Canad. Math. Bull. Vol. **25** (1), 1982

INTEGRAL COMPARISON THEOREMS FOR SCALAR RICCATI EQUATIONS AND APPLICATIONS

BY

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ABSTRACT. Comparison theorems are developed for the pair of first order Riccati equations (1) $r'+p^{-1}r^2+q=0$ and (2) $z'+p_1^{-1}z^2+q_1=0$. The comparisons are of an integral type and involve an auxiliary function μ. Applications are given to disconjugacy theory for self-adjoint equations of the second and fourth order.

1. **Introduction.** Consider the pair of second order equations

$$(py')'+qy=0 \tag{1.1}$$

and

$$(p_1y')+q_1y=0 \tag{1.2}$$

where p, q, p_1, q_1 are continuous on an interval I of the real line with $p, p_1>0$ on I. The Riccati equations corresponding to (1.1) and (1.2) are

$$r'+p^{-1}r^2+q=0, \qquad r=\frac{py'}{y} \tag{1.3}$$

and

$$z'+p_1^{-1}z^2+q_1=0, \qquad z=\frac{p_1y'}{y}. \tag{1.4}$$

We shall be interested in comparison theorems of integral type-results which will guarantee the existence of a continuous solution of (1.4) on I when it is known that (1.3) has a continuous solution on I. The existence of a solution of (1.3) on I is, of course, equivalent to the disconjugacy of (1.1) on I ([4]); (that is, no solution of (1.1) has more than one zero on I). Many criteria have been developed relating the oscillation and disconjugacy of (1.1) and (1.2)—we refer to [4], [9], [11], [14] and the references therein—and often the methods involve use of (1.3) and (1.4). A very well-known criterion—aside from the Sturm comparison theorem—is the Hille–Wintner comparison Theorem ([5], [13]). We refer to [1] for a recent extension of the Hille–Wintner theorem.

*Research supported by NSERC Grant A-7673.

Received by the editors May 16, 1980.

AMS(MOS) subject classification (1980): Primary 34C10, 34C11.

Key words and phrases: disconjugacy, Riccati equations, comparison theorems.

Additional criteria of integral type may also be found in [7], [10] and [12], and it is this latter type which will be of primary interest here. In addition to the fact that the results obtained below are new except for certain special cases, another motivation for the criteria which we develop is that they afford a connection to the comparison theory for the fourth order self-adjoint equations

(1.5) $$(py'')''-qy=0$$

and

(1.6) $$(p_1y'')''-q_1y=0.$$

In Section 2 we state the results for both the finite and infinite interval case and compare with some known results. Examples are also given illustrating the applicability of the results obtained. Section 3 is devoted to the fourth order case. The proofs of the results of Section 2 are given in Section 4.

2. In the first two theorems we assume that $I=[a, b]$ or $[a, b)$ with $a<b\leq+\infty$ and that there exists $\mu\in C^1(I)$, $\mu>0$ on I, such that $p\mu\in C^1(I)$ and the following condition holds:

(H_1) $$\int_a^t \mu^2q_1\,dt\leq\int_a^t \mu^2q\,dt \qquad a\leq t<b$$

THEOREM 2.1. *Assume condition* (H_1) *holds, where* μ, $p\mu'\in C^1(I)$, $\mu>0$, *and* $I=[a, b)$, $a<b\leq+\infty$, *let* $0<p\leq p_1$ *on I and assume there exists a solution r of* (1.3) *on I such that*

(2.1) $$\mu(t)r(t)\leq p(t)\mu'(t), \qquad t\geq a,$$

and

(2.2) $$\mu(a)(\mu'(a)p(a)-\mu(a)r(a))+\int_a^t \mu(\mu'p)'\,dt+\int_a^t \mu^2q_1\,dt\geq 0, \qquad t\geq a.$$

Then (1.4) *has a solution z on I.*

The next theorem replaces the integral condition (2.2) by a pointwise condition. If $p\equiv p_1\equiv 1$, it reduces to a result of [10].

THEOREM 2.2. *Let I be as in Theorem* 2.1, *assume condition* (H_1) *holds where* $\mu>0$, $\mu\in C^1(I)$, *let* $0<p\leq p_1$ *on I and assume there exists a solution r of* (1.3) *on I such that* (2.1) *holds and*

(2.3) $$2\mu(t)(\mu'(t)p(t))'+2p^{-1}(t)[\mu'(t)p(t)-\mu(t)r(t)]^2+\mu^2(t)(q_1(t)+q(t))\geq 0, \qquad t\in I$$

Then (1.4) *has a solution z on I.*

In Theorems 2.1 and 2.2 we assume that $I=[a, b]$ or $[a, b)$ where $a<b\leq$

$+\infty$. In the next two results, we assume that $I=(a, b]$ or $[a, b]$ where $-\infty \le a < b < +\infty$, and we suppose that $\mu \in C^1(I)$, $\mu > 0$ such that the following condition holds:

$$(H_2) \qquad \int_t^b \mu^2 q_1 \, dt \le \int_t^b \mu^2 q \, dt, \qquad a < t \le b$$

THEOREM 2.3. *Assume condition* (H_2) *holds, let* μ, $p\mu' \in C^1(I)$, $\mu > 0$ *where* $I=(a, b]$ *or* $[a, b]$, $-\infty \le a < b < +\infty$ *let* $0 < p \le p_1$ *on I, and assume there exists a solution r of* (1.3) *on I such that*

$$(2.4) \qquad \mu(t)r(t) \ge p(t)\mu'(t), \qquad a < t \le b$$

and

$$(2.5) \quad \mu(b)(\mu(b)r(b) - \mu'(b)p(b)) + \int_t^b \mu(\mu' p)' \, dt + \int_t^b \mu^2 q_1 \, dt \ge 0, \qquad a > t \le b.$$

