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ON THE TRANSCENDENCE OF CERTAIN REAL NUMBERS

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Abstract

In this article, we prove the transcendence of certain infinite sums and products by applying the subspace theorem. In particular, we extend the results of Hančl and Rucki [‘The transcendence of certain infinite series’, *Rocky Mountain J. Math.* **35** (2005), 531–537].

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1. Introduction

There are several methods to prove the transcendence of an infinite series. Using Mahler’s method [8], one can prove the transcendence of certain infinite sums and products. In 2001, Adhikari *et al.* [2] proved the transcendence of certain infinite series by an application of Baker’s theory of linear forms in logarithms of algebraic numbers. In the same year, Hančl [4] and Nyblom [9] (see also [10]) studied the transcendence of infinite series by invoking Roth’s theorem. In 2004, using the subspace theorem, Adamczewski *et al.* [1] proved a transcendence criterion for a real number based on its b-ary expansion.

In 1974, Erdős and Straus [3] studied the linear independence of certain Cantor series expansions. In particular, they proved the following result.

Theorem 1.1 [3]. *Let $Q = (b_n)_{n\geq 1}$ be a sequence of positive integers with $b_n \geq 2$ for all integers $n \geq 1$ and let $\delta > \frac{1}{3}$ be any positive real number. Suppose that for all sufficiently large values of N,*

$$(b_1 b_2 \cdots b_N)^{\delta} \leq b_{N+1}.$$

Then the real numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{\sigma(n)}{b_1 b_2 \cdots b_n}, \quad \sum_{n=1}^{\infty} \frac{\phi(n)}{b_1 b_2 \cdots b_n}, \quad \sum_{n=1}^{\infty} \frac{d_n}{b_1 b_2 \cdots b_n}$$

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are $\mathbb{Q}$-linearly independent. Here $\sigma(n) = \sum_{d|n} d$, $\phi(n)$ denotes the Euler totient function and $(d_n)_n$ is any sequence of integers such that $|d_n| < n^{(1/2)-\delta}$ for all large n and $d_n \neq 0$ for infinitely many n.

Since $b_n > n^{(1/2)+\delta}$ for all large n, Theorem 1.1 follows from [3, Theorem 3.7]. We prove the following extension of Theorem 1.1.

Theorem 1.2. *Let $Q = (b_n)_{n\geq 1}$ be a sequence of positive integers with $b_n \geq 2$ for all integers $n \geq 1$ and let $\delta > \frac{1}{3}$ be any positive real number. Suppose that for all sufficiently large values of N,*

$$\sigma(N+1)(b_1 b_2 \cdots b_N)^{\delta} \leq b_{N+1}. \tag{1.1}$$

Then at least one of the real numbers

$$\beta_1 = \sum_{n=1}^{\infty} \frac{\sigma(n)}{b_1 b_2 \cdots b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{\phi(n)}{b_1 b_2 \cdots b_n}, \quad \beta_3 = \sum_{n=1}^{\infty} \frac{d_n}{b_1 b_2 \cdots b_n}$$

is transcendental.

In 2005, Hančl and Rucki [7] gave sufficient conditions under which an infinite sum is transcendental. We mention one of their results here.

Theorem 1.3 [7]. *Let $\delta > 0$ be a real number. Let $(b_n)_n$ and $(c_n)_n$ be sequences of positive integers such that*

$$\limsup_{n\to\infty} \frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{2+\delta}} \frac{1}{c_{n+1}} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \frac{c_n}{c_{n+1}} > 1.$$

Then the real number $\alpha = \sum_{n=1}^{\infty} c_n/b_n$ is transcendental.

We extend the results in [7] and study the transcendence of certain infinite products.

