A LATTICE ANALOGUE OF A THEOREM OF CHOW

Dedicated to the memory of Hanna Neumann

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Let K be a skewfield, E a left vector space over K, r an integer ≥ 1 and $G_r(E)$ the set of all r-dimensional subspaces of E, called the Grassmannian of index r. The function $d(A, B) = r - \dim(A \cap B)$ is a distance on $G_r(E)$. If K' is a skewfield and E' a left vector space over K', then any semilinear isomorphism $u: E \to E'$ (relative to an isomorphism $K \to K'$) induces a distance preserving bijection $G_r(u): G_r(E) \to G_r(E')$. When E has finite dimension n and $2r = n$, another example of such a mapping is obtained by taking $K' = K^{op}$, $E' = E^*$ and defining $w_r: G_r(E) \to G_r(E^*)$ to be $w_r(A) = \{f \in E^* \mid f(A) = 0\}$.

In 1949, Chow [2] proved the following converse:

THEOREM. *Suppose E and E' both have finite dimension n and $2 \leq r \leq n - 2$. Let ϕ be a distance preserving bijection $G_r(E) \to G_r(E')$. Then if $2r \neq n$ there exists a semilinear isomorphism $u: E \to E'$ such that $\phi = G_r(u)$, while in the case $2r = n$, ϕ may also be of the form $G_r(v)w_r$, where v is a semilinear isomorphism $E^* \to E'$.*

The proof consists of a reduction to the case $r = 1$, followed by an application of the fundamental theorem of projective geometry. The purpose of this note is to place the first part of Chow's argument in a lattice-theoretic setting and apply it to some other examples.

Let L be a lattice with 0; we assume that L is *discrete* i.e. that for all a, $b \in L$, the supremum of lengths of chains from a to b is finite. Such a lattice has a height function $h: L \to \mathbb{N}$, defined by $h(a) = \sup$. of lengths of chains from 0 to a. The supremum of all numbers $h(a)$, which may be infinite, is denoted by $h(L)$. We shall assume furthermore that L is atomic and modular; the height function then satisfies the identity

$$h(a \vee b) + h(a \wedge b) = h(a) + h(b).$$

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For every integer $r \geq 1$, the set L_r of all elements of height r in L is a metric space under the distance $d(a, b) = r - h(a \wedge b)$.

Atoms $p, q, r \in L_1$ are called *collinear* if they are all contained in some element of L_2. If L' is a lattice with the same properties as L, a bijective mapping $u: L_1 \to L'_1$ is called a *collineation* provided that atoms $p, q, r \in L_1$ are collinear if and only if their images in L'_1 are collinear.

LEMMA 1. *A collineation $u: L_1 \to L'_1$ induces a distance preserving bijection $u_r: L_r \to L'_r$ by defining $u_r(a) = u(p_1) \vee \cdots \vee u(p_r)$, where $a = p_1 \vee \cdots \vee p_r$ is some representation of a as a join of atoms.*

Granted this lemma, our principal result can be stated as follows:

THEOREM. *Suppose L and L' are discrete atomic modular lattices such that $h(L) = h(L')$ and $2 \leq r \leq h(L) - 2$. Let ϕ be a distance preserving bijection $L_r \to L'_r$. Then if $2r \neq h(L)$ there exists a collineation $u: L_1 \to L'_1$ such that $\phi = u_r$, while in the case $2r = h(L)$, ϕ may also be of the form wv_r, where w is a particular exceptional mapping $L_r \to L'_r$ and v is a collineation $L_1 \to L_1$.*

Taking L (L') to be the lattice of all finite dimensional subspaces of E (E'), one recovers essentially the theorem of Chow, with the added observation that it remains valid for infinite dimensional spaces (in an even simpler form). Another interesting example is obtained when L is the lattice $F(S)$ of all finite subsets of a set S; in this case, the theorem simplifies to read as follows:

COROLLARY. *If S and S' are sets, any distance preserving bijection $\phi: F_r(S) \to F_r(S')$ is induced by a bijection $u: S \to S'$ unless both S and S' are finite of cardinality n, and $2r = n$, when ϕ may also be of the form $\phi(A) = u(S \setminus A)$.*

