Bull. Aust. Math. Soc. **97** (2018), 406–411
doi:10.1017/S0004972718000072

ON p-PARTS OF CONJUGACY CLASS SIZES OF FINITE GROUPS

YONG YANG✉ and GUOHUA QIAN

(Received 4 September 2017; accepted 16 January 2018; first published online 28 March 2018)

Abstract

Let G be a finite group. Let $\mathrm{cl}(G)$ be the set of conjugacy classes of G and let $\mathrm{ecl}_p(G)$ be the largest integer such that $p^{\mathrm{ecl}_p(G)}$ divides $|C|$ for some $C \in \mathrm{cl}(G)$. We prove the following results. If $\mathrm{ecl}_p(G) = 1$, then $|G : F(G)|_p \le p^4$ if $p \ge 3$. Moreover, if G is solvable, then $|G : F(G)|_p \le p^2$.

2010 *Mathematics subject classification*: primary 20D20; secondary 20D10.

Keywords and phrases: conjugacy class sizes.

1. Introduction

We first fix the notation. Let n be a positive integer and p be a prime. We may write $n = p^a m$, where $p \nmid m$. We use $n_p = p^a$ to denote the p-part of the integer n.

It is interesting to study how arithmetic conditions on the invariants of a finite group affect the group structure. Let G be a finite group, P a Sylow p-subgroup of G and $\mathrm{Irr}(G)$ the set of irreducible complex characters of G. It is reasonable to expect that the degrees of irreducible characters of G restrict those of P. Let $e_p(G)$ be the largest integer such that $p^{e_p(G)}$ divides $\chi(1)$ for some $\chi \in \mathrm{Irr}(G)$. The fundamental Ito–Michler theorem [5] asserts that $e_p(G) = 0$ if and only if $P \lhd G$ and P is abelian. In particular, this implies that $|G : F(G)|_p = 1$, where $F(G)$ is the Fitting subgroup of G. A natural generalisation of the Ito–Michler theorem is the following result of Lewis *et al.* [3]: if G is solvable and $e_p(G) = 1$, then $|G : F(G)|_p \le p^2$. In [2], Lewis *et al.* studied a similar problem for arbitrary finite groups and showed that if G is finite and $e_p(G) = 1$, then $|G : F(G)|_p \le p^4$.

Results for character degrees may have dual results for conjugacy class sizes. The dual of the Ito–Michler theorem on the set of conjugacy class sizes is that every conjugacy class of G has p'-size if and only if a Sylow p-subgroup of G is central. In this paper, we study the dual versions of the results of Lewis *et al.* on the set of conjugacy class sizes and show that analogues of the main results of [3] and [2] hold.

The project was supported by NSFC (Nos. 11671063 and 11471054), the Natural Science Foundation Project of CSTC (cstc2016jcyjA0065) and the NSF of Jiangsu Province (No. BK20161265). The first author was also supported by a grant from the Simons Foundation (No. 499532).

Let G be a finite group and p be a prime number. We say that an element $x \in G$ is p-regular if the order of x is not a multiple of p. Let $\mathrm{clsize}_{\mathrm{preg}}(G)$ denote the set of conjugacy class sizes of p-regular elements of G. Inspired by the results in [6], we can prove a little more by considering the conjugacy class sizes of the p-regular elements in G. The following results and more will be proved in Section 2.

Theorem 1.1. *Let G be a finite solvable group and suppose that p^2 does not divide $|x^G|$ for every p-regular element $x \in G$. Then $|G : F(G)|_p \le p^2$. In particular, if $P \in \mathrm{Syl}_p(G)$, then P' is subnormal in G.*

Theorem 1.2. *Let G be a finite group and suppose that p^2 does not divide $|x^G|$ for every p-regular element $x \in G$. Then $|G : F(G)|_p \le p^4$ if $p \ge 3$ and $|G : F(G)|_p \le p^2$ if $p = 2$.*

Theorem 1.1 is a consequence of Theorem 2.2 and Theorem 1.2 is Theorem 2.11.

2. Proof of the main results

We will use the following results very often in the proofs.

Lemma 2.1. *Let N be a normal subgroup of G.*

(1) *If $x \in N$, then $|x^N|$ divides $|x^G|$.*
(2) *If $x \in G$, then $|(xN)^{G/N}|$ divides $|x^G|$.*

We first observe that the condition that p^2 does not divide $|x^G|$ for every p-regular element $x \in G$ is inherited by all the normal subgroups of G and all the quotient groups of G. Since the assertion for normal subgroups follows easily from Lemma 2.1(1), we will just explain the assertion for quotient groups. Let $N \lhd G$ and T be a p-regular class of G/N. Then there is a p-regular element $xN \in G/N$ such that $T = (xN)^{G/N}$. We may write $x = yz$, where y is a p'-element, z is a p-element and $yz = zy$. If $H = \langle x \rangle N$, then $|H/N|$ is a p'-number and so $z \in N$. Thus, $xN = yN$ and $T = (yN)^{G/N}$. From this, $|T| \mid |y^G|$ and the result follows.

Theorem 2.2. *Suppose that G is a solvable group and p is a prime. If $a_p \le p$ for all $a \in \mathrm{clsize}_{\mathrm{preg}}(G)$, then a Sylow p-subgroup of $G/F(G)$ has order at most p^2.*

Proof. If N is normal in G, then N and G/N inherit the hypothesis. In particular, $G/O_p(G)$ inherits the hypothesis, so we can assume that $O_p(G) = 1$. We wish to show that a Sylow p-subgroup P of G has order at most p^2. We know that P is elementary abelian by [6, Lemma 3].

Let $K = O_{p'}(G)$. Since G is p-solvable and a Sylow p-subgroup is abelian, G/K has a normal Sylow p-subgroup Q/K. Let P be a Sylow p-subgroup of G, so $Q = KP$. The action of Q/K on the classes of K has all orbits of size 1 or p and thus the same is true for the action of P on the classes of K. Since $|P|$ and $|K|$ are coprime, the actions of P on the classes of K and on $\mathrm{Irr}(K)$ are permutation isomorphic, so the P-orbits on $\mathrm{Irr}(K)$ all have size 1 or p.

Now let χ be an irreducible character of $Q = KP$ and let θ be an irreducible constituent of the restriction χ_K. Let T be the stabiliser of θ in Q, so that $|Q : T|$ is the size of the p-orbit of θ and hence $|Q : T|$ is 1 or p. Now $\chi = \eta^Q$, where η is some irreducible character of T. But K is a normal Hall subgroup of T, so θ has an extension ψ to T and, by Gallagher's theorem, $\eta = \psi\beta$, where β is an irreducible character of T/K, which is abelian since P is abelian. Thus, $\beta(1) = 1$, so $\eta(1) = \psi(1) = \theta(1)$ is not divisible by p, and $\chi(1) = |Q : T|\,\eta(1)$, so the p-part of $\chi(1)$ is $|Q : T|$, which is 1 or p.

We know that no irreducible character of Q has degree divisible by p^2, so, by the main result of [3], p^3 does not divide $|Q|$ and the theorem is proved. □

Remark 2.3. Theorem 1.1 is a straightforward consequence of Theorem 2.2. Once we know that $|G : F(G)|_p \leq p^2$, since p-groups of order at most p^2 are abelian, it is easily seen that $P' \leq F(G)$ and thus P' is subnormal in G.

Proposition 2.4. *Let p be an odd prime and let G be a p-solvable group that satisfies $a_p \leq p$ for all $a \in \mathrm{clsize}_{\mathrm{preg}}(G)$. Suppose that $G = O^{p'}(G)$ and $O_p(G) = 1$. Suppose also that G admits a minimal normal subgroup $N = T_1 \times \cdots \times T_n \cong T^n$, where T is a nonabelian simple group. Then $|G|_p \leq p$.*

Proof. Let P be a Sylow p-subgroup of G. Then P is elementary abelian by [6, Lemma 3] and thus the p-length of G is 1. Let $K = O_{p'}(G)$. Since $G = O^{p'}(G)$, it follows that $G = PK$.

The action of G/K on the classes of K has all orbits of size 1 or p and so the same is true for the action of P on the classes of K. Since $|P|$ and $|K|$ are coprime, the actions of P on the classes of K and on $\mathrm{Irr}(K)$ are permutation isomorphic, so the P-orbits on $\mathrm{Irr}(K)$ all have size 1 or p. Since P is abelian, it follows by a similar argument to that in the proof of Theorem 2.2 that no irreducible character of Q has degree divisible by p^2. Thus, $|G|_p \leq p$ by [2, Proposition 2.4]. □

Theorem 2.5. *Let p be an odd prime and let G be a p-solvable finite group. If p^2 does not divide $|g^G|$ for any p-regular element g of G, then the following statements hold:*

(1) *$|G/\mathrm{sol}(G)|_p \leq p$ and $|G/F(G)|_p \leq p^3$;*
(2) *either $|G/F(G)|_p \leq p$ or $F^*(O^{p'}(G)) = F(O^{p'}(G))$.*

Proof. Let $L = O^{p'}(G)$. Then $|L|_p = |G|_p$, $\mathrm{sol}(L) \leq \mathrm{sol}(G)$, $F(L) \leq F(G)$ and $F(\mathrm{sol}(L)) \leq F(\mathrm{sol}(G))$. Furthermore, p^2 does not divide $|g^L|$ for any p-regular element g of L. Hence, we may replace G by L and assume that $G = O^{p'}(G)$.

To see (1), set $R = \mathrm{sol}(G)$. Note that $|G/R|_p \leq p$ by Proposition 2.4 and that $|R/F(R)|_p \leq p^2$ by Theorem 2.2. Since $F(R) = F(G)$, we conclude that

$$|G/F(G)|_p \leq |G/R|_p |R/F(R)|_p \leq p^3.$$

For (2), write $U = \Phi(F^*(G))$. Since $F^*(G/U) = F^*(G)/U$ and $F(G/U) = F(G)/U$, we may assume that $U = 1$. Now $F^*(G)$ is a direct product of simple groups. Assume that $F^*(G) > F(G)$. Then $G/F(G)$ admits a nonabelian minimal normal subgroup and Proposition 2.4 implies that $|G/F(G)|_p \leq p$. □

https://doi.org/10.1017/S0004972718000072 Published online by Cambridge University Press

Lemma 2.6 [2, Lemma 3.1]. *Let S be a nonabelian simple group and let p be a prime dividing $|S|$. Then $|S|_p > |\mathrm{Out}(S)|_p$.*

Lemma 2.7. *Let S be a nonabelian simple group and $p \geq 3$ be a prime divisor of $|S|$. Then there exists a p-regular element $x \in S$ such that $|x^S|_p^2 > |\mathrm{Aut}(S)|_p$.*

Proof. For a simple group of Lie type and any prime p, or for an alternating group and $p \geq 5$, there is a p-block of defect 0. Hence, there is a conjugacy class $\mathrm{cl}_G(x)$ of p-defect 0 and $|G|_p$ divides $|\mathrm{cl}_G(x)|$. Clearly, x is p-regular since otherwise $|\mathrm{cl}_G(x)|_p < |G|_p$. The result now follows from Lemma 2.6.

Thus, we only need to consider the alternating groups A_n and $p = 3$.

Assume n is odd. Set $\alpha = (1, 2, \ldots, n)$. Then $\alpha \in A_n$ and $|\mathrm{cl}_{S_n}(\alpha)| = (n-1)!$. Thus, $|\mathrm{cl}_{A_n}(\alpha)|$ is a multiple of $\frac{1}{2}(n-1)!$. Set $\beta = (1, 2, \ldots, n-2)$. Then again $\beta \in A_n$ and $|\mathrm{cl}_{S_n}(\beta)| = n!/(2(n-2))$. Thus, $|\mathrm{cl}_{A_n}(\beta)|$ is a multiple of $\frac{1}{2}n!/(2(n-2))$. If $3 \nmid n$, then the class of α satisfies the condition. If $3 \mid n$, then $3 \nmid n-2$ and the class of β satisfies the condition.

Assume n is even. Set $\alpha = (1, 2, \ldots, n-1)$. Then $\alpha \in A_n$ and $|\mathrm{cl}_{S_n}(\alpha)| = n!/(n-1)$. Thus, $|\mathrm{cl}_{A_n}(\alpha)|$ is a multiple of $n!/(2(n-1)$. Set $\beta = (1, 2, \ldots, n-3)$. Then $\beta \in A_n$ and $|\mathrm{cl}_{S_n}(\beta)| = n!/(6(n-3))$. Thus, $|\mathrm{cl}_{A_n}(\beta)|$ is a multiple of $n!/(2 \cdot 6(n-3)$. If $3 \nmid n-1$, then the class of α satisfies the condition. If $3 \nmid n$, then the class of α satisfies the condition. If $3 \mid n-1$, then $3 \nmid n-3$ and the class of β satisfies the condition.

For the sporadic groups, the result can be checked by using [1]. □

Lemma 2.8. *Let S be a nonabelian simple group and let G be an almost simple group with $S \leq G \leq \mathrm{Aut}(S)$. Let p be an odd prime. Suppose that p divides $|S|$ and $a_p \leq p$ for all $a \in \mathrm{clsize}_{\mathrm{preg}}(G)$. Then $|G|_p = p$.*

Proof. This follows from Lemmas 2.6 and 2.7. □

Proposition 2.9. *Let p be an odd prime and let G be a finite non-p-solvable group with a trivial p-solvable radical and $O^{p'}(G) = G$. If $a_p \leq p$ for all $a \in \mathrm{clsize}_{\mathrm{preg}}(G)$, then G is a nonabelian simple group with $|G|_p = p$.*

Proof. Since the p-solvable radical is trivial, it follows that $F(G) = 1$ and, thus, $F^*(G) = E(G) = T_1 \times T_2 \times \cdots \times T_k$, where $T_1, \ldots, T_k$ are nonabelian simple groups with p dividing $|T_i|$ and $C_G(F^*(G)) = 1$. For each $i \in \{1, \ldots, k\}$, we have $p \mid |T_i|$ and T_i is nonabelian and simple. Thus, by Lemma 2.7, there exists a p-regular element $x_i \in T_i$ with $p \mid |x_i^{T_i}|$. Let $x = x_1 \times \cdots \times x_k \in F^*(G)$. Then $|x^G|$ is divisible by p^k. This forces $k = 1$. Hence, $F^*(G)$ is a nonabelian simple group and G is an almost simple group with socle $F^*(G)$. The result now follows from Lemma 2.8. □

Theorem 2.10. *Let p be an odd prime and suppose that v is a positive integer and G is a group such that every section H of G that is p-solvable satisfies $|H : F(H)|_p \leq p^v$. If $a_p \leq p$ for all $a \in \mathrm{clsize}_{\mathrm{preg}}(G)$, then $|G : F(G)|_p \leq p^{1+v}$.*

https://doi.org/10.1017/S0004972718000072 Published online by Cambridge University Press

Proof. If G is p-solvable, then $|G : F(G)|_p \le p^{\nu} < p^{1+\nu}$ and we are done. So, we assume that G is not p-solvable. If $L = O^{p'}(G)$, then $O^{p'}(L) = L$ and, by [2, Lemma 3.3], $|G : F(G)|_p = |L : F(L)|_p$. By induction on $|G|$, one may assume that $G = O^{p'}(G)$. Let R_p be the p-solvable radical of G. Then G/R_p has a trivial p-solvable radical and $O^{p'}(G/R_p) = G/R_p$. By Proposition 2.9, G/R_p is a nonabelian simple group with $|G/R_p|_p = p$. Since $F(G) \le R_p$, we have $F(R_p) = F(G) \cap R_p = F(G)$. Now R_p is a p-solvable group and is normal in G and $a_p \le p$ for all $a \in \mathrm{clsize}_{\mathrm{preg}}(R_p)$, so we obtain $|G : F(G)|_p = |G : R_p|_p \cdot |R_p : F(R_p)|_p \le p \cdot p^{\nu} = p^{1+\nu}$. □

Theorem 2.11. *Let G be a finite group and suppose that p^2 does not divide $|x^G|$ for every p-regular element $x \in G$. Then $|G : F(G)|_p \le p^4$ if $p \ge 3$ and $|G : F(G)|_p \le p^2$ if $p = 2$.*

Proof. If $p = 2$, then G is solvable by a result in [4] and the theorem follows by Theorem 2.2. If $p \ge 3$, then the theorem follows by Theorems 2.5 and 2.10. □

Remark 2.12. We provide an example to show that the bound we obtained for the solvable groups is the best possible. Indeed, we claim that [3, Example 3.3] will work.

In that example, we have $G = HV$, where $H \le \Gamma(V)$, and furthermore we may write $G = A \ltimes K$, where A is an elementary abelian group of order p^2 and K is a p'-group. Consider the action of A on $\mathrm{Irr}(K)$ and $\mathrm{cl}(K)$. The stabiliser of A on every nonprincipal $\psi \in \mathrm{Irr}(K)$ has a normal subgroup of order p (since $p^2 \nmid \chi(1) \in cd(G)$) and the same holds for the conjugacy classes of K.

Let K be the Hall p'-subgroup of H. Then KV is a normal Hall p'-subgroup of G. We note that KV is a Frobenius group. Pick a p-regular element $x \in G$. There are now two possibilities. First, if $x \in V$, then $p \mid |\mathbf{C}_G(x)|$. Otherwise, we may assume that a conjugate of x is in K and so we may assume that $x \in K$. Thus, $p \mid |\mathbf{C}_H(x)| \mid |\mathbf{C}_G(x)|$. In both cases, $p^2 \nmid |x^G|$ for any p-regular element $x \in G$.

Acknowledgements

We would like to thank Professor Martin Isaacs for fruitful discussions which greatly simplified the proofs of some results in the paper. The authors are also grateful to the referee for valuable suggestions which improved the manuscript.

References

[1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups* (Oxford University Press, New York, 1985).

[2] M. Lewis, G. Navarro, P. H. Tiep and H. P. Tong-Viet, 'p-parts of character degrees', *J. Lond. Math. Soc. (2)* **92** (2015), 483–497.

[3] M. Lewis, G. Navarro and T. R. Wolf, 'p-parts of character degrees and the index of the Fitting subgroup', *J. Algebra* **411** (2014), 182–190.

[4] X. Liu, Y. Wang and H. Wei, 'Notes on the length of conjugacy classes of finite groups', *J. Pure Appl. Algebra* **196** (2005), 111–117.

[5] G. Michler, 'A finite simple group of Lie type has p-blocks with different defects, $p \neq 2$', *J. Algebra* **104**(2) (1986), 220–230.
[6] G. Qian and Y. Wang, 'A note on the conjugacy class sizes of finite groups', *Acta Math. Sci. Chin. Ser.* **52**(1) (2009), 125–130.

YONG YANG, Department of Mathematics,
Texas State University, 601 University Drive,
San Marcos, TX 78666, USA
and
Key Laboratory of Group and Graph Theories and Applications,
Chongqing University of Arts and Sciences,
Chongqing, PR China
e-mail: yang@txstate.edu

GUOHUA QIAN, Department of Mathematics,
Changshu Institute of Technology, Changshu, JiangSu 215500,
PR China
e-mail: ghqian2000@163.com

https://doi.org/10.1017/S0004972718000072 Published online by Cambridge University Press