In order to state the main results, we first fix some notation. Let $\delta > 0$ and $\epsilon > 0$ be given real numbers. For any given integer $m \geq 2$, let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$, be a collection of sequences of nonzero integers. Consider the following two conditions on a sequence $(b_n)_n$ of positive integers:

$$\limsup_{n\to\infty} \frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{1+\delta}} \frac{1}{c_{i,n+1}} = \infty, \tag{1.2}$$

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \frac{c_{i,n}}{c_{i,n+1}} > 1, \tag{1.3}$$

holding in both cases for all $i \in \{1, 2, \ldots, m\}$. We may now state our results.

Theorem 1.4. *For any given integer $m \geq 2$, let $\delta > 1/m$ be a real number. Let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$, and $(b_n)_n$ be sequences of positive integers satisfying (1.2) and (1.3). Then either at least one of the real numbers*

$$\beta_1 = \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{c_{2,n}}{b_n}, \ldots, \beta_m = \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

is transcendental or $1, \beta_1, \beta_2, \ldots, \beta_m$ are $\mathbb{Q}$-linearly dependent.

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The following corollary shows that Theorem 1.4 extends Theorem 1.3.

Corollary 1.5. *Let $\delta > \frac{1}{2}$ be a real number and let $(b_n)_n$ be a sequence of positive integers such that $b_1 = 2$ and*

$$b_{n+1} = (b_1 b_2 \cdots b_n + 1)^2 \quad \textit{for all integers } n \geq 1.$$

Then the real numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{b_n} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{d(n)}{b_n}$$

are $\mathbb{Q}$-linearly independent. Here $d(n) = \sum_{d|n} 1$.

First, note that if $\frac{1}{2} < \delta < 1$, then $(b_1 b_2 \cdots b_n)^{1+\delta} < b_{n+1}$ in the statement of Corollary 1.5 and $(b_1 b_2 \cdots b_n)^{2+\delta} > b_{n+1}$ for any choice of $\delta > 0$ and for all sufficiently large values of n. Therefore, we cannot conclude the transcendence of either of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{b_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{d(n)}{b_n}$$

from Theorem 1.3. On the other hand, by taking $c_{1,n} = 1$ and $c_{2,n} = d(n)$ in Theorem 1.4, we see that at least one of the real numbers

$$\sum_{n=1}^{\infty} \frac{1}{b_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{d(n)}{b_n}$$

is transcendental.

The conclusion of Theorem 1.4 can be strengthened to show that at least one of the β_i's is transcendental under additional assumptions on the growth rate of the sequences $(c_{i,n})_n$ and $(b_n)_n$. More precisely, we have the following theorem.

Theorem 1.6. *For any given integer $m \geq 2$, let $\delta > 1/m$ be a real number. Let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$, and $(b_n)_n$ be sequences of positive integers satisfying (1.2) and (1.3). Further, suppose that*

$$1 \leq \liminf_{n\to\infty} b_n^{1/(m+1)^n} < \limsup_{n\to\infty} b_n^{1/(m+1)^n} < \infty,$$
$$\lim_{n\to\infty} c_{i,n}/c_{j,n} = 0 \quad \textit{for all } i, j \in \{1, 2, \ldots, m\} \textit{ with } i > j.$$

Then at least one of the real numbers

$$\beta_1 = \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{c_{2,n}}{b_n}, \ldots, \beta_m = \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

is transcendental.

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Using the same notation as in (1.2) and (1.3), we consider two more conditions on the sequence of positive integers (b_n) and the collection $(c_{i,n})_n$, $i = 1, 2, \dots, m$, of sequences of nonzero integers:

$$\limsup_{n\to\infty} \frac{b_{n+1}}{(b_1 b_2 \dots b_n)^{1+\delta+1/\epsilon}} \frac{1}{c_{i,n+1}} = \infty, \tag{1.4}$$

$$\sqrt[1+\epsilon]{\frac{b_{n+1}}{c_{i,n+1}}} \geq \sqrt[1+\epsilon]{\frac{b_n}{c_{i,n}}} + 1, \tag{1.5}$$

holding in both cases for all $i \in \{1, 2, \dots, m\}$.

