{\partial\Omega}(x)\leqslant \tilde{c}_2(R^2-d_0(x)^2)^2$, where

$$\tilde{c}_2 = \min\left\{\frac{\varepsilon_0}{3\tilde{C}_1}, \frac{\varepsilon_0^2\tilde{c}_1^2}{9\tilde{C}_2^2}\right\}.$$

Then, we can easily show that (A.19) is valid. The rest of the proof is the same as in case 1 with obvious modifications so we omit it. The proof is complete. □

References

1 D. R. Adams and L. I. Hedberg. *Function spaces and potential theory* (New York: Springer, 1996).

2 A. Ancona. Negatively curved manifolds, elliptic operators and the Martin boundary. *Ann. Math.* **125** (1987), 495–536.

3 G. Barbatis, K. T. Gkikas and A. Tertikas. Heat and Martin kernel estimates for Schrödinger operators with critical Hardy potentials. *Math. Ann.* (2023). doi:10.1007/s00208-023-02693-9.

4 M. Bhakta, M. Marcus and P.-T. Nguyen. Boundary value problems for semilinear Schrödinger equations with singular potentials and measure data. *Math. Ann.* (2023). doi:10.1007/s00208-023-02764-x.

5 M.-F. Bidaut-Véron, G. Hoang, Q.-H. Nguyen and L. Véron. An elliptic semilinear equation with source term and boundary measure data: the supercritical case. *J. Funct. Anal.* **269** (2015), 1995–2017.

6 M. F. Bidaut-Véron and L. Vivier. An elliptic semilinear equation with source term involving boundary measures: the subcritical case. *Rev. Mat. Iberoam.* **16** (2000), 477–513.

7 H. Brezis and M. Marcus. Hardy's inequalities revisited. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **25** (1997), 217–237.

8 H. Chen and L. Véron. Boundary singularities of semilinear elliptic equations with Leray-Hardy potential. *Commun. Contemp. Math.* **24** (2022), 2150051.

9 H. Chen and F. Zhou. Isolated singularities for elliptic equations with Hardy operator and source nonlinearity. *Discrete Contin. Dyn. Syst.* **38** (2018), 2945–2964.

10 G. Dal Maso. On the integral representation of certain local functionals. *Ric. Mat.* **32** (1983), 85–113.

11 J. Dávila and L. Dupaigne. Comparison results for PDEs with a singular potential. *Proc. R. Soc. Edinburgh Sect. A* **133** (2003), 61–83.

12 J. Dávila and L. Dupaigne. Hardy-type inequalities. *J. Eur. Math. Soc.* **6** (2004), 335–365.

13 L. Dupaigne and G. Nedev. Semilinear elliptic PDE's with a singular potential. *Adv. Differ. Equ.* **7** (2002), 973–1002.

14 M. Fall and F. Mahmoudi. Weighted Hardy inequality with higher dimensional singularity on the boundary. *Calc. Var. Partial Differ. Equ.* **50** (2014), 779–798.

15 M. M. Fall. Nonexistence of distributional super solutions of a semilinear elliptic equation with Hardy potential. *J. Funct. Anal.* **264** (2013), 661–690.

16 D. Feyel and A. de la Pradelle. Topologies fines et compactifications associées certains espaces de Dirichlet. *Ann. Inst. Fourier (Grenoble)* **27** (1977), 121–146.

17 S. Filippas, L. Moschini and A. Tertikas. Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains. *Commun. Math. Phys.* **273** (2007), 237–281.

18 K. T. Gkikas and P.-T. Nguyen. On the existence of weak solutions of semilinear elliptic equations and systems with Hardy potentials. *J. Differ. Equ.* **266** (2019), 833–875.

19 K. Gkikas and P.-T. Nguyen. Semilinear elliptic Schrödinger equations with singular potentials and absorption terms. To appear in *J. London Math. Soc.*, arXiv:2203.01266.

20 K. Gkikas and P.-T. Nguyen. Semilinear elliptic Schrödinger equations involving singular potentials and source terms. *Nonlinear Anal.* **238** (2024), 113403.

21 K. T. Gkikas and L. Véron. Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials. *Nonlinear Anal.* **121** (2015), 469–540.

22 A. Gmira and L. Véron. Boundary singularities of solutions of some nonlinear elliptic equations. *Duke Math. J.* **64** (1991), 271–324.

23 N. J. Kalton and I. E. Verbitsky. Nonlinear equations and weighted norm inequality. *Trans. Am. Math. Soc.* **351** (1999), 3441–3497.

24 M. Marcus. Estimates of Green and Martin kernels for Schrödinger operators with singular potential in Lipschitz domains. *Ann. Inst. Henri. Poincaré Anal. Non Linéaire* **36** (2019), 1183–1200.

25 M. Marcus. Boundary value problems with signed measure data for semilinear Schrödinger equations, https://arxiv.org/abs/2305.10370.

26 M. Marcus, V. J. Mizel and Y. Pinchover. On the best constant for Hardy's inequality in $\mathbb{R}^n$. *Trans. Am. Math. Soc.* **350** (1998), 3237–3255.

27 M. Marcus and V. Moroz. Moderate solutions of semilinear elliptic equations with Hardy potential under minimal restrictions on the potential. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **18** (2018), 39–64.

28 M. Marcus and P.-T. Nguyen. Moderate solutions of semilinear elliptic equations with Hardy potential. *Ann. Inst. Henri. Poincaré Anal. Non Linéaire* **34** (2017), 69–88.

29 M. Marcus and P.-T. Nguyen. Schrödinger equations with singular potentials: linear and nonlinear boundary value problems. *Math. Ann.* **374** (2019), 361–394.

30 M. Marcus and L. Véron. *Nonlinear second order elliptic equations involving measures.* De Gruyter Ser. Nonlinear Anal. Appl., Vol. 21 (Berlin: De Gruyter, 2014), xiv+248 pp.

31 M. Marcus and L. Véron. Boundary trace of positive solutions of supercritical semilinear elliptic equations in dihedral domains. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **15** (2016), 501–542.

32 M. Marcus and L. Véron. Removable singularities and boundary trace. *J. Math. Pures Appl.* **80** (2001), 879–900.

33 P.-T. Nguyen. Semilinear elliptic equations with Hardy potential and subcritical source term. *Calc. Var. Partial Differ. Equ.* **56** (2017), 44.

34 L. Véron. *Singularities of solutions of second order quasilinear equations*. Pitman Res. Notes Math. Ser., Vol. 353 (Harlow: Longman, 1996), viii+377 pp.

35 E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Math. Ser., No. 30 (Princeton, NJ: Princeton University Press, 1970), xiv+290 pp.