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ON THE SPEED OF CONVERGENCE OF DISCRETE PICKANDS CONSTANTS TO CONTINUOUS ONES

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Abstract

In this manuscript, we address open questions raised by Dieker and Yakir (2014), who proposed a novel method of estimating (discrete) Pickands constants $\mathcal{H}_\alpha^\delta$ using a family of estimators $\xi_\alpha^\delta(T)$, $T > 0$, where $\alpha \in (0, 2]$ is the Hurst parameter, and $\delta \geq 0$ is the step size of the regular discretization grid. We derive an upper bound for the discretization error $\mathcal{H}_\alpha^0 - \mathcal{H}_\alpha^\delta$, whose rate of convergence agrees with Conjecture 1 of Dieker and Yakir (2014) in the case $\alpha \in (0, 1]$ and agrees up to logarithmic terms for $\alpha \in (1, 2)$. Moreover, we show that all moments of $\xi_\alpha^\delta(T)$ are uniformly bounded and the bias of the estimator decays no slower than $\exp\{-\mathcal{C}T^\alpha\}$, as T becomes large.

Keywords: Fractional Brownian motion; Pickands constants; Monte Carlo simulation; discretization error

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1. Introduction

For any $\alpha \in (0, 2]$ let $\{B_\alpha(t), t \in \mathbb{R}\}$ be a fractional Brownian motion (fBm) with Hurst parameter $H = \alpha/2$; that is, $B_\alpha(t)$ is a centered Gaussian process with covariance function given by

$$\mathrm{cov}(B_\alpha(t), B_\alpha(s)) = \frac{|t|^\alpha + |s|^\alpha - |t-s|^\alpha}{2}, \qquad t, s \in \mathbb{R},\ \alpha \in (0, 2].$$

In this manuscript we consider the classical Pickands constant defined by

$$\mathcal{H}_\alpha := \lim_{S\to\infty} \frac{1}{S}\mathbb{E}\left\{\sup_{t\in[0,S]} e^{\sqrt{2}B_\alpha(t) - t^\alpha}\right\} \in (0, \infty), \qquad \alpha \in (0, 2]. \tag{1}$$

The constant $\mathcal{H}_\alpha$ was first defined by Pickands [31, 32] to describe the asymptotic behavior of the maximum of stationary Gaussian processes. Since then, Pickands constants have played an important role in the theory of Gaussian processes, appearing in various asymptotic results related to the supremum; see the monographs [33, 34]. In [22], it was recognized that the discrete Pickands constant can be interpreted as an *extremal index* of a Brown–Resnick process. This new realization motivated the generalization of Pickands constants beyond the realm of

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Gaussian processes. For further references the reader may consult [13, 14], which give an excellent account of the history of Pickands constants, their connection to the theory of max-stable processes, and the most recent advances in the theory.

Although it is omnipresent in the asymptotic theory of stochastic processes, to date, the value of $\mathcal{H}_\alpha$ is known only in two very special cases: $\alpha = 1$ and $\alpha = 2$. In these cases, the distribution of the supremum of process B_α is well known: B_1 is a standard Brownian motion, while B_2 is a straight line with random, normally distributed slope. When $\alpha \notin \{1, 2\}$, one may attempt to estimate the numerical value of $\mathcal{H}_\alpha$ from the definition (1) using Monte Carlo methods. However, there are several problems associated with this approach:

(i) Firstly, the Pickands constant $\mathcal{H}_\alpha$ in (1) is defined as a limit as $S \to \infty$, so one must approximate it by choosing some (large) S. This results in a bias in the estimation, which we call the *truncation error*. The truncation error has been shown to decay faster than S^{-p} for any $p < 1$; see [12, Corollary 3.1].

(ii) Secondly, for every $\alpha \in (0, 2)$, the variance of the truncated estimator blows up as $S \to \infty$; that is,

$$\lim_{S\to\infty} \operatorname{var}\left\{\frac{1}{S}\sup_{t\in[0,S]} \exp\{\sqrt{2}B_\alpha(t) - t^\alpha\}\right\} = \infty.$$

This can easily be seen by considering the second moment of $\frac{1}{S}\exp\{\sqrt{2}B_\alpha(S) - S^\alpha\}$. This directly affects the *sampling error* (standard deviation) of the crude Monte Carlo estimator. As $S \to \infty$, one needs more and more samples to prevent its variance from blowing up.