Theorem 1.7. *For any given integer $m \geq 2$, let δ and ϵ be positive real numbers such that $\delta\epsilon/(1+\epsilon) > 1/m$. Let $(c_{i,n})_n$, $i = 1, 2, \dots, m$, and $(b_n)_n$ be sequences of positive integers satisfying (1.4) and (1.5). Then at least one of the real numbers*

$$\beta_1 = \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{c_{2,n}}{b_n}, \dots, \beta_m = \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

is transcendental or $1, \beta_1, \beta_2, \dots, \beta_m$ are $\mathbb{Q}$-linearly dependent.

The conclusion of Theorem 1.7 can be strengthened to show that at least one of the β_i's is transcendental under some additional assumptions on the growth of the sequences $(c_{i,n})_n$ and $(b_n)_n$. More precisely, we have the following theorem.

Theorem 1.8. *For any given integer $m \geq 2$, let δ and ϵ be positive real numbers such that $\delta\epsilon/(1+\epsilon) > 1/m$. Let $(c_{i,n})_n$, $i = 1, 2, \dots, m$, and $(b_n)_n$ be sequences of positive integers satisfying (1.4) and (1.5). Further, suppose that*

$$1 \leq \liminf_{n\to\infty} b_n^{1/(m+1)^n} < \limsup_{n\to\infty} b_n^{1/(m+1)^n} < \infty,$$

$$\lim_{n\to\infty} c_{i,n}/c_{j,n} = 0 \quad \text{for all } i, j \in \{1, 2, \dots, m\} \text{ with } i > j.$$

Then at least one of the real numbers

$$\beta_1 = \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{c_{2,n}}{b_n}, \dots, \beta_m = \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

is transcendental.

Finally, we give the following result for infinite products.

Theorem 1.9. *For any given integer $m \geq 2$, let $\delta > 1/m$ be a real number. Let $(c_{i,n})_n$, $i = 1, 2, \dots, m$, and $(b_n)_n$ be sequences of positive integers satisfying all the hypotheses of Theorem 1.6. Suppose that $c_{i,n} \leq b_n$ for all $n \geq 1$ and $i = 1, 2, \dots, m$. Then at least one of the real numbers*

$$\beta_1 = \prod_{n=1}^{\infty} \left(1 + \frac{c_{1,n}}{b_n}\right), \quad \beta_2 = \prod_{n=1}^{\infty} \left(1 + \frac{c_{2,n}}{b_n}\right), \dots, \beta_m = \prod_{n=1}^{\infty} \left(1 + \frac{c_{m,n}}{b_n}\right)$$

is transcendental.

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2. Preliminaries

The following theorem is a well-known corollary of the subspace theorem (see for instance [11, page 176]).

THEOREM 2.1 [11]. *For any given integer $m \geq 2$, let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be real numbers. Let $\delta > 0$ be a real number such that $\delta > 1/m$. Suppose that there exist infinitely many $(m+1)$-tuples $(p_{n1}, p_{n2}, \ldots, p_{nm}, q_n)$ of integers satisfying $q_n \neq 0$ and*

$$\left|\alpha_i - \frac{p_{in}}{q_n}\right| < \frac{1}{q_n^{1+\delta}} \quad \text{for } 1 \leq i \leq m.$$

Then either the real numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly dependent or at least one of the α_i is transcendental.

The following result of Hančl [5] will also be useful.

THEOREM 2.2 [5]. *For a given integer $m \geq 2$, let $(b_n)_n$ be a sequence of positive integers such that*

$$1 \leq \liminf_{n\to\infty} b_n^{1/(m+1)^n} < \limsup_{n\to\infty} b_n^{1/(m+1)^n} < \infty \quad \text{and} \quad b_n \geq n^{1+\epsilon}$$

for all large n and for some $\epsilon > 0$. Let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$, be a collection of sequences of positive integers such that, for $1 \leq i < j \leq m$,

$$\lim_{n\to\infty} \frac{c_{i,n}}{c_{j,n}} = 0,$$

$$c_{i,n} < 2^{(\log b_n)^\alpha} \quad \text{for some fixed } \alpha > 0 \text{ and for all large enough } n.$$

Then the real numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \ldots, \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

are $\mathbb{Q}$-linearly independent.

Hančl *et al.* [6] proved the following theorem for infinite products.

THEOREM 2.3 [6]. *Let (b_n) be a sequence as in Theorem 2.2. For any given integer $m \geq 2$, let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$, be a collection of sequences of positive integers such that, for $1 \leq i < j \leq m$,*

$$\lim_{n\to\infty} \frac{c_{i,n}}{c_{j,n}} = 0,$$

$$c_{i,n} < b_n^{1/\log^{1+\epsilon}\log b_n} \quad \text{for all large enough } n.$$

Then the real numbers

$$1, \quad \prod_{n=1}^{\infty}\left(1 + \frac{c_{1,n}}{b_n}\right), \ldots, \prod_{n=1}^{\infty}\left(1 + \frac{c_{m,n}}{b_n}\right)$$

are $\mathbb{Q}$-linearly independent.

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3. Proofs of the theorems

Proof of Theorem 1.2. We define sequences $(\beta_{1,N})_N$, $(\beta_{2,N})_N$ and $(\beta_{3,N})_N$ of rational numbers as follows. For each integer $N \geq 1$ and for $i = 1, 2$ and 3,

$$\beta_{i,N} = \sum_{n=1}^{N} \frac{f_i(n)}{b_1 b_2 \ldots b_n} = \frac{p_{i,N}}{b_1 b_2 \ldots b_N},$$

where the $p_{i,N}$ are positive integers and $f_1(n) = \sigma(n)$, $f_2(n) = \phi(n)$, $f_3(n) = d_n$. By (1.1) and using the fact that $\sigma(N+1) > d_{N+1}$ and $\sigma(N+1) > \phi(N+1)$,

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{1}{(b_1 b_2 \cdots b_N)^{1+\delta'}},$$

for all sufficiently large N and for some $\delta' > \frac{1}{3}$.

Put $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, $q_N = b_1 b_2 \cdots b_N$ and $p_{iN} = p_{i,N}$ for $1 \leq i \leq 3$ in Theorem 2.1 with N sufficiently large. Then, either $1, \beta_1$, β_2 and β_3 are $\mathbb{Q}$-linearly dependent or at least one of them is transcendental. By Theorem 1.1, we know that $1, \beta_1, \beta_2$ and β_3 are $\mathbb{Q}$-linearly independent. Therefore, we conclude that one of β_1, β_2 and β_3 is transcendental. This proves the assertion. □

Proof of Theorem 1.4. For each integer i with $1 \leq i \leq m$, we define the sequence $(\beta_{i,N})_N$ of rational numbers by

$$\beta_{i,N} = \sum_{n=1}^{N} \frac{c_{i,n}}{b_n} = \frac{p_{i,N}}{b_1 b_2 \cdots b_N}, \quad N \geq 1,$$

where the $p_{i,N}$ are positive integers. By (1.3), there exists a real number $A > 1$ and a positive constant N_0 such that, for all positive integers $N > N_0$,

$$\frac{1}{A} \cdot \frac{c_{i,N}}{b_N} > \frac{c_{i,N+1}}{b_{N+1}}.$$

Therefore, inductively, for every N with $N > N_0$,

$$\frac{1}{A^p} \cdot \frac{c_{i,N}}{b_N} > \frac{c_{i,N+p}}{b_{N+p}}$$

for any natural number p. Hence, for all sufficiently large positive integers N,

$$\begin{aligned}\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| &= \left|\sum_{n=1}^{\infty} \frac{c_{i,n}}{b_n} - \sum_{n=1}^{N} \frac{c_{i,n}}{b_n}\right| = \left|\sum_{n=N+1}^{\infty} \frac{c_{i,n}}{b_n}\right| \\ &= \frac{c_{i,N+1}}{b_{N+1}} + \frac{c_{i,N+2}}{b_{N+2}} + \cdots \\ &< \frac{c_{i,N+1}}{b_{N+1}}\left(1 + \frac{1}{A} + \frac{1}{A^2} + \cdots\right) = \frac{c_{i,N+1}}{b_{N+1}} \frac{A}{A-1}.\end{aligned}$$