Then (1.4) *has a solution on I.*

Analogous to Theorem 2.2 we have

THEOREM 2.4. *Let I be as in Theorem* 2.3, *assume condition* (H_2) *holds, where* $\mu > 0$, μ, $p\mu' \in C^1(I)$, *let* $0 < p \le p_1$ *on I and assume there exists a solution r of* (1.3) *on I such that* (2.4) *and* (2.3) *hold. Then* (1.4) *has a solution on I.*

The next result is the analogue of Theorem 2.3 for the case $I=(a, +\infty)$ or $[a, +\infty)$, $-\infty < a < +\infty$. If $\mu \equiv 1$, q, $q_1 \ge 0$, it reduces to the Hille-Wintner Theorem (cf. [5], [13])

THEOREM 2.5. *Let* μ, $p\mu' \in C^1(I)$, $\mu > 0$, *where* $I=(a, +\infty)$, or $[a, +\infty)$, $-\infty < a < +\infty$, *let* $0 < p \le p_1$ *on I and assume there exists a solution r of* (1.3) *on I such that*

$$(2.6) \qquad \mu(t)r(t) \ge p(t)\mu'(t), \qquad t \in I$$

and

$$(2.7) \qquad L + \int_t^\infty \mu(\mu' p)' \, dt + \int_t^\infty \mu^2 q_1 \ge 0, \qquad t \in I,$$

where it is assumed that the integrals in (2.7) *converge and* $L \equiv \lim_{t\to\infty} \mu(t) \times (\mu(t)r(t) - \mu'(t)p(t))$ *is assumed to exist. Finally, let the following condition hold:*

$$(2.8) \qquad \int_t^\infty \mu^2 q_1 \, dt \le \int_t^\infty \mu^2 q \, dt, \qquad t \in I.$$

Then (1.4) *has a solution on I.*

Recall that the Euler equation

$$(2.9) \qquad (t^\alpha y') + \beta t^{\alpha-2} y = 0, \qquad 0 < a \le t < +\infty$$

is disconjugate in $[a, +\infty)$ if and only if $(1-\alpha)^2 \leq 4\beta$. In the critical case $4\beta = (1-\alpha)^2$, the solutions of (2.9) are $y = C_1 t^\lambda + C_2 t^\lambda \ln|t|$, where $\lambda = (1-\alpha)/2$. In this case, with $r = py'/y = t^{\alpha-1}$ and with the choice $\mu(t) = t^\gamma$, condition (2.1) ($\mu r \leq p\mu'$) holds provided $\gamma \geq \lambda$, and condition (2.2) becomes, after some simplification,

$$\frac{\eta}{2} a^\eta + \frac{\gamma(\eta-\gamma)}{\eta}(t^\eta - a^\eta) + \int_a^t s^{2\gamma} q_1(s)\, ds \geq 0, \qquad t \geq a \tag{2.10}$$

where $\eta = 2\gamma + \alpha - 1 > 0$. If $\gamma = (1-\alpha)/2$, then (2.2) becomes

$$\int_a^t s^{2r} q_1(s)\, ds \geq \gamma^2 \ln\left|\frac{t}{a}\right|, \qquad t \geq a. \tag{2.10)'$$

On the other hand, condition (2.3) is equivalent to

$$\eta^2 + t^{2-\alpha} q_1(t) \geq \frac{(1-\alpha)^2}{4}, \qquad t \geq a. \tag{2.11}$$

where $\eta = 2\gamma + \alpha - 1 \geq 0$. In view of the above, we may state the following Corollary of Theorem 2.1 and 2.2.

COROLLARY 2.6. *Equation* (1.2) *is disconjugate on* $[a, +\infty)$, $a > 0$, *provided* $p_1(t) \geq t^\alpha$, $t \geq a$ *and there exists* $\gamma > (1-\alpha)/2$ *such that either* (2.10) *or* (2.11) *holds and such that*

$$\int_a^t s^{2\gamma} q_1(s)\, ds \leq \frac{(1-\alpha)^2}{4\eta}(t^\eta - a^\eta), \qquad t \geq a, \tag{2.12}$$

where $\eta = 2\gamma + \alpha - 1 > 0$.

For various choices of α, γ it is not difficult to see that (2.2) and (2.3) are indeed independent. For example, if $0 < \frac{3}{4}(1-\alpha) < \gamma < 1 - \alpha < 2\eta$, then (2.11) becomes

$$t^{2-\alpha} q_1(t) \geq -\left(\eta^2 - \frac{(1-\alpha)^2}{4}\right), \qquad t \geq a \tag{2.13}$$

which implies

$$\int_a^t s^{2\gamma} q_1(s)\, ds \geq -\left(\eta - \frac{(1-\alpha)^2}{4\eta}\right)(t^\eta - a^\eta), \qquad t \geq a \tag{2.14}$$

so that the integral on the left side of (2.14) can assume arbitrarily large negative values. However, since $\eta - \gamma = \gamma + \alpha - 1 < 0$, condition (2.10) would require $\int_a^t s^{2\gamma} q_1(s)\, ds \geq kt^\eta$ for all large t and some $k > 0$. Nevertheless, it is also clear that one can have $\lim_{t\to\infty} \inf t^{2-\alpha} q_1(t) = -\infty$ and still satisfy the integral condition (2.10).

Similarly, from Theorem 2.5 one obtains

COROLLARY 2.7. *Equation* (1.2) *is disconjugate on* $[a, +\infty)$, $a > 0$, *provided* $p_1(t) \geq t^\alpha$, $t \geq a$ *and there exists* $\gamma < (1-\alpha)/2$ *such that*

$$\gamma(\gamma+\delta) \leq \delta t^\delta \int_t^\infty s^{2\gamma} q_1(s)\, ds \leq \frac{(1-\alpha)^2}{4}, \qquad t \geq a \tag{2.15}$$

where $\delta = 1 - \alpha - 2\gamma > 0$.