(iii) Finally, there are no methods available for the exact simulation of $\sup_{t\in[0,S]} \exp\{\sqrt{2}B_\alpha(t) - t^\alpha\}$ for $\alpha \notin \{1, 2\}$. One must therefore resort to some method of approximation. Typically, one would simulate fBm on a regular δ-grid, i.e. on the set $\delta\mathbb{Z}$ for $\delta > 0$; cf. Equation (2) below. This approximation leads to a bias, which we call the *discretization error*.

In the following, for any fixed $\delta > 0$ we define the discrete Pickands constant

$$\mathcal{H}_\alpha^\delta := \lim_{S\to\infty} \frac{1}{S}\mathbb{E}\left\{\sup_{t\in[0,S]_\delta} e^{\sqrt{2}B_\alpha(t)-t^\alpha}\right\}, \quad \alpha \in (0, 2], \tag{2}$$

where, for $a, b \in \mathbb{R}$ and $\delta > 0$, $[a, b]_\delta = [a, b] \cap \delta\mathbb{Z}$. Additionally, we set $0\mathbb{Z} = \mathbb{R}$, so that $\mathcal{H}_\alpha^0 = \mathcal{H}_\alpha$. In light of the discussion in item (iii) above, the discretization error equals $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$. We should note that the quantity $\mathcal{H}_\alpha^\delta$ is well defined and $\mathcal{H}_\alpha^\delta \in (0, \infty)$ for $\delta \geq 0$. Moreover, $\mathcal{H}_\alpha^\delta \to \mathcal{H}_\alpha$ as $\delta \to 0$, which means that the discretization error diminishes as the size of the gap of the grid goes to 0. We refer to [13] for the proofs of these properties.

In recent years, [23] proposed a new representation of $\mathcal{H}_\alpha^\delta$, which does not involve the limit operation. They show [23, Proposition 3] that for all $\delta \geq 0$ and $\alpha \in (0, 2]$,

$$\mathcal{H}_\alpha^\delta = \mathbb{E}\left\{\xi_\alpha^\delta\right\}, \quad \text{where} \quad \xi_\alpha^\delta := \frac{\sup_{t\in\delta\mathbb{Z}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}}{\delta\sum_{t\in\delta\mathbb{Z}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}}. \tag{3}$$

For $\delta = 0$ the denominator in the fraction above is replaced by $\int_{\mathbb{R}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}\mathrm{d}t$. In fact, the denominator can be replaced by $\eta\sum_{t\in\eta\mathbb{Z}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}$ for any η, which is an integer multiple

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of δ; see [13, Theorem 2]. While one would ideally estimate $\mathcal{H}_\alpha$ using ξ_α^0, this is unfortunately infeasible since there are no exact simulation methods for ξ_α^δ (see also item (iii) above). For that reason, the authors define the 'truncated' version of the random variable ξ_α^δ, namely

$$\xi_\alpha^\delta(T) := \frac{\sup_{t\in[-T,T]_\delta} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}}{\delta \sum_{t\in[-T,T]_\delta} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}},$$

where for $\delta = 0$ the denominator of the fraction is replaced by $\int_{-T}^{T} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha} \mathrm{d}t$. For any $\delta, T \in (0, \infty)$, the estimator $\xi_\alpha^\delta(T)$ is a functional of a fractional Brownian motion on a finite grid, and as such it can be simulated exactly; see e.g. [24] for a survey of methods of simulation of fBm. A side effect of this approach is that the new estimator induces both the truncation and the discretization errors described in items (i) and (iii) above.