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Choose $M > A/(A-1)$. Then, by (1.2), there exist infinitely many integers N such that

$$\frac{1}{M(b_1 b_2 \cdots b_N)^{1+\delta}} > \frac{c_{i,N+1}}{b_{N+1}}.$$

Hence,

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{c_{i,N+1}}{b_{N+1}} \frac{A}{A-1} \le \frac{1}{(b_1 b_2 \cdots b_N)^{1+\delta}}$$

for infinitely many positive integers N.

By taking $\alpha_i = \beta_i$ and $p_{in} = p_{i,n}$ for $1 \le i \le m$ in Theorem 2.1, we see that either $1, \beta_1, \beta_2, \ldots, \beta_m$ are $\mathbb{Q}$-linearly dependent or at least one β_i is transcendental. □

Proof of Theorem 1.6. By Theorem 1.4, either $1, \beta_1, \beta_2, \ldots, \beta_m$ are $\mathbb{Q}$-linearly dependent or at least one β_i is transcendental. Since the sequences $(c_{i,n})_n$ and $(b_n)_n$ satisfy the hypotheses of Theorem 2.2, $1, \beta_1, \beta_2, \ldots, \beta_m$ are $\mathbb{Q}$-linearly independent. Therefore, we conclude that at least one β_i is transcendental. This proves the theorem. □

Proof of Theorem 1.7. For each integer i with $1 \le i \le m$, we define the sequence $(\beta_{i,N})_N$ of rational numbers by

$$\beta_{i,N} = \sum_{n=1}^{N} \frac{c_{i,n}}{b_n} = \frac{p_{i,N}}{b_1 b_2 \ldots b_N} \quad \text{for } N \ge 1,$$

where the $p_{i,N}$ are positive integers. By (1.5) and mathematical induction, for all sufficiently large integers N and every integer r,

$$\sqrt[1+\epsilon]{\frac{b_{N+r}}{c_{i,N+r}}} \ge \sqrt[1+\epsilon]{\frac{b_N}{c_{i,N}}} + r.$$

Hence

$$\frac{b_{N+r}}{c_{i,N+r}} \ge \left(\sqrt[1+\epsilon]{\frac{b_N}{c_{i,N}}} + r\right)^{1+\epsilon}. \tag{3.1}$$

Now, for all real $x > 1$,

$$\sum_{s=0}^{\infty} \frac{1}{(x+s)^{1+\epsilon}} < \int_{x-1}^{\infty} \frac{dy}{y^{1+\epsilon}} = \frac{1}{\epsilon(x-1)^{\epsilon}}. \tag{3.2}$$

By (3.1) and (3.2), for infinitely many N,

$$\begin{aligned}\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| &= \left|\sum_{n=1}^{\infty} \frac{c_{i,n}}{b_n} - \sum_{n=1}^{N} \frac{c_{i,n}}{b_n}\right| = \left|\sum_{n=N+1}^{\infty} \frac{c_{i,n}}{b_n}\right| = \frac{c_{i,N+1}}{b_{N+1}} + \frac{c_{i,N+2}}{b_{N+2}} + \cdots \\ &\le \left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}}\right)^{-(1+\epsilon)} + \left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}} + 1\right)^{-(1+\epsilon)} + \cdots \\ &< \frac{1}{\epsilon}\left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}} - 1\right)^{-\epsilon}.\end{aligned}$$

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Since $\lim_{n\to\infty}(b_n/c_{i,n}) = \infty$ by (1.4), there exists a positive constant C which does not depend on N such that

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{1}{\epsilon}\left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}} - 1\right)^{-\epsilon} < \frac{C}{\epsilon}\left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}}\right)^{-\epsilon} = \frac{C}{\epsilon}\left(\frac{c_{i,N+1}}{b_{N+1}}\right)^{\epsilon/(1+\epsilon)}.$$

Choose $M > C/\epsilon$. Then by (1.4), there are infinitely many integers N such that

$$\frac{1}{M(b_1 b_2 \cdots b_N)^{1+\delta+1/\epsilon}} > \frac{c_{i,N+1}}{b_{N+1}}.$$

This implies that

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{1}{(b_1 b_2 \cdots b_n)^{1+\delta\epsilon/(1+\epsilon)}}$$

for infinitely many positive integers N. The rest of the proof is the same as the proof of Theorem 1.6. □

Proof of Theorem 1.8. The proof follows the same lines as that of Theorem 1.6. □

Proof of Theorem 1.9. For each integer i with $1 \le i \le m$, we define the sequence $(\beta_{i,N})_N$ of rational numbers by

$$\beta_{i,N} = \prod_{n=1}^{N}\left(1 + \frac{c_{i,n}}{b_n}\right) = \frac{p_{i,N}}{b_1 b_2 \ldots b_N} \quad \text{for } N \ge 1,$$

where the $p_{i,N}$ are positive integers. Consider

$$\begin{aligned}\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| &= \prod_{n=1}^{\infty}\left(1 + \frac{c_{i,n}}{b_n}\right) - \prod_{n=1}^{N}\left(1 + \frac{c_{i,n}}{b_n}\right) \\ &= \prod_{n=1}^{N}\left(1 + \frac{c_{i,n}}{b_n}\right)\left(\prod_{n=N+1}^{\infty}\left(1 + \frac{c_{i,n}}{b_n}\right) - 1\right). \end{aligned} \tag{3.3}$$

By the hypothesis, for all sufficiently large values of N,

$$\prod_{n=N+1}^{\infty}\left(1 + \frac{c_{i,n}}{b_n}\right) < 1 + 2\sum_{n=N+1}^{\infty}\frac{c_{i,n}}{b_n}.$$

Thus, by (3.3),

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \ldots b_N}\right| < 2\prod_{n=1}^{N}\left(1 + \frac{c_{i,n}}{b_n}\right)\left(\sum_{n=N+1}^{\infty}\frac{c_{i,n}}{b_n}\right).$$

By a similar argument to that in the proof of Theorem 1.6, from (1.3), we conclude that for all sufficiently large positive integers N,

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| \le 2\prod_{n=1}^{N}\left(1 + \frac{c_{i,n}}{b_n}\right)\left(\sum_{n=N+1}^{\infty}\frac{c_{i,n}}{b_n}\right) < 2\prod_{n=1}^{N}\left(1 + \frac{c_{i,n}}{b_n}\right)\frac{c_{i,N+1}}{b_{N+1}}\frac{A}{A-1}.$$

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Hence, by (1.2),

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \prod_{n=1}^{N}\left(1 + \frac{c_{i,n}}{b_n}\right)\frac{1}{(b_1 b_2 \cdots b_N)^{1+\delta}}, \tag{3.4}$$

for infinitely many values of N. By the hypothesis of the theorem, $c_{i,n}/b_n \le 1$ for $n \ge 1$, so

$$\prod_{n=1}^{N}\left(1 + \frac{c_{i,n}}{b_n}\right) < 2^N$$

for all integers $N \ge 1$. Therefore, by (3.4),

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{2^N}{(b_1 b_2 \ldots b_N)^{1+\delta}}.$$

Since the sequence $(b_n)_n$ grows like a doubly exponential sequence, we can find δ' with $1/m < \delta' < \delta$ such that

$$\frac{2^N}{(b_1 b_2 \cdots b_N)^{1+\delta}} < \frac{1}{(b_1 b_2 \cdots b_N)^{1+\delta'}}.$$

Therefore, for $1 \le i \le m$,

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{1}{(b_1 b_2 \cdots b_N)^{1+\delta'}}$$

for infinitely many values of N. The rest of the proof follows as for the proofs of Theorems 1.4 and 1.6. □

Proof of Corollary 1.5. Suppose that these numbers are $\mathbb{Q}$-linearly dependent. Then, there exist integers z_0, z_1 and z_2 not all zero such that

$$z_0 + z_1 \sum_{n=1}^{\infty} \frac{1}{b_n} + z_2 \sum_{n=1}^{\infty} \frac{d(n)}{b_n} = 0.$$

This is equivalent to

$$z_0 + z_1 \sum_{n=1}^{N} \frac{1}{b_n} + z_2 \sum_{n=1}^{N} \frac{d(n)}{b_n} = -\left(z_1 \sum_{n=N+1}^{\infty} \frac{1}{b_n} + z_2 \sum_{n=N+1}^{\infty} \frac{d(n)}{b_n}\right).$$

By multiplying by $b_1 b_2 \ldots b_N$ on both sides,

$$b_1 b_2 \cdots b_N\left(z_0 + z_1 \sum_{n=1}^{N} \frac{1}{b_n} + z_2 \sum_{n=1}^{N} \frac{d(n)}{b_n}\right) = -b_1 \cdots b_N\left(z_1 \sum_{n=N+1}^{\infty} \frac{1}{b_n} + z_2 \sum_{n=N+1}^{\infty} \frac{d(n)}{b_n}\right). \tag{3.5}$$

Note that the left-hand side of this equation is an integer.

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Claim. The quantity

$$\left|-b_1b_2\cdots b_N\Big(z_1\sum_{n=N+1}^{\infty}\frac{1}{b_n}+z_2\sum_{n=N+1}^{\infty}\frac{d(n)}{b_n}\Big)\right|\to 0 \quad \text{as } N\to\infty.$$

To prove the claim, observe first that $d(n)=O(n)$ and so

$$\begin{aligned}\left|-b_1b_2\cdots b_N\Big(z_2\sum_{n=N+1}^{\infty}\frac{d(n)}{b_n}\Big)\right| &\le |z_2|\Big(\frac{d(N+1)}{b_{N+1}}+\frac{d(N+2)}{b_{N+2}}+\cdots\Big)\\ &< \frac{1}{b_1b_2\cdots b_N}\Big(\frac{N+1}{b_1b_2\cdots b_N}+\frac{N+2}{(b_1b_2\ldots b_N)^2+\cdots}\Big)\\ &< \frac{C}{b_1b_2\cdots b_N}.\end{aligned}$$

Hence,

$$\left|-b_1b_2\cdots b_N\Big(z_2\sum_{n=N+1}^{\infty}\frac{d(n)}{b_n}\Big)\right|\to 0 \quad \text{as } N\to\infty. \tag{3.6}$$

Similarly,

$$\left|-b_1b_2\cdots b_N\Big(z_1\sum_{n=N+1}^{\infty}\frac{1}{b_n}\Big)\right|\to 0 \quad \text{as } N\to\infty. \tag{3.7}$$

The claim therefore follows from (3.6) and (3.7).

Since the left-hand side of (3.5) is an integer, it follows that

$$Q_N := z_0+z_1\sum_{n=1}^{N}\frac{1}{b_n}+z_2\sum_{n=1}^{N}\frac{d(n)}{b_n}=0$$

for all sufficiently large values of N. Now, for all sufficiently large values of N, $Q_N=Q_{N-1}=0$ and so

$$Q_N-Q_{N-1}=\frac{z_1+z_2d(N)}{b_N}=0 \iff \frac{1}{d(N)}=-\frac{z_2}{z_1}$$

for all sufficiently large values of N. This implies that the sequence $(d(n))_n$ is eventually constant, which contradicts the fact that it has at least two limit points. □

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