EXAMPLE 2.8. We illustrate here a further class of equations whose disconjugate behavior may be inferred from Corollary 2.6. Let $p(t) \equiv p_1(t) \equiv t^\alpha$, $q(t) \equiv ((1-\alpha)^2/4)t^{\alpha-2}$ and $q_1(t) \equiv q(t)(1 - \sin(t-a))$, $t \geq a > 0$. Here, if $\alpha \leq 0$, then with the choice $\gamma = 1 - \alpha/2$, we have

$$\int_a^t s^{2\gamma} q_1(s)\, ds = \frac{(1-\alpha)^2}{4}(t - a + \cos(t-a) - 1) \leq \frac{(1-\alpha)^2}{4}(t-a), \qquad t \geq a \tag{2.16}$$

so that (2.12) holds ($\eta = 1$). Furthermore, (2.11) holds if $4 \geq (1-\alpha)^2$, i.e. $\alpha \geq -1$. Therefore, Corollary 2.6 implies that (1.2) is disconjugate on $[a, +\infty)$, $a > 0$, for $-1 \leq \alpha \leq 0$, $p_1(t)$, $q_1(t)$ as above. This may be extended to all $\alpha > 0$ (on perhaps a smaller subinterval $[a_1, \infty)$, $a_1 > a$) if one notices that (2.10) is equivalent (again with $\eta = 1$, $\gamma = 1 - \alpha/2$)

$$\frac{a}{2} + \tfrac{1}{4}(2\alpha - \alpha^2)(t-a) + \frac{(1-\alpha)^2}{4}((t-a) + \cos(t-a) - 1) \geq 0, \qquad t \geq a \tag{2.17}$$

which, in turn is equivalent to

$$2a + (t-a) + (1-\alpha)^2(\cos(t-a) - 1) \geq 0, \qquad t \geq a. \tag{2.18}$$

Therefore, given any $\alpha > 0$, (2.21) holds for large enough $a > 0$ so that Corollary 2.6 again implies that (1.2) is disconjugate on $[a, +\infty)$.

EXAMPLE 2.9. To illustrate Corollary 2.7 for a specific class, let $p(t) \equiv t^\alpha$, $q(t) \equiv ((1-\alpha)^2/4)t^{\alpha-2}$ and let $q_1(t) \equiv k_1 t^{\alpha-2} + k_2 t^\beta \sin t$, $t \geq a > 0$. In this case, with $\delta = 1 - \alpha - 2\gamma > 0$, and with $\beta \leq \alpha - 1$ we have

$$\delta t^\delta \int_t^\infty s^{2\gamma} q_1(s)\, ds = k_1 + k_2 \delta (\cos t) t^{2\gamma+\beta+\delta} + 0(t^{2\gamma+\beta+\delta-1}), \qquad t \to \infty.$$

Therefore, if $\beta < \alpha - 1$, then $2\gamma + \beta + \delta = \beta + 1 - \alpha < 0$ and so (2.15) holds (eventually) for arbitrary k_2 and any k_1 with $\gamma(\gamma+\delta) < k_1 < (1-\alpha)^2/4$. If $\beta = \alpha - 1$, then (2.15) holds eventually in case $\gamma(\gamma+\delta) < k_1 \pm k_2\delta < (1-\alpha)^2/4$. Thus, to summarize, with $p_1(t) \geq t^\alpha$ and $q_1(t) = k_1 t^{\alpha-2} + k_2 t^\beta \sin t$, equation (1.2) is eventually disconjugate in case:

(i) $\gamma<(1-\alpha)/2$, $\beta<\alpha-1$, $-[\gamma^2+\gamma(\alpha-1)]<k_1<(1-\alpha)^2/4$ and k_2 arbitrary; or in case:

(ii) $\gamma<(1-\alpha)/2$, $\beta=\alpha-1$, and k_1, k_2 satisfy

$$-[\gamma^2+\gamma(\alpha-1)]<k_1\pm k_2(1-\alpha-2\gamma)<\frac{(1-\alpha)^2}{4}.$$

3. In this section we wish to apply the previous results to the fourth order equations

$$(py'')''-qy=0 \tag{1.5}$$

and

$$(p_1y'')''-q_1y=0 \tag{1.6}$$

where p, q, p_1, $q_1\in C[a,\infty)$. Furthermore, we shall assume throughout that

$$0<p(t)\le p_1(t),\qquad q(t)>0,\qquad q_1(t)>0,\qquad t\in[a,+\infty), \tag{3.1}$$

and

$$\int_a^\infty p_1^{-1}(s)\,ds=+\infty. \tag{3.2}$$

It is known (cf. [8]) that (1.5) is disconjugate on an interval I (i.e. no nontrivial solution of (1.5) has more than three zeros on I) iff (1.5) is (2, 2) disconjugate on I (i.e. no nontrivial solution of (1.5) has a pair of consecutive double zeros). This is, of course, a consequence of the positivity assumptions on the coefficients. Moreover, Elias [2, Lemmas 3 and 6] has shown that if $\int^\infty p^{-1}(s)\,ds=+\infty$, $p(t)>0$, $q(t)>0$, then (1.5) is (2, 2) disconjugate on $[a,+\infty)$ iff there exists a solution y such that

$$y>0,\qquad y'>0,\qquad y''>0,\qquad (py'')'<0,\qquad (py'')''>0,\qquad t\ge a. \tag{3.3}$$

It is this characterization of disconjugacy along with the following Lemma which we shall need to establish our comparison results for (1.5) and (1.6). For completeness, we include the proof. (See also [3]).

LEMMA 3.1. *Assume* $p(t)>0$, $q(t)>0$ *and* $\int^\infty p^{-1}(s)\,ds=+\infty$. *Then equation* (1.5) *is disconjugate on* $[a,+\infty)$ *iff there exists a positive function* $\sigma\in C^1[a,+\infty)$ *such that both of the equations*

$$\begin{cases}(pu')'+\sigma u=0\\(\sigma v')'+qv=0\end{cases} \tag{3.4}$$

are disconjugate on $[a,+\infty)$.

Proof. Theorem 6.3 of [8] is the sufficiency part. That is, if there exists $\sigma\in C^1[a,+\infty)$, $\sigma>0$ such that (3.4) is disconjugate, then (1.5) is disconjugate.

On the other hand, if (1.5) is disconjugate, then there exists a solution y satisfying (3.3) so that letting

$$\sigma \equiv \frac{-(py')'}{y'} > 0, \qquad t \geq a \tag{3.5}$$

we see that $u = y'$ and $v = y$ are positive solutions of (3.4) and hence both equations are disconjugate on $[a, +\infty)$.

We may now prove the following comparison theorems for (1.5) and (1.6)

THEOREM 3.2. *Let* (3.1), (3.2) *hold, assume equation* (1.5) *is disconjugate on* $[a, +\infty)$ *with* y *a solution of* (1.5) *satisfying* (3.3) *and let* σ *be defined by* (3.5). *Assume further that there exists* $\mu \in C^1[a, +\infty)$ *such that* $\mu > 0$, $\mu'\sigma \in C^1[a, +\infty)$ *and*

$$\mu y' \leq \mu' y, \qquad t \geq a \tag{3.6}$$

and

$$\frac{\mu(a)\sigma(a)}{y(a)}(\mu'(a)y(a) - \mu(a)y'(a)) + \int_a^t \mu(\mu'\sigma)'\,ds + \int_a^t \mu^2 q_1\,ds \geq 0, \qquad t \geq a. \tag{3.7}$$

Finally, assume

$$\int_a^t \mu^2 q_1\,ds \leq \int_a^t \mu^2 q\,ds, \qquad t \geq a. \tag{3.8}$$

Then equation (1.6) *is disconjugate on* $[a, +\infty)$.

Proof. The proof follows immediately from Theorem 2.1, the previous Lemma, and the Sturm comparison Theorem. That is, since $p_1 \geq p > 0$ and with σ given by (3.5), it follows from the Sturm comparison theorem [11] that

$$(p_1 u')' + \sigma u = 0 \tag{3.9}$$

is also disconjugate on $[a, +\infty)$. We now apply Theorem 2.1 to the pair of equations

$$(\sigma v')' + qv = 0 \tag{3.10}$$

and

$$(\sigma v')' + q_1 v = 0. \tag{3.11}$$

With $r = \sigma v'/v = \sigma y'/y$, which is a solution of the Riccati equation corresponding to (3.10), it follows that condition (2.1) (with $p = \sigma$) is equivalent to condition (3.6). Similarly, condition (2.2) is equivalent to (3.7) and condition

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(H_1) is (3.8). Therefore, equation (3.11) is disconjugate on $[a, +\infty)$ so that by Lemma 3.1, it follows that equation (1.6) is disconjugate on $[a, +\infty)$.

THEOREM 3.3. *Let all hypotheses of the previous theorem hold with condition* (3.7) *replaced by*

$$2\mu(\mu'\sigma)'+2\sigma^{-1}(\mu'\sigma-\mu r)^2+(q_1+q)\mu^2\geq 0, \qquad t\geq a. \tag{3.12}$$

Then equation (1.6) *is disconjugate on* $[a, +\infty)$.

Proof. We use Theorem 2.2 instead of 2.1, noting that condition (2.3) is equivalent to (3.12). The proof is the same as Theorem 3.2.

Similarly, applying Theorem 2.5 we obtain

THEOREM 3.4. *Let* (3.1), (3.2) *hold, assume equation* (1.5) *is disconjugate on* $[a, +\infty)$ *with* y *a solution of* (1.5) *satisfying* (3.3), *and let* σ *be defined by* (3.5). *Assume further that* μ, $\mu'\sigma\in C^1[a, +\infty)$, $\mu>0$ *such that*

$$\mu y'\geq \mu' y, \qquad t\geq a \tag{3.13}$$

and

$$L+\int_t^\infty \mu(\mu'\sigma)'\,ds+\int_t^\infty \mu^2 q_1\,ds\geq 0, \qquad t\geq a \tag{3.14}$$

where

$$L\equiv\lim_{t\to\infty}\frac{\mu(t)\sigma(t)}{y(t)}(\mu(t)y'(t)-\mu'(t)y(t))\geq 0$$

is assumed to exist, along with the integrals appearing in (3.14). *Finally, assume*

$$\int_t^\infty \mu^2 q_1\,ds\leq\int_t^\infty \mu^2 q\,ds. \tag{3.15}$$

Then equation (1.6) *is disconjugate on* $[a, +\infty)$.

Proof. The proof proceeds as in Theorem 3.2. One need only verify that with $r=\sigma y'/y$, conditions (2.6) and (2.7) are equivalent to (3.13) and (3.14), respectively, and (2.8) is (3.15).

In the same manner as in Section 2, one can illustrate the theorems of this section by choosing the critical case of the fourth order Euler equation

$$(t^{\alpha+2}y'')''-\frac{(1-\alpha^2)^2}{16}t^{\alpha-2}y=0, \qquad t\geq a>0. \tag{3.16}$$

The solution $y(t)=t^{(1-\alpha)/2}$ satisfies condition (3.3) since $\int^\infty t^{-\alpha-2}\,dt=+\infty$ requires $\alpha<-1$ (i.e., condition (3.2)). A calculation yields

$$\sigma=\frac{(1+\alpha)^2}{4}t^\alpha, \qquad r=\frac{\sigma y'}{y}=\frac{(1-\alpha^2)(1+\alpha)}{8}t^{\alpha-1}.$$

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We may now state the following Corollaries. The proofs follow directly from Theorems 3.2, 3.3, 3.4, respectively, with $\mu(t)\equiv t^{\gamma}$ in each case.

COROLLARY 3.5. *If* $p_1(t)\geq t^{\alpha+2}$, $q_1(t)>0$, $t\geq a>0$, *and there exists* $\gamma>(1-\alpha)/2$ *such that*

$$\frac{\eta(1+\alpha)^2}{8}a^{\eta}+\frac{\gamma(1+\alpha)^2(\eta-\gamma)}{4\eta}(t^{\eta}-a^{\eta})+\int_a^t s^{2\gamma}q_1(s)\,ds\geq 0, \qquad t\geq a \tag{3.17}$$

and

$$\int_a^t s^{2\gamma}q_1(s)\,ds\leq\frac{(1-\alpha^2)^2}{16}(t^{\eta}-a^{\eta}), \qquad t\geqslant a, \tag{3.18}$$

where $\eta=2\gamma+\alpha-1>0$ *and* $\alpha<-1$, *then equation* (1.6) *is disconjugate on* $[a,+\infty)$.

Replacing the integral condition (3.17) by the pointwise condition (3.12) of Theorem 3.3 one obtains

COROLLARY 3.6. *Let all hypotheses of Corollary* 3.5 *hold with* (3.17) *replaced by*

$$\frac{\gamma(1+\alpha)^2}{2}(\eta-\gamma)+\frac{\eta^2(1+\alpha)^2}{8}+\frac{(1-\alpha^2)^2}{16}+t^{2-\alpha}q_1(t)\geq 0, \qquad t\geq a. \tag{3.19}$$

Then equation (1.6) *is disconjugate on* $[a,+\infty)$.

Finally, we state the following Corollary which follows from Theorem 3.4.

COROLLARY 3.7. *If* $p_1(t)\geq t^{\alpha+2}$, $q_1(t)>0$, $t\geq a>0$ *and there exists* $\gamma<(1-\alpha)/2$ *such that*

$$\frac{(1+\alpha)^2}{4}\gamma(\gamma+\delta)\leq\delta t^{\delta}\int_t^{\infty}s^{2\gamma}q_1(s)\,ds\leq\frac{(1-\alpha^2)^2}{16}, \qquad t\geq a. \tag{3.20}$$

where $\delta=1-\alpha-2\gamma>0$, *then equation* (1.6) *is disconjugate on* $[a,+\infty)$. (*Here one need only note that* $L=0$ *in* (3.14) *since* $\gamma>(1-\alpha)/2$ *and that* (3.14), (3.15) *are equivalent to* (3.20)).

REMARK 3.10. It is well known (cf. [8]) that if $0<p\leq p_1$, $0<q_1\leq q$, $t\geq a$, then disconjugacy of (1.5) implies disconjugacy of (1.6). Other pointwise comparison results have been given in [6] and [8] and the reader may find a discussion of these in [11]. Hille–Wintner type comparison theorems for (1.5) and (1.6) were also obtained by the author in [3].

4. The proofs of the results of section 2 will now be given.

Proof of Theorem 2.1. In the Riccati equation (1.3) make the change of variables $W \equiv \mu' p - \mu r$ so that

$$\begin{aligned}\mu W' &= \mu(\mu' p)' - \mu(\mu r)' = \mu(\mu' p)' - \mu^2 r' - \mu\mu' r\\ &= \mu(\mu' p)' - \mu\mu' r + \mu^2(q + p^{-1} r^2)\\ &= \mu(\mu' p)' + \mu^2 q - \mu\mu'\left(\frac{\mu' p - W}{\mu}\right) + \mu^2 p^{-1}\left(\frac{\mu' p - W}{\mu}\right)^2\\ &= \mu(\mu' p)' + \mu^2 q + p^{-1} W^2 - \mu' W,\end{aligned}$$

after some simplification. Therefore, integrating by parts and rearranging we obtain

$$\mu(t)W(t) = \mu(a)W(a) + \int_a^t \mu(\mu' p)'\,dt + \int_a^t p^{-1} W^2\,dt + \int_a^t \mu^2 q\,dt. \tag{4.1}$$

Now define the sequence of successive approximations $\{V_n\}_{n=0}^{\infty}$ by

$$\begin{aligned}\mu(t)V_0(t) &= \mu(a)W(a) + \int_a^t \mu(\mu' p)'\,dt + \int_a^t \mu^2 q_1\,dt, \qquad t \ge a\\ \mu(t)V_{n+1}(t) &= \mu(a)W(a) + \int_a^t \mu(\mu' p)'\,dt + \int_a^t p^{-1} V_n^2\,dt + \int_a^t \mu^2 q_1\,dt,\\ &\qquad n = 0, 1, 2, \ldots, t \ge a.\end{aligned} \tag{4.2}$$

Condition (2.2) implies that $V_0(t) \ge 0$, $t \ge a$ and since $\mu(t)(V_{n+2}(t) - V_{n+1}(t)) = \int_a^t p^{-1}(V_{n+1}^2 - V_n^2)\,dt \ge 0$ we see by induction that $V_{n+2}(t) \ge V_{n+1}(t) \ge 0$ for all $n \ge 0$. Furthermore, by induction we can also show that $V_n(t) \le W(t)$ for all $n \ge 0$ and $t \ge a$. Thus, $\mu(t)(W(t) - V_0(t)) = \int_a^t p^{-1} W^2\,dt + \int_a^t (q - q_1)\mu^2\,dt \ge 0$ and the assumption that $W(t) \ge V_n(t)$ implies that $\mu(t)(W(t) - V_{n+1}(t)) = \int_a^t p^{-1}(W^2 - V_n^2)\,dt + \int_a^t (q - q_1)\mu^2\,dt \ge 0$. Therefore by the Monotone convergence theorem and Dini's theorem, $\{V_n\}_{n=0}^{\infty}$ converges, uniformly on compact subintervals, to a solution $V = V(t)$ of the equation

$$\mu(t)V(t) = \mu(a)W(a) + \int_a^t \mu(\mu' p)'\,dt + \int_a^t p^{-1} V^2\,dt + \int_a^t \mu^2 q_1\,dt, \qquad t \ge a. \tag{4.3}$$

Defining $z(t) \equiv (\mu' p - V)/\mu$, $t \ge a$, it follows that z is a solution of $z' + p^{-1} z^2 + q_1 = 0$ on I. But since $p_1 \ge p > 0$, it follows (by the Sturm comparison theorem) that (1.4) also has a solution on I.

Proof of Theorem 2.2. Let the hypotheses of Theorem 2.2 hold with, however, the additional assumption that the inequality in (2.3) is strict and that

$$\mu(a)r(a) < p(a)\mu'(a). \tag{4.4}$$

We assume also that $p_1 \equiv p$ since the result for $p_1 \ge p$ follows by the Sturm theorem. An approximation and convergence argument will give Theorem 2.2

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in its full generality. In (1.3) and (1.4) make the change of variables $U(t)=\mu(t)r(t)$ and $V(t)=\mu(t)z(t)$, respectively, to get

$$\mu U'=\mu' U-p^{-1}U^2-q\mu^2 \tag{4.5}$$

and

$$\mu V'=\mu' V-p^{-1}V^2-q_1\mu^2. \tag{4.6}$$

Let $U(a)<V(a)<\mu'(a)p(a)$ and let V solve (4.6) on a maximal right interval of existence $[a, t_0)$. We claim that $t_0=b$ and that on $[a, b)$ we have

$$U(t)\le V(t)\le 2\mu'(t)p(t)-U(t). \tag{4.7}$$

Suppose first that $V(t)>2\mu'(t)p(t)-U(t)$ for some $t\in[a, t_0)$. Then there exists $t_1>a$ such that $V(t_1)=2\mu'(t_1)p(t_1)-U(t_1)$ and $V'(t_1)\ge 2(\mu'p)'(t_1)-U'(t_1)$. Also from equation (4.6) we have

$$\begin{cases} \mu(t_1)V'(t_1)=\mu'(t_1)V(t_1)-p^{-1}(t_1)V^2(t_1)-q_1(t_1)\mu^2(t_1) \\ \qquad =-p^{-1}(t_1)\left(V(t_1)-\dfrac{p(t_1)\mu'(t_1)}{2}\right)^2+\dfrac{p(t_1)(\mu'(t_1))^2}{4}-q_1(t_1(\mu^2(t_1) \\ \qquad =-p^{-1}(t_1)(\tfrac{3}{2}\mu'(t_1)p(t_1)-U(t_1))^2+\dfrac{p(t_1)(\mu'(t_1))^2}{4}-q_1(t_1)\mu^2(t_1) \end{cases} \tag{4.8}$$

Therefore, from equation (4.5) and (4.8) we get

$$\begin{aligned} \mu(t_1)(V'(t_1)+U'(t_1)) &= -p^{-1}(t_1)[(\tfrac{3}{2}\mu'(t_1)p(t_1)-U(t_1))^2+(U(t_1)-\tfrac{1}{2}\mu'(t_1)p(t_1))^2] \\ &\quad +\frac{p(t_1)(\mu'(t_1))^2}{2}-\mu^2(t_1)(q_1(t_1)+q(t_1)) \\ &= -p^{-1}(t_1)[2U^2(t_1)-4\mu'(t_1)p(t_1)U(t_1)+\tfrac{5}{2}(\mu'(t_1)p(t_1))^2] \\ &\quad +\frac{p(t_1)(\mu'(t_1)^2}{2}-\mu^2(t_1)(q_1(t_1)+q(t_1)) \\ &= -p^{-1}(t_1)[2U^2(t_1)-4\mu'(t_1)p(t_1)U(t_1)] \\ &\quad -2(\mu'(t_1))^2p(t_1)-\mu^2(t_1)(q_1(t_1)+q(t_1)). \end{aligned} \tag{4.9}$$

From (2.3) (with the strict inequality) we have

$$\begin{aligned} 2\mu(t_1)(\mu'p)'(t_1) &> -\mu^2(t_1)(q_1(t_1)+q(t_1))-2p^{-1}(t_1)[U(t_1)-\mu'(t_1)p(t_1)]^2 \\ &= -\mu^2(t_1)(q_1(t_1)+q(t_1))-2p^{-1}(t_1) \\ &\quad \times[U^2(t_1)-2\mu'(t_1)p(t_1)U(t_1)]-2(\mu'(t_1))^2p(t_1). \end{aligned} \tag{4.10}$$

Therefore, from (4.9) and (4.10) we conclude that

$$V'(t_1)+U'(t_1)<2(\mu'p)'(t_1), \tag{4.11}$$

which is contradictory to our assumption that $V'(t_1)\ge 2(\mu'p)'(t_1)-U'(t_1)$. It

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follows that $V(t) \le 2\mu'(t)p(t) - U(t)$ on $[a, t_0)$. We show next that $V(t) \ge U(t)$ on $[a, t_0)$. Now on $[a, t_0)$, an integration of (4.5), (4.6) gives

$$\mu(t)(V(t)-U(t)) = \mu(a)(V(a)-U(a)) + \int_a^t \mu^2(q-q_1)\,dt + \int_a^t p^{-1}[(2V\mu'p - V^2) - (2U\mu'p - U^2)]\,dt. \tag{4.12}$$

If there exists $t_2 \in (a, t_0)$ such that $U(t_2) = V(t_2)$ and $U(t) < V(t) < 2\mu'(t)p(t) - U(t)$ on $[a, t_2)$, then

$$\begin{cases} 0 < (V(t)-U(t))[2\mu'(t)p(t) - (U(t)+V(t))] \\ \quad = 2\mu'(t)p(t)(V(t)-U(t)) + U^2(t) - V^2(t) \\ \quad = (2\mu'(t)p(t)V(t) - V^2(t)) - (2\mu'(t)p(t)U(t) - U^2(t)). \end{cases} \tag{4.13}$$

Hence, in (4.12) with $t = t_2$, the left side is zero and from condition (H_1) and (4.13) the right side is positive. This condition shows that $V(t) \ge U(t)$ on $[a, t_0)$. Therefore, since (4.7) holds on $[a, t_0)$, it follows that $t_0 = b$ (since the only way that a solution of (4.6) can fail to be continuable is if $\lim_{t\to t_0} |V(t)| = +\infty$). This proves the theorem with the added assumption (4.4) and with strict inequality in condition 2.3.

Next we suppose that (2.1), (2.3) hold as given. Let $V(t)$ be a solution of (4.6) on a right maximal interval of existence $[a, t_0)$ with

$$U(a) = \mu(a)r(a) \le V(a) \le \mu'(a)p(a). \tag{4.14}$$

We claim that $V(t)$ satisfies

$$U(t) \le V(t) \le 2\mu'(t)p(t) - U(t) \tag{4.15}$$

on $[a, b)$. If not, let $c \in [a, b)$ such that (4.15) holds on $[a, c)$ and let c be the largest such number. For each $n = 1, 2, \ldots$ let $U_n(t)$ be the solution of (4.5) satisfying $U_n(a) = U(a) - 1/n$ and let $V_n(t)$ be the solution of (4.6) satisfying $V_n(a) = U(a) - 1/2n$. If $\delta > 0$ is such that $c + \delta < t_0$, then we may choose a subsequence of $\{U_n\}$, $\{V_n\}$, which we again relabel $\{U_n\}$, $\{V_n\}$, such that $\lim_{n\to\infty} U_n(t) = U(t)$ and $\lim_{n\to\infty} V_n(t) = V(t)$, uniformly on $[a, c+\delta]$. Now since $r_n(t) = U_n(t)/\mu(t)$ is a solution of (1.3) and since

$$r_n(a) = \frac{U_n(a)}{\mu(a)} < \frac{U(a)}{\mu(t)} = r(a),$$

it follows that the difference $\sigma(t) = r(t) - r_n(t)$ satisfies $\sigma' + p^{-1}(r + r_n)\sigma = 0$ and hence $\sigma(t) = \sigma(a)\exp(\int_a^t p^{-1}(r + r_n)\,dt) > 0$ on $[a, c+\delta]$. Thus, since $0 \le \mu'(t)p(t) - U(t) < \mu'(t)p(t) - U_n(t)$ on $[a, c+\delta]$, it follows using (2.3) that

$$\begin{aligned} 0 &\le \mu^2(t)(q_1(t)+q(t)) + 2\mu(t)(\mu'(t)p(t))' + 2p^{-1}(t)[\mu'(t)p(t) - U(t)]^2 \\ &< \mu^2(t)(q_1(t)+q(t)) + 2\mu(t)(\mu'(t)p(t))' + 2p^{-1}(t)[\mu'(t)p(t) - U_n(t)]^2. \end{aligned}$$

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Therefore, by the first part of the Theorem it follows that for each $n \geq 1$ we have $U_n(t) \leq V_n(t) \leq 2\mu'(t)p(t) - U_n(t)$ on $[a, c+\delta]$, so that $U(t) \leq V(t) 2\mu'(t)p(t) = U(t)$ on $[a, c+\delta]$, contradicting the choice of c. Therefore, $U(t) \leq V(t) \leq 2\mu'(t)p(t) - U(t)$ on $[a, t_0)$ and we must have $t_0 = b$. This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. Let $\hat{W}(t) = \mu(t)r(t) - \mu'(t)p(t)$ so that from the Riccati equation (1.3) we obtain for $a < t \leq b$, (after an integration by parts, as in the proof of Theorem 2.3)

$$\mu(t)\hat{W}(t) = \mu(b)\hat{W}(b) + \int_t^b \mu(\mu' p)'\, dt + \int_t^b p^{-1}\hat{W}^2\, dt + \int_t^b \mu^2 q\, dt. \tag{4.16}$$

In much the same manner as in Theorem 2.1, we define the sequence of successive approximations $\{\hat{V}_n\}_{n=0}^{\infty}$ by

$$\begin{cases} \mu(t)\hat{V}_0(t) = \mu(b)\hat{W}(b) + \int_t^b \mu(\mu' p)'\, dt + \int_t^b \mu^2 q_1\, dt, & a < t \leq b \\ \mu(t)\hat{V}_{n+1}(t) = \mu(t)\hat{V}_0(t) + \int_t^b p^{-1}\hat{V}_n^2\, dt, & a < t \leq b,\ n = 0, 1, 2, \ldots \end{cases} \tag{4.17}$$

Condition (2.5) implies that $\hat{V}_0(t) \geq 0$ on $(a, b]$ and an induction argument shows that

$$0 \leq \hat{V}_n(t) \leq \hat{V}_{n+1}(t) \leq \hat{W}(t), \qquad a < t \leq b,$$

for all $n \geq 1$. Therefore, the sequence $\{\hat{V}_n\}$ converges monotonically and uniformly (on compacta) to a solution $\hat{V}(t)$ of

$$\mu(t)\hat{V}(t) = \mu(b)\hat{W}(b) + \int_t^b \mu(\mu' p)'\, dt + \int_t^b p^{-1}\hat{V}^2\, dt + \int_t^b \mu^2 q_1\, dt \tag{4.18}$$

so that $z(t) \equiv [\hat{V}(t) + \mu'(t)p(t)]/\mu(t)$ is a solution of (1.4) on $(a, b]$.

Proof of Theorem 2.4. As in the proof of Theorem 2.2, we assume the hypotheses of Theorem 2.4 hold with, however, the additional assumption that strict inequality holds in (2.4) at $t = b$ and throughout I in (2.3). Thus,

$$\mu(b)r(b) > p(b)\mu'(b) \tag{4.19}$$

and

$$2\mu(t)(\mu'(t)p(t))' + 2p^{-1}(t)[\mu'(t)p(t) - \mu(t)r(t)]^2 + \mu^2(t)(q_1(t) + q(t)) > 0 \tag{4.20}$$

are assumed to hold. We also assume $p \equiv p_1$, the more general result $p \leq p_1$ follows as in Theorem 2.2. With the same change of variables as in Theorem 2.2, $U(t) = \mu(t)r(t)$, $V(t) = \mu(t)z(t)$, we obtain equations (4.5) and (4.6) for $U(t)$ and $V(t)$, respectively. Let $U(b) > V(b) > \mu'(b)p(b)$ and let V solve (4.6) on a left maximal interval of existence $(t_0, b]$. We claim that $t_0 = a$ and that

$$U(t) \geq V(t) \geq 2\mu'(t)p(t) - U(t) \text{ on } (a, b]. \tag{4.21}$$

If not, suppose there exists $t \in (t_0, b]$ with $V(t) < 2\mu'(t)p(t) - U(t)$ and choose $t_1 \in (t_0, b)$ such that $V(t_1) = 2\mu'(t_1)p(t_1) - U(t_1)$ and $V'(t_1) \geq 2(\mu'p)'(t_1) - U'(t_1)$. From (4.6) we get as in Theorem 2.2

$$\mu(t_1)V'(t_1) = -p^{-1}(t_1)(\tfrac{3}{2}\mu'(t_1)p(t_1) - U(t_1))^2 + \frac{p(t_1)(\mu'(t_1))^2}{4} - q_1(t_1)\mu^2(t_1) \tag{4.22}$$

so from equation (4.5) and (4.22) we get

(4.23)

$$\begin{aligned}\mu(t_1)(V'(t_1)+U'(t_1)) &= -p^{-1}(t_1)[(\tfrac{3}{2}\mu'(t_1)p(t_1)-U(t_1))^2+(U(t_1)-\tfrac{1}{2}\mu'(t_1)p(t_1))^2]\\ &\quad+\frac{p(t_1)(\mu'(t_1))^2}{2}-\mu^2(t_1)(q_1(t_1)+q(t_1))\\ &= -p^{-1}(t_1)(2U^2(t_1)-4\mu'(t_1)p(t_1)U(t_1))-2(\mu'(t_1))^2p(t_1)\\ &\quad-\mu^2(t_1)(q_1(t_1)+q(t_1)).\end{aligned}$$

From (4.20) and (4.23) we now conclude that

$$(V'(t_1) + U'(t_1)) < 2(\mu'p)'(t_1), \tag{4.24}$$

which is a contradiction. Hence, $V(t) \geq 2\mu'(t)p(t) - U(t)$ on $(t_0, b]$. Next we claim that $V(t) \leq U(t)$ on $(t_0, b]$. Now on $(t_0, b]$, an integration of (4.5), (4.6) gives

$$\begin{aligned}\mu(b)(V(b)-U(b)) &= \mu(t)(V(t)-U(t)) + \int_t^b \mu^2(q-q_1)\,dt\\ &\quad+\int_t^b p^{-1}[(2V\mu'p - V^2)-(2U\mu'p-U^2)]\,dt.\end{aligned} \tag{4.25}$$

If there exists $t_0 < t_2 < b$ such that $V(t_2) = U(t_2)$ and $2\mu'(t)p(t) - U(t) < V(t) < U(t)$ on $(t_0, b]$, then on $(t_0, b]$

$$\begin{aligned}0 &< (V(t)-U(t))[2\mu'(t)p(t)-(U(t)+V(t))]\\ &= (2\mu'(t)p(t)V(t)-V^2(t))-(2\mu'(t)p(t)U(t)-U^2(t))\end{aligned}$$

so that in (4.25) with $t = t_2$ the left side is negative and the right side is positive. This contradiction shows that $V(t) \leq U(t)$ on $(t_0, b]$ and it follows that $t_0 = a$. Now a convergence and approximation argument analogous to that in the latter portion of the proof of Theorem 2.2 shows that the Theorem holds without the additional assumptions (4.19) and (4.20). This completes the proof of Theorem 2.4.

Proof of Theorem 2.5. The proof is essentially the same as the proof of Theorem 2.3. With $\hat{W}(t)=\mu(t)r(t)-\mu'(t)p(t)$ we see that $\hat{W}(t)$ satisfies the equation

$$\mu(t)\hat{W}(t)=L+\int_t^\infty \mu(\mu'p)'\,dt+\int_t^\infty p^{-1}\hat{W}^2\,dt+\int_t^\infty \mu^2 q\,dt, \qquad a<t<\infty \tag{4.26}$$

Therefore, as in the proof of Theorem 2.3, we define the sequence of successive approximations $\{\hat{V}_n\}_{n=0}^\infty$ by

$$\begin{cases} \mu(t)\hat{V}_0(t)=L+\int_t^\infty \mu(\mu'p)'\,dt+\int_t^\infty \mu^2 q\,dt, & a<t<\infty \\ \mu(t)\hat{V}_{n+1}(t)=\mu(t)\hat{V}_0(t)+\int_t^\infty p^{-1}\hat{V}_n^2\,dt, & n=0,1,2\ldots, \quad a<t<\infty. \end{cases} \tag{4.27}$$

Condition (2.7) implies that $\hat{V}_0(t)\geq 0$ on (a,∞) and by induction one shows that $0\leq \hat{V}_n(t)\leq \hat{V}_{n+1}(t)\leq \hat{W}(t)$ in (a,∞) so that $\{\hat{V}_n\}$ converges monotonically and uniformly on compact subintervals to a solution $\hat{V}(t)$ by

$$\mu(t)\hat{V}(t)=L+\int_t^\infty \mu(\mu'p)'\,dt+\int_t^\infty p^{-1}\hat{V}^2\,dt+\int_t^\infty \mu^2 q_1\,dt, \qquad a<t<\infty \tag{4.28}$$

so that $z(t)\equiv(\hat{V}(t)+\mu'(t)p(t))/\mu(t)$ is a solution of (1.4) on (a,∞). This completes the proof of Theorem 2.5.

The proofs of Corollaries 2.6 and 2.7 have essentially been given already in the comments preceding them.

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