In this manuscript we rigorously show that the estimator $\xi_\alpha^\delta(T)$ is well suited for simulation. In Theorem 1, we address the conjecture stated by the inventors of the estimator ξ_α^δ about the asymptotic behavior of the discretization error between the continuous and discrete Pickands constant for a fixed $\alpha \in (0, 2]$:

[23, Conjecture 1] For all $\alpha \in (0, 2]$ it holds that $\lim\limits_{\delta\to 0} \dfrac{\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta}{\delta^{\alpha/2}} \in (0, \infty)$.

We establish that the conjecture is *true* when $\alpha = 1$ and is *not true* when $\alpha = 2$; see Corollary 1 below, where the exact asymptotics of the discretization error are derived in these two special cases.

Furthermore, in Theorem 1(i) we show that

$$\limsup_{\delta\to 0} \delta^{-\alpha/2}(\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta) \in \left[0, \frac{\mathcal{H}_\alpha\sqrt{\pi}}{(1-2^{-1-\alpha/2})\sqrt{4-2^\alpha}}\right]$$

for $\alpha \in (0, 1)$, and in Theorem 1(ii) we show that $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$ is upper-bounded by $\delta^{\alpha/2}$ up to logarithmic terms for $\alpha \in (1, 2)$ and all $\delta > 0$ small enough. These results support the claim of the conjecture for all $\alpha \in (0, 2)$.

Secondly, we consider the truncation and sampling errors induced by $\xi_\alpha^\delta(T)$. In Theorem 2 we derive a uniform upper bound for the tail of the probability distribution of ξ_α^δ which implies that all moments of ξ_α^δ exist and are uniformly bounded in $\delta \in [0, 1]$. In Theorem 3 we establish that for any $\alpha \in (0, 2)$ and $p \geq 1$, the difference $|\mathbb{E}(\xi_\alpha^\delta(T))^p - \mathbb{E}(\xi_\alpha^\delta)^p|$ decays no slower than $\exp\{-\mathcal{C}T^\alpha\}$, as $T \to \infty$, uniformly for all $\delta \in [0, 1]$. This implies that the truncation error of the Dieker–Yakir estimator decays no slower than $\exp\{-\mathcal{C}T^\alpha\}$, and combining this with Theorem 2, we have that $\xi_\alpha^\delta(T)$ has a uniformly bounded sampling error, i.e.

$$\sup_{(\delta,T)\in[0,1]\times[1,\infty)} \mathrm{var}\left\{\xi_\alpha^\delta(T)\right\} < \infty. \tag{4}$$

Although arguably the most celebrated, Pickands constants are not the only constants appearing in the asymptotic theory of Gaussian processes and related fields. Depending on the setting, other constants may appear, including Parisian Pickands constants [15, 16, 27], sojourn Pickands constants [18, 20], Piterbarg-type constants [3, 28, 33, 34], and generalized Pickands constants [11, 21]. As with the classical Pickands constants, the numerical values of these constants are typically known only in the case $\alpha \in \{1, 2\}$. To approximate them, one can try the discretization approach. We believe that, using techniques from the proof of Theorem 1(ii), one could derive upper bounds for the discretization error which are exact up to logarithmic terms; see, e.g., [6].

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The manuscript is organized as follows. In Section 2 we present our main results and discuss their extensions and relationship to other problems. The rigorous proofs are presented in Section 3, while some technical calculations are given in the appendix.

2. Main results

In the following, we give an upper bound for $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$ for all $\alpha \in (0, 1) \cup (1, 2)$ for small $\delta > 0$.

Theorem 1. *The following hold:*

(i) *For any $\alpha \in (0, 1)$ and $\varepsilon > 0$, for all $\delta > 0$ sufficiently small,*

$$\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \le \frac{\mathcal{H}_\alpha \sqrt{\pi}(1+\varepsilon)}{(1-2^{-1-\alpha/2})\sqrt{4-2^\alpha}} \cdot \delta^{\alpha/2}.$$

(ii) *For every $\alpha \in (1, 2)$ there exists $\mathcal{C} > 0$ such that for all $\delta > 0$ sufficiently small,*

$$\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \le \mathcal{C}\delta^{\alpha/2} |\log \delta|^{1/2}.$$

While for the proof of the case $\alpha \in (1, 2)$ we were able to use general results from the theory of Gaussian processes, in the case $\alpha \in (0, 1)$ we needed to come up with more precise tools in order to skip the $|\log \delta|^{1/2}$ part in the upper bound. Therefore, the proofs in these two cases are very different from each other. Unfortunately, the proof in case (i) cannot be extended to case (ii) because of the switch from positive to negative correlations between the increments of fBm; see also Remark 1.

In the following two results we establish an upper bound for the survival function of ξ_α^δ and for the truncation error discussed in item (i) in Section 1. These two results combined imply that the sampling error of $\xi_\alpha^\delta(T)$ is uniformly bounded in $(\delta, T) \in [0, 1] \times [1, \infty)$; cf. Equation (4).

Theorem 2. *For any $\alpha \in (0, 2)$, $\delta \in [0, 1]$, and $\varepsilon > 0$ for sufficiently large x, T, we have*

$$\max\left(\mathbb{P}\left\{\xi_\alpha^\delta(T) > x\right\}, \mathbb{P}\left\{\xi_\alpha^\delta > x\right\}\right) \le e^{-\frac{\log^2 x}{4+\varepsilon}}.$$

Moreover, there exist positive constants $\mathcal{C}_1, \mathcal{C}_2$ such that for all $x, T > 0$ and $\delta \ge 0$,

$$\max\left(\mathbb{P}\left\{\xi_\alpha^\delta(T) > x\right\}, \mathbb{P}\left\{\xi_\alpha^\delta > x\right\}\right) \le \mathcal{C}_1 e^{-\mathcal{C}_2 \log^2 x}.$$

Evidently, Theorem 2 implies that all moments of ξ_α^δ are finite and uniformly bounded in $\delta \in [0, 1]$ for any fixed $\alpha \in (0, 2)$.

Theorem 3. *For any $\alpha \in (0, 2)$ and $p > 0$ there exist postive constants $\mathcal{C}_1, \mathcal{C}_2$ such that*

$$\left|\mathbb{E}\left\{(\xi_\alpha^\delta(T))^p\right\} - \mathbb{E}\left\{(\xi_\alpha^\delta)^p\right\}\right| \le \mathcal{C}_1 e^{-\mathcal{C}_2 T^\alpha}$$

for all $(\delta, T) \in [0, 1] \times [1, \infty)$.

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2.1. Case $\alpha \in \{1, 2\}$.

In this scenario, the explicit formulas for $\mathcal{H}_1^\delta$ (see, e.g., [19, 29]) and $\mathcal{H}_2^\delta$ (see, e.g., [14, Equation (2.9)]) are known. They are summarized in the proposition below, with Φ being the cumulative distribution function of a standard Gaussian random variable.

Proposition 1. *It holds that*

(i) $\mathcal{H}_1 = 1$ *and* $\mathcal{H}_1^\delta = \left(\delta \exp\left\{2\sum_{k=1}^{\infty} \frac{\Phi(-\sqrt{\delta k/2})}{k}\right\}\right)^{-1}$ *for all* $\delta > 0$, *and*

(ii) $\mathcal{H}_2 = \frac{1}{\sqrt{\pi}}$, *and* $\mathcal{H}_2^\delta = \frac{2}{\delta}\left(\Phi(\delta/\sqrt{2}) - \frac{1}{2}\right)$ *for all* $\delta > 0$.

Relying on the above, we can provide the exact asymptotics of the discretization error, as $\delta \to 0$. In the following, ζ denotes the Euler–Riemann zeta function.

Corollary 1. *It holds that*

(i) $\lim_{\delta\to 0} \frac{\mathcal{H}_1 - \mathcal{H}_1^\delta}{\sqrt{\delta}} = -\frac{\zeta(1/2)}{\sqrt{\pi}}$, *and*

(ii) $\lim_{\delta\to 0} \frac{\mathcal{H}_2 - \mathcal{H}_2^\delta}{\delta^2} = \frac{1}{12\sqrt{\pi}}$.

2.2. Discussion

We believe that finding the exact asymptotics of the speed of the discretization error $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$ is closely related to the behavior of fBm around the time of its supremum. We motivate this by the following heuristic:

$$\begin{aligned}
\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta &= \mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha} - \sup_{t\in\delta\mathbb{Z}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}}{\delta\sum_{t\in\delta\mathbb{Z}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}}\right\} \\
&\approx \mathbb{E}\left\{\Delta(\delta)\cdot\frac{\sup_{t\in\delta\mathbb{Z}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}}{\delta\sum_{t\in\delta\mathbb{Z}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha}}\right\}, \\
&\approx \mathbb{E}\{\Delta(\delta)\}\cdot\mathcal{H}_\alpha^\delta,
\end{aligned}$$

where $\Delta(\delta)$ is the difference between the suprema on the continuous and discrete grids, i.e. $\Delta(\delta) := \sup_{t\in\mathbb{R}}\{\sqrt{2}B_\alpha(t) - |t|^\alpha\} - \sup_{t\in\delta\mathbb{Z}}\{\sqrt{2}B_\alpha(t) - |t|^\alpha\}$. The first approximation above is due to the mean value theorem, and the second approximation is based on the assumption that $\Delta(\delta)$ and ξ_α^δ are asymptotically independent as $\delta \to 0$. We believe that $\Delta(\delta) \sim \mathcal{C}\delta^{\alpha/2}$ by self-similarity, where $\mathcal{C} > 0$ is some constant, which would imply that $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \sim \mathcal{C}\mathcal{H}_\alpha\delta^{\alpha/2}$. This heuristic reasoning can be made rigorous in the case $\alpha = 1$, when $\sqrt{2}B_\alpha(t) - |t|^\alpha$ is a Lévy process (Brownian motion with drift). In this case, the asymptotic behavior of functionals such as $\mathbb{E}\{\Delta(\delta)\}$, as $\delta \to 0$, can be explained by the weak convergence of trajectories around the time of supremum to the so-called *Lévy process conditioned to be positive*; see [26] for more information on this topic. In fact, Corollary 1(i) can be proven using the tools developed in [5]. To the best of the authors' knowledge, there are no such results available for a general fBm.

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However, it is worth mentioning that recently [2] considered the related problem of penalizing fractional Brownian motion for being negative.

A problem related to the asymptotic behavior of $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$ was considered in [7, 8], who showed that $\mathbb{E}\sup_{t\in[0,1]} B_\alpha(t) - \mathbb{E}\sup_{t\in[0,1]_\delta} B_\alpha(t)$ decays like $\delta^{\alpha/2}$ up to logarithmic terms. We should emphasize that in Theorem 1, in the case $\alpha \in (0, 1)$, we were able to establish that the upper bound for the discretization error decays *exactly* like $\delta^{\alpha/2}$. In light of the discussion above, we believe that the result and the method of proof of Theorem 1(i) could be useful in further research related to the discretization error for fBm.

Monotonicity of Pickands constants. Based on the definition (2), it is clear that for any $\alpha \in (0, 2)$, the sequence $\{\mathcal{H}_\alpha^\delta, \mathcal{H}_\alpha^{2\delta}, \mathcal{H}_\alpha^{4\delta}, \ldots\}$ is decreasing for any fixed $\delta > 0$. It is therefore natural to speculate that $\delta \mapsto \mathcal{H}_\alpha^\delta$ is a decreasing function. The explicit formulas for $\mathcal{H}_1^\delta$ and $\mathcal{H}_2^\delta$ given in Proposition 1 allow us to give a positive answer to this question in these cases.

Corollary 2. *For all $\delta \geq 0$, $\mathcal{H}_1^\delta$ and $\mathcal{H}_2^\delta$ are strictly decreasing functions with respect to δ.*

3. Proofs

For $\alpha \in (0, 2)$ define

$$Z_\alpha(t) = \sqrt{2} B_\alpha(t) - |t|^\alpha, \quad t \in \mathbb{R}.$$

Assume that all of the random processes and variables we consider are defