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NON-WANDERING POINTS AND THE DEPTH OF A GRAPH MAP

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Abstract

Let $f : G \to G$ be a continuous map of a graph and let $d(A)$ denote the derived set (or limit points) of $A \subset G$. We prove that $d(\Omega(f)) \subset \Lambda(f)$ and the depth of f is at most three. We also prove that if f is piecewise monotone or has zero topological entropy, then the depth of f is at most two. Furthermore, we obtain some results on the topological structure of $\Omega(f)$.

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1. Introduction

By a *graph* we mean a finite one-dimensional polyhedron which is not necessarily connected and has no isolated points. Recently, there is a growing interest to study the dynamics of a *graph map*, that is, a continuous map of a graph, as this kind of research is related to the study of the dynamics of a surface homeomorphism and the structure of attractors of a diffeomorphism, see for instance [4] and [10]. In this paper we study the depth of a continuous map of a graph and the topological structure of the non-wandering set. To be more precise, we introduce some notation.

Let X be a compact metric space and $f : X \to X$ be continuous. The *topological entropy* of f is denoted by $h(f)$ (see [1] for the definition and basic properties). For $x \in X$, $\{x, f(x), f^2(x), \dots\}$ is called the *orbit* of x. The set of *periodic points, recurrent points, ω-limit points* for some $x \in X$ and *non-wandering points* of f (for the definitions see [1]) are denoted by $P(f)$, $R(f)$, $\omega(x, f)$ and $\Omega(f)$. Set $\Lambda(f) = \bigcup_{x\in X} \omega(x, f)$. It is known that $P(f) \subset R(f) \subset \Lambda(f) \subset \Omega(f)$. Note

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that we use $\operatorname{int}(A)$, $\partial(A)$ and $\overline{A}$ to denote the interior, boundary and the closure of a subset A of X. A graph map $f : G \to G$ is *piecewise monotone* if there is a finite subset A of G such that for the closure of each connected component C of $G \setminus A$, $f|_C : C \to f(C)$ is a homeomorphism.

Let $\Omega_1 = \Omega(f)$. For each non-limit ordinal number $\alpha \geq 2$ let $\Omega_\alpha = \Omega(f|_{\Omega_{\alpha-1}})$. If α is a limit ordinal number define $\Omega_\alpha = \bigcap_{\beta<\alpha} \Omega_\beta$. Then there is a countable ordinal number c such that $\Omega_c = \Omega_\alpha$ for each $\alpha > c$ and $\Omega_c = \overline{R(f)}$. The minimal c with the above property is called the *depth* of f.

It is known [9] that for an interval map $d(\Omega(f)) \subset \Lambda(f)$ and this result can be generalized to a tree map easily by the same method. In [8] Sharkovskii studied the depth of an interval map and showed that it is at most 2 (for some other proofs of the result see [3, 11]) and in [13] the author showed that the depth of a tree map is at most 3. In [5] Kato showed that for each countable ordinal number α there is a continuous map of a dendrite (respectively, a disk) such that the depth of the map is α. Note that the depth of a flow on a 2-dimensional closed manifold is at most 3, [7]. In this paper we prove that if $f : G \to G$ is a graph map, then $d(\Omega(f)) \subset \Lambda(f)$ and the depth of f is at most 3. We also prove that if f is piecewise monotone or has zero topological entropy, then the depth of f is at most 2. Furthermore, we show that: (1) $\Omega(f) \setminus \overline{R(f)}$ is countable and no-where dense, (2) for each connected component C of $G \setminus \overline{R(f)}$: (a) there are only finitely many points from $C \cap \Omega(f)$ with infinite orbits, (b) if f has zero topological entropy, then $C \cap \Omega(f)$ is finite.

We remark that it is easy to construct interval maps with depths 1 or 2 and it is still an open question if there is a graph map with depth 3.

2. Preliminary

In this section we obtain some results which are used in the next section. We start with some notation.

Let G be a graph. For $x \in G$ and a sequence of connected neighbourhoods $\{V_i\}$ of x with $\operatorname{diam}(V_i) \to 0$, $\min\{\#(\partial(V_i)) : i \in \mathbb{N}\}$ is denoted by $\operatorname{Val}_G(x)$ and is called the *valence* of x (in G), where $\#(A)$ is the number of elements of a finite subset A of G. If $\operatorname{Val}_G(x) = 1$, x is called an *endpoint* of G; if $\operatorname{Val}_G(x) > 2$, x is called a *branch point* of G. We use $e(G)$ and $b(G)$ to denote the set of endpoints of G and the set of branch points of G respectively. A finite set $v(G) \supset b(G) \cup e(G)$ is a set of *vertices* of G if for each simple closed curve S in G, $S \cap v(G) \subset b(G) \cup e(G)$ when $\#(S \cap (b(G) \cup e(G))) \geq 3$ and $\#(S \cap v(G)) = 3$ when $\#(S \cap (b(G) \cup e(G))) < 3$, that is, we add some artificial points with valence 2 as vertices. In this way each *edge* (the closure of some connected component of $G \setminus v(G)$) is homeomorphic to $[0, 1]$ and if I and J are two edges of G then either $I \cap J = \emptyset$ or $I \cap J$ is a set consisting of one point. A *tree* is a graph containing no simple closed curve. A *star* is either a tree with only one branch point or an arc.

For a continuous map f of a compact metric space, it is known that $\Omega(f)$ is closed, $f(D(f)) = D(f)$ for each $D \in \{P, R, \Lambda\}$, $f(\Omega(f)) \subset \Omega(f)$ and $R(f) \neq \emptyset$. Moreover, $D(f^n) = D(f)$ for each $D \in \{P, R, \Lambda\}$ and $n \in \mathbb{N}$.

Let G be a graph and $f : G \to G$ be a continuous map. For $x \in G$ we define $P_G(x, f) = \bigcap_{U \in \mathscr{U}} \overline{\mathrm{Orb}(U, f)}$, where $\mathscr{U}$ is the set of all neighbourhood of x and $\mathrm{Orb}(U, f) = \bigcup_{i=0}^{\infty} f^i(U)$. One can check that $f|_{P_G(x,f)}$ is surjective if $x \in \Omega(f)$. For simplicity we write $P(x)$ instead of $P_G(x, f)$, if no confusion rises. The following simple lemma [2] is useful.

LEMMA 2.1. *Let $x \in \Omega(f)$. Then $P(x)$ has only the following three possibilities*:

(1) *$P(x)$ is a cycle.*
(2) *$P(x) = \bigcup_{i=1}^{n} M_i$, where each M_i is closed and connected, and $f(M_1) = M_2$, $\dots, f(M_n) = M_1$.*
(3) *$P(x) = \bigcap_{n \geq 0} M_n$, where for each n, $M_n = \bigcup_{i=1}^{k_n} M_n^i$ and each M_n^i is closed and connected. Furthermore, $f(M_n^1) \subset M_n^2, \dots, f(M_n^{k_n}) \subset M_n^1$, $M_1 \supset M_2 \supset \cdots$ and $k_n \to +\infty$.*

For $x \in \Omega(f)$ and $i \in \{1, 2, 3\}$, we say that x is *of type* (i) if $P(x)$ satisfies Lemma 2.1 (i). Let M be an invariant subgraph of G and let

$$E(M, f) = \left\{x \in M : \text{ for any open subset } U \text{ of } M \text{ with } x \in U, \overline{\mathrm{Orb}(U, f)} = M\right\}.$$

We say that f is *transitive* if there exists $x \in X$ such that $X = \omega(x, f)$. If f is transitive, then $X = \overline{R(f)}$. We also need the following two lemmas from [2].

LEMMA 2.2. *Assume that M is an invariant subgraph of G and $E = E(M, f)$ is infinite. Then $E = d(E)$ and $f|_E$ is transitive.*

LEMMA 2.3. *Let G be a graph and $f : G \to G$ be continuous. Then $\overline{R(f)} \subset \Lambda(f)$. Furthermore, if $x \in \Lambda(f) \setminus \overline{R(f)}$, then x is of type* (3).

For a continuous map f of $I = [0, 1]$, it is known that in each component of $I \setminus \overline{R(f)}$ there is at most one point from $\Lambda(f)$. It is easy to construct a continuous map f of S^1 (the unit circle) such that there exists a component of $S^1 \setminus \overline{R(f)}$ containing two points from $\Lambda(f)$. Generally, we have the following corollary.

COROLLARY 2.4. *Let G be a graph and $f : G \to G$ be a continuous map. Then in each component C of $G \setminus \overline{R(f)}$, there are at most $c(G)$ points from $\Omega(f)$ which are* *of type* (3), *and hence there are at most* $c(G)$ *points from* $\Lambda(f) \cap C$, *where* $c(G)$ *is only dependent on the topology of* G. *Thus* $d(\Lambda(f)) \subset \overline{R(f)}$ *and* $\Lambda(f)$ *is closed.*

PROOF. Let C be a connected component of $G \setminus \overline{R(f)}$ and $x \in C \cap \Omega(f)$ which is of type (3). Then $P(x) = \bigcap_{n \geq 1} M_n$, where for each n, $M_n = \bigcup_{i=1}^{k_n} M_n^i$, and each M_n^i is closed and connected. Furthermore, $f(M_n^1) \subset M_n^2, \dots, f(M_n^{k_n}) \subset M_n^1$ and $M_1 \supset M_2 \supset \cdots$ with $k_n \to \infty$. Assume by relabeling as necessary that $x \in M_n^1$ for each $n \geq 1$. If for some $n \in \mathbb{N}$, $M_n^1 \subset C$, then there exists $y \in R(f^{k_n}|_{M_n^1}) \subset R(f) \cap C$ as $f^{k_n}(M_n^1) \subset M_n^1$, a contradiction. Hence for each n, M_n^1 contains x and some point of ∂C. Let $H(x)$ be the connected component of $P(x)$ containing x. Then $H(x) \supset \{x, x'\}$ for some $x' \in \partial(C)$. As $f^i(H(x)) \cap H(x) = \emptyset$ for each $i \in \mathbb{N}$, $\operatorname{int}(H(x)) \cap \Omega(f) = \emptyset$. This implies that if E is an edge of G, then there are at most two points from $C \cap E \cap \Omega(f)$ which are of type (3). Consequently, the number of points in $C \cap \Omega(f)$ which are of type (3) is at most $c(G)$, which is only dependent on the topology of G. By Lemma 2.3, there are at most $c(G)$ points from $C \cap \Lambda(f)$.

Noting that $\lim_i \operatorname{diam}(C_i) = 0$ if $C_1, C_2, \dots$ are connected components of $G \setminus \overline{R(f)}$ and $\Lambda(f) \supset \overline{R(f)}$ (Lemma 2.3), we have $d(\Lambda(f)) \subset \overline{R(f)}$ and $\Lambda(f)$ is closed. □

3. The depth of a graph map

In this section we show that the depth of a graph map is at most 3. To this end, first we prove $d(\Omega(f)) \subset \Lambda(f)$ and then use the result to get the conclusion. We start with the following definition. An interval J (a subset of a graph which is homeomorphic to a connected subset of the real line and with some given orientation) contained in some edge of G is of *increasing type* (*decreasing type*) if $J \cap P(f) = \emptyset$ and if for each $x \in J$ and $n \in \mathbb{N}$, $f^n(x) \in J$ implies that $f^n(x) > x$ ($f^n(x) < x$).

THEOREM 3.1. *Let* G *be a graph and* $f : G \to G$ *be a continuous map. Then* $d(\Omega(f)) \subset \Lambda(f)$.

PROOF. Let $C_1, C_2, \dots$ be connected components of $G \setminus \overline{R(f)}$. As G is a finite graph we have $\lim_i \operatorname{diam}(C_i) = 0$. Since $\overline{R(f)} \subset \Lambda(f)$, to prove $d(\Omega(f)) \subset \Lambda(f)$ it suffices to show that if $\{x_n : n \in \mathbb{N}\} \subset C_{i_0} \cap \Omega(f)$ and $\lim_n x_n = x \in C_{i_0} \cap \Omega(f)$ for some $i_0 \in \mathbb{N}$, then $x \in \Lambda(f)$.

Let V_x be a small connected closed neighbourhood of x contained in C_{i_0} such that $V_x \cap v(T) \subset \{x\}$. Without loss of generality we assume that for all $n \in \mathbb{N}$, $x_n \in b_1$, where b_1 is one of the connected components of $V_x \setminus \{x\}$. Give an orientation of b_1 such that $x' < x_1 < x_2 < \cdots < \lim_n x_n = x$, where x' is an endpoint of b_1. By Corollary 2.4, there are finitely many points (of $\{x_n : n \in \mathbb{N}\}$) which are of type (3). Thus we assume each x_n is of type (2). Hence for each n, $P(x_n) = \bigcup_{i=1}^{k_n} M_n^i$ such that each M_n^i is connected, closed and $f(M_n^1) = M_n^2, \ldots, f(M_n^{k_n}) = M_n^1$. As before, we assume that $x_n \in M_n^1$. As $M_n^1 \not\subset C_{i_0}$ and $x_n \in M_n^1$, we have $[x_1, x_n] \subset M_n^1$ or $[x_n, x] \subset M_n^1$. $\square$

CLAIM 1. *There is no subsequence* $\{x_{n_i}\}_1^\infty$ *of* $\{x_n\}_1^\infty$ *such that* $P(x_{n_j}) \supset \{x_{n_i} : i \in \mathbb{N}\}$ *for each* $j \in \mathbb{N}$.

PROOF OF CLAIM 1. Assume on the contrary that there is a subsequence $\{x_{n_i}\}_1^\infty$ of $\{x_n\}_n^\infty$ such that $P(x_{n_j}) \supset \{x_{n_i} : i \in \mathbb{N}\}$ for each $j \in \mathbb{N}$. Note that if $y_1, y_2 \in G$ are such that $P(y_1)$ contains a neighbourhood of y_2 and $P(y_2)$ contains a neighbourhood of y_1, then $P(y_1) = P(y_2)$. Hence there exist a subgraph M of G and infinitely many points $\{y_i\}_1^\infty$ from $\{x_{n_i} : i \in \mathbb{N}\} \cap M$ (say, $\{x_{n_{2i}} : i \in \mathbb{N}\}$) such that $P_M(y_i, f|_M) = M$ for each $i \in \mathbb{N}$. By Lemma 2.2, $d(E) = E$ and $f|_E$ is transitive, where $E = E(M, f)$. Hence $y_i \in \overline{R(f|_E)} \subset \overline{R(f)}$ for each $i \geq 1$, a contradiction. This ends the proof of Claim 1. $\square$

By Claim 1, without loss of generality we may assume (by taking a subsequence) that either

(a) for each $j \in \mathbb{N}$, $x_q \notin P(x_j)$ for each $q \geq j + 1$ or
(b) for each $j \in \mathbb{N}$, $x_q \notin P(x_{j+1})$ for each $1 \leq q \leq j$.

CLAIM 2. (a) *is impossible.*

PROOF OF CLAIM 2. Assume that (a) holds. As $M_n^1 \not\subset C_{i_0}$ we have that $M_1^1 \subset M_2^1 \ldots$ and $k_{n+1}|k_n$ for each $n \in \mathbb{N}$. Thus without loss of generality we suppose that $k = k_1 = k_2 = \cdots$.

Let $M = \bigcup_{i=1}^\infty M_i^1$. Then $f^k(M) = M$ and $f^k(\overline{M}) = \overline{M}$. Let $N_1 = \overline{M} \setminus M$ and N_2 be the set of points q of M with the property that for each $n \in \mathbb{N}$ there exists a neighbourhood V of q (in M) such that $V \cap M \not\subset M_i$ for each $i \geq n$. As G is a finite graph, N_1 is finite. Since in each edge E of G there are at most two points from N_2, N_2 is also finite.

Let $z \in N_1$. Then there exists $z' \in \overline{M}$ such that $f^k(z') = z$. It is obvious that $z' \notin M$. Hence $z' \in N_1$. This implies that $f^k(N_1) = N_1$ and $N_1 \subset P(f^k) = P(f)$.

Now let $z \in N_2$. Then without loss of generality we may assume that for each n there exists an endpoint z_n of M_n with $z_n \to z$ and $z_n \notin M_{n-1}$. Hence for each n there exists $z_n' \in \overline{M_n} \setminus M_{n-1}$ such that $f^k(z_n') = z_n$. Without loss of generality assume that $z_n' \to z'$. As $f^k(N_1) = N_1$ we have that $z' \notin N_1$. If $z' \in M \setminus N_2$, then there exists some n_0 such that $z' \in M_{n_0}$ and $z_n' \in M_{n_0}$ for each $n \geq n_1$ with $n_1 \in \mathbb{N}$, a contradiction. Hence $z' \in N_2$. As $f^k(z') = z$, this implies that $f^k(N_2) = N_2$ and $N_2 \subset P(f^k) = P(f)$.

It is obvious that $x \in N_1 \cup N_2 \subset P(f)$, a contradiction. This ends the proof of Claim 2. □

Hence we have situation (b). It is easy to see that $P(x_n) \supset P(x_{n+1})$ and $k_n | k_{n+1}$ for each $n \geq 1$. Let $g = f^{k_2}$ and $K = M_2^1$. Then $x_i \in \Omega(f^{k_2}) = \Omega(g)$ and $g(K) = K$ for each $i \geq 3$.

CLAIM 3. $[x_2, x]$ *is of increasing type for* g.

PROOF OF CLAIM 3. Let $a = \min\{y \in M_2^1 : y \in b_1\}$. If $[a, x]$ is not of increasing type, then there exist $m \in \mathbb{N}$ and $b \in (a, x]$ such that $g^m(b) < b$. As $g^m(a) > a$, there exists $y \in (a, b)$ such that y is a fixed point of g^m (as $g : K \to K$). That is, y is a periodic point of f, a contradiction. □

For each $n \geq 3$, as $x_n \in \Omega(g)$, there are $p_n \in \mathbb{N}$ and $y_n \in (x_{n-1}, x_{n+1})$ such that $g^{p_n}(y_n) \in (x_{n-1}, x_{n+1})$. Then we have $g^{p_n}([y_n, x]) \supset [g^{p_n}(y_n), x]$ for each $n \in \mathbb{N}$ as $[x_2, x]$ is of increasing type and $g^{p_n}(x) \notin [x_2, x)$. Take y_n such that y_n and $g^{p_n}(y_n)$ are closed to x_n with $y_n, g^{p_n}(y_n) < y_{n+1}$. Let

$$
\begin{aligned}
F_0 &= [y_3, x], \\
F_1 &= F_0 \cap g^{-w_3}([y_4, x]), \dots, \\
F_i &= F_{i-1} \cap g^{-w_{i+2}}([y_{i+3}, x]),
\end{aligned}
$$

where $w_i = \sum_{j=3}^{i} p_j$. It is easy to see that each F_i is closed and nonempty, and

$$F_0 \supset F_1 \supset F_2 \dots .$$

Hence $F = \bigcap_{i=1}^{\infty} F_i \neq \emptyset$. For each $z \in F$, we have

$$g^{w_i}(z) \in [y_{i+1}, x], \quad i = 3, 4, \dots .$$

That is, $x \in \Lambda(g) = \Lambda(f^{k_2}) = \Lambda(f)$.

Now we are ready to prove the following theorem.

THEOREM 3.2. *Let* G *be a graph and* $f : G \to G$ *be continuous. Then* $\Omega_3 = \overline{R(f)}$. *That is, the depth of* f *is at most three.*

PROOF. As $\overline{R(f)} \subset \Omega(f)$ we know that $\Omega_3 \supset \overline{R(f)}$. If there exists $x \in \Omega_3 \setminus \overline{R(f)}$, then there exists a connected component C of $G \setminus \overline{R(f)}$ such that $x \in C$ and there are $x_i \in C$ with $x_i \in d(\Omega(f))$ and $\lim x_i = x$. By Theorem 3.1, $x_i \in \Lambda(f)$ for $i \in \mathbb{N}$. That is, there are infinitely many elements of $\Lambda(f) \cap C$, contradicting Corollary 2.4. Hence $\Omega_3 \subset \overline{R(f)}$. Thus $\Omega_3 = \overline{R(f)}$. □

REMARK 3.3. In fact we can prove that $\Omega_2 \subset (b(G) \cap \Lambda(f)) \cup \overline{R(f)}$.

Using Theorem 3.1 we can also prove the following results.

THEOREM 3.4. *Let G be a graph and $f : G \to G$ be continuous. Then $\Omega(f) \setminus \overline{R(f)}$ is countable and nowhere dense.*

PROOF. Let $C_1, C_2, \dots$ be the connected components of $G \setminus \overline{R(f)}$. To show $\Omega(f) \setminus \overline{R(f)}$ is countable and nowhere dense we only need to show that if $x \in C_i \cap \Omega(f)$, then either x is isolated in $\Omega(f)$, or x is not isolated and there is a nonempty connected subset A of C_i such that $x \in \overline{A} \setminus A$ and $A \cap \Omega(f) = \emptyset$.

Assume that x is not isolated. Then by Theorem 3.1, $x \in \Lambda(f)$. According to Lemma 2.3, x is of type (3). Let $H(x)$ be a connected component of $P(x)$ containing x. Then $\mathrm{int}(H(x)) \cap \Omega(f) = \emptyset$ and $x \in H(x)$. Thus $A = \mathrm{int}(H(x))$ is the set we need. □

THEOREM 3.5. *Let G be a graph and $f : G \to G$ be continuous. Then in each connected component of $G \setminus \overline{R(f)}$ there are only finitely many non-wandering points with infinite orbits.*

PROOF. Let C be a connected component of $G \setminus \overline{R(f)}$. Assume that $\{x_n, n \in \mathbb{N}\} \subset C \cap \Omega(f)$ with infinite orbits and $\lim_n x_n = x \in \overline{C}$.

Let V_x be a small connected closed neighbourhood of x contained in C such that $V_x \cap v(T) \subset \{x\}$. Without loss of generality we assume that $x_n \in b_1$, $n \in \mathbb{N}$, where b_1 is one of the connected components of $V_x \setminus \{x\}$. Give an orientation of b_1 such that $x' < x_1 < x_2 < \cdots < \lim_n x_n = x$, where x' is an endpoint of b_1.

According to the proof of Theorem 3.1 (before the statement of Claim 2) we may assume each x_i is of type (2). Moreover, either (a) or (b) holds:

(a) For each $j \in \mathbb{N}$, $x_q \notin P(x_j)$ for each $q \geq j+1$.
(b) For each $j \in \mathbb{N}$, $x_q \notin P(x_{j+1})$ for each $1 \leq q \leq j$.

For each $n \in \mathbb{N}$ suppose $P(x_n) = \bigcup_{i=1}^{k_n} M_n^i$ such that each M_n^i is connected, closed and $f(M_n^1) = M_n^2, \dots, f(M_n^{k_n}) = M_n^1$. Furthermore, we assume that $x_n \in M_n^1$.

Let $K = M_3^1$ and $g = f^{k_3}$. In case (a), some point in $[x_3, x_4)$ is an endpoint of K and $x_1, x_2 \in \Omega(g)$. In case (b), some point in $(x_2, x_3]$ is an endpoint of K and $x_i \in \Omega(g)$ for each $i \geq 4$. Using the same proof as in the interval case (see [12]) one readily shows that x_2 has finite orbit in case (a) and x_4 has finite orbit in case (b), a contradiction. □

4. The depths of piecewise monotone and zero entropy graph maps

In this section we shall show that for piecewise monotone and zero entropy graph maps the depths of them are at most two.

THEOREM 4.1. *Let G be a graph and $f : G \to G$ be continuous and piecewise monotone. Then $\Lambda(f) = \overline{R(f)}$ and $\Omega(f|_{\Omega(f)}) = \overline{R(f)}$.*

PROOF. Assume the contrary. That is, there is $x \in \Lambda(f) \setminus \overline{R(f)}$. By Lemma 2.3, x is of type (3). Hence $P(x) = \bigcap_{n\geq 0} M_n$, where for each n, $M_n = \bigcup_{i=1}^{k_n} M_n^i$ and each M_n^i is closed and connected. Furthermore, $f(M_n^1) \subset M_n^2, \dots, f(M_n^{k_n}) \subset M_n^1$ and $M_1 \supset M_2 \supset \cdots$ with $k_n \to +\infty$. As G is a finite graph and $k_n \to \infty$ there exist $n \in \mathbb{N}$ and $1 \leq k \leq k_n$ such that M_n^k is homeomorphic to $[0, 1]$. It is easy to see that $x \in \Lambda(f|_{M_n})$ as x has infinite orbit.

As $f(\Lambda(f|_{M_n})) = \Lambda(f|_{M_n})$, there exist $1 \leq i \leq k_n$ and $x_i \in \Lambda(f|_{M_n}) \cap M_n^k$ such that $f^i(x_i) = x$. Since $x \in \Lambda(f) \setminus \overline{R(f)}$, we have $x_i \in \Lambda(f) \setminus \overline{R(f)}$. As $f^{k_n}|_{M_n^k}$ is piecewise monotone, by [1, page 81] we have $x_i \in \overline{R(f^{k_n}|_{M_n^k})}$. Thus $x \in \overline{R(f^{k_n}|_{M_n^k})} \subset \overline{R(f)}$, a contradiction.

Hence by Theorem 3.1 it is easy to see that $\Omega(f|_{\Omega(f)}) = \overline{R(f)}$. □

To prove the next theorem we need a lemma from [6].

LEMMA 4.2 ([6]). *Let T be a tree and $f : T \to T$ be continuous. Then $h(f) = 0$ if and only if for each $x \in \Omega(f) \setminus P(f)$, $\omega(x, f) \cap P(f) = \emptyset$.*

By Lemma 4.2 we know that if a continuous map $f : T \to T$ of a tree T has zero topological entropy, then for each $x \in \Omega(f) \setminus P(f)$, the orbit of x is infinite.

THEOREM 4.3. *Let G be a graph and $f : G \to G$ be continuous with zero topological entropy. Then in each component of $G \setminus \overline{R(f)}$ there exist at most finitely many non-wandering points. Consequently, $\Omega(f|_{\Omega(f)}) = \overline{R(f)}$.*

PROOF. Assume that there is a connected component C of $G \setminus \overline{R(f)}$ such that $C \cap \Omega(f)$ is infinite. Then there are distinct $x_1, x_2, \dots \in C \cap \Omega(f)$ and $x \in \overline{C}$ such that $\lim_i x_i = x$ and an edge B of G with $x, x_i \in B$. Give an orientation of B such that $x_1 < x_2 < \cdots < x$.

As the proof of Theorem 3.1 (before the statement of Claim 2) is valid in our situation, we may assume each x_i is of type (2). Moreover, we have the following two cases

(a) for each $j \in \mathbb{N}$, $x_q \notin P(x_j)$ for each $q \geq j + 1$; or
(b) for each $j \in \mathbb{N}$, $x_q \notin P(x_{j+1})$ for each $1 \leq q \leq j$.

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For each $n \in \mathbb{N}$ suppose $P(x_n) = \bigcup_{i=1}^{k_n} M_n^i$ such that each M_n^i is connected, closed and $f(M_n^1) = M_n^2, \ldots, f(M_n^{k_n}) = M_n^1$. Furthermore, we assume that $x_n \in M_n^1$.

In case (a), it is clear that $k_{n+1} | k_n$ for each $n \in \mathbb{N}$. As M_n^1 is a proper subset of M_{n+1}^1 for each $n \in \mathbb{N}$ and G is a finite graph, we see that there exists $n_0 \in \mathbb{N}$ such that M_n^1 is homeomorphic to M_{n+2}^1 for each $n \geq n_0$. Collapsing $M_{n_0}^1$ in $M_{n_0+2}^1$ we get a star S. Let $g_1 = f^{k_{n_0+2}}|_{M_{n_0+2}^1}$, $p : M_{n_0+2}^1 \to S$ be the projection and $g : S \to S$ be the induced map of g_1.

In case (b), it is clear that $k_n | k_{n+1}$ for each $n \in \mathbb{N}$. As M_{n+1}^1 is a proper subset of M_n^1 for each $n \in \mathbb{N}$ and G is a finite graph, we see that there exists $n_0 \in \mathbb{N}$ such that M_n^1 is homeomorphic to M_{n+2}^1 for each $n \geq n_0$. Collapsing $M_{n_0+2}^1$ in $M_{n_0}^1$ we get a star S. Let $g_1 = f^{k_{n_0}}|_{M_{n_0}^1}$, $p : M_{n_0}^1 \to S$ be the projection and $g : S \to S$ be the induced map of g_1.

Hence, in both cases we have $p(x_{n+1}) \in \Omega(g) \setminus \overline{P(g)}$ and $p(x_{n+1})$ is an eventually periodic point of g. By Lemma 4.2, $h(g) > 0$. Hence

$$h(f) = \frac{1}{k_{n_0+2}} h(f^{k_{n_0+2}}) \geq \frac{1}{k_{n_0+2}} h(g) > 0$$

in case (a) and

$$h(f) = \frac{1}{k_{n_0}} h(f^{k_{n_0}}) \geq \frac{1}{k_{n_0}} h(g) > 0$$

in case (b), a contradiction. This proves that in each component of $G \setminus \overline{R(f)}$ there are only finitely many points from $\Omega(f)$. Thus $\Omega(f|_{\Omega(f)}) \subset \overline{R(f)}$, and hence $\Omega(f|_{\Omega(f)}) = \overline{R(f)}$. □

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