It is curious to note that, just as the fundamental theorem of projective geometry is used [3, p. 93] in determining the automorphisms of $GL_n(K)$, so can a special case of the above corollary be applied to the automorphisms of finite or infinite symmetric groups S_X. Indeed, one first shows by some means that for card$(X) \neq 6$,the set of transpositions is left invariant by an automorphism and then remarks that the distance between transpositions (regarded as elements of $F_2(X)$) is $0, 1$ or 2 according to whether the order of their product is $1, 3$ or 2. Since automorphisms preserve order, the corollary shows that for card$(X) \neq 4$ the automorphism is induced, at least on transpositions, by conjugation with an element $u \in S_X$. The argument is then concluded as in [4].

Finally, we remark that if L satisfies $h(L) < \infty$ in addition to the other hypotheses, a theorem of Birkhoff [1, p. 93] says that L must be a product of a boolean algebra with projective geometries. In this case, Chow's theorem covers the simple Desarguean factors, while the corollary above takes care of the boolean algebra.

We now turn to the proofs. Elements $a, b \in L_r$ are called *adjacent* if $d(a, b) = 1$.

LEMMA 2. *If* $d(a, b) = k$, *there exists a sequence* $c_0, \cdots, c_k$ *of elements in* L_r *such that* $c_0 = 0$, $c_k = b$ *and* c_{i-1} *is adjacent to* c_i *for* $1 \leqq i \leqq k$.

PROOF. An argument of Birkhoff [1, p. 88] shows that a lattice satisfying our assumptions is relatively complemented. Therefore $a = (a \wedge b) \vee x$, $b = (a \wedge b) \vee y$ for some x and y disjoint from $a \wedge b$ and from each other. If $x = p \vee x'$, $y = q \vee y'$ for some atoms p and q, the element $c_{k-1} = (a \wedge b) \vee p \vee y'$ satisfies $d(a, c_{k-1}) = k - 1$ and $d(c_{k-1}, b) = 1$; the argument is now concluded by induction.

LEMMA 3. *A bijection* $\phi: L_r \rightarrow L'_r$ *preserves distance if and only if both* ϕ *and* ϕ^{-1} *preserve adjacence.*

PROOF. Suppose $d(a, b) = k$. Then, with the notation of lemma 2, the sequence $\phi(c_0), \cdots, \phi(c_k)$ is such $\phi(c_{i-1})$ and $\phi(c_i)$ are adjacent for $1 \leqq i \leqq k$. The triangle inequality implies $d(\phi(a), \phi(b)) \leqq d(a, b)$. (Actually, this inequality in turn implies that ϕ preserves adjacence, for if $d(a, b) = 1$, $d(\phi(a), \phi(b))$ cannot be 0 since ϕ is injective and must therefore be 1.) Since ϕ^{-1} also preserves adjacence, we conclude that $d(\phi^{-1}(a), \phi^{-1}(b)) \leqq d(a, b)$ and together the two inequalities imply that ϕ preserves distance.

PROOF OF LEMMA 1. If $a \in L_r$ is written as a join $p_1 \vee \cdots \vee p_r$ of r atoms, the latter are necessarily independent. We show by induction that an atom $q \leqq p \vee \cdots \vee p_r$ if and only if $u(q) \leqq u(p_1) \vee \cdots \vee u(p_r)$. This is clear if $q = p_r$ or if $q \leqq p_1 \vee \cdots \vee p_{r-1}$; otherwise, $q \vee p_r$ intersects $p_1 \vee \cdots \vee p_{r-1}$ in an atom t. We must have $q \leqq q \vee p_r = t \vee p_r$, and thus $u(q) \leqq u(t) \vee u(p_r)$ since u is a collineation; however, the induction assumption shows that $u(t) \leqq u(p_1) \vee \cdots \vee u(p_{r-1})$. The converse follows by applying the same argument to u^{-1}. We conclude in particular that the atoms $u(p_1), \cdots, u(p_r)$ are independent so that $u(p_1) \vee \cdots \vee u(p_r) \in L'_r$; secondly, if $p'_1 \vee \cdots \vee p'_r$ is another representation of a as a join of atoms, $u(p_i) \leqq u(p'_1) \vee \cdots \vee u(p_r)$ so that $u(p'_1) \vee \cdots \vee u(p'_r) = u(p_1) \vee \cdots \vee u(p_r)$. Thus u_r is well-defined, with inverse $(u^{-1})_r$. If $a, b \in L_r$ are adjacent then, since $u_r(a)$ and $u_r(b)$ clearly contain $u_{r-1}(a \wedge b)$, they must be adjacent or equal, but the latter is impossible. We conclude by lemma 3 that u_r preserves distance.

We now come the the theorem itself. The first step is to determine the structure of a maximal set M of pairwise adjacent elements of L_k. Since $k < h(L)$, M must have at least two elements a and b. If there exists some $c \in M$ which does not contain $a \wedge b$, then every element of M is contained in $a \vee b$. Indeed, if $d \in M$ does not contain $a \wedge b$, then $a \wedge d \neq b \wedge d$ so that $d = (a \wedge d) \vee (b \wedge d) \leqq (a \vee b) \wedge d$ i.e. $d \leqq a \vee b$. If, on the other hand, $d \in M$ does contain $a \wedge b$, then $c \wedge d$ must be distinct from $a \wedge d = a \wedge b$ so that $d = (c \wedge d) \vee (a \wedge d)$

$\leqq (c \vee a) \wedge d = (a \vee b) \wedge d$ and again $d \leqq a \vee b$. This argument shows that M is either of the form $C_x = \{a \in L_r \,|\, a \geqq x\}$ for some $x \in L_{r-1}$ or $C^y = \{a \in L_r \,|\, a \leqq y\}$ for some $y \in L_{r+1}$, since such sets consist of pairwise adjacent elements. If a set C_x is not maximal, it is contained in another set of the form C_y or C^y. The first alternative is clearly impossible and the second is ruled out by the condition $r \leqq h(L) - 2$. Indeed, if p is an atom not contained in y, then $p \vee x \in C_x \setminus C^y$. Similarly, the condition $r \geqq 2$ shows that sets of the form C^y are maximal.

Now suppose that $\phi \colon L_r \to L'_r$ is a distance preserving bijection. If $x \in L_{r-1}$, $\phi(C_x)$ is a maximal set of pairwise adjacent elements and thus of the form C_y or C^y. If the first alternative holds for a particular $x \in L_{r-1}$, then it holds for all elements of L_{r-1}. To see this, it suffices to remark that if $x \neq x'$, $C_x \cap C_{x'}$ consists of at most one element $(x \vee x')$, while an intersection of a set C_y with a set $C^{y'}$ has at least two elements, namely $y \vee p$ and $y \vee q$, where p and q are distinct atoms contained in y' but not in y.

It follows that ϕ induces either a bijection $\psi \colon L_{r-1} \to L'_{r-1}$ or $L_{r-1} \to L'_{r+1}$. Since x and x' are adjacent precisely when $C_x \cap C_{x'} \neq \varnothing$ or $C^x \cap C^{x'} \neq \varnothing$, it follows that both ψ and ψ^{-1} preserve adjacence and hence distance. Unless $h(L) < \infty$ and $2r = h(L)$, the second alternative is impossible. If $2r < h(L)$, the distance between elements of L_{r-1} is at most $r - 1$, while the distance between elements L'_{r+1} can be as high as r. If $2r > h(L)$, the second distance is at most $h(L) - r - 1$, while the first can be $h(L) - r$.

The original mapping ϕ can be recovered from ψ through the equation $\psi(C^x) = C^{\phi(x)}$, which also shows that even if $2(r - 1)$ happens to be $h(L)$, ψ still maps sets of the form C_x to sets of the same form and thus induces a mapping $L_{r-2} \to L'_{r-2}$. Eventually, we obtain a collineation $u \colon L_1 \to L'_1$ such that $\phi = u_r$.

If there does exist, when $2r = h(L)$, an exceptional mapping w which induces a mapping $L_{r-1} \to L'_{r+1}$ then for every other such mapping ϕ, $w^{-1}\phi$ induces a mapping $L_{r-1} \to L_{r-1}$ and thus comes from a collineation $v \colon L_1 \to L_1$. This completes the proof.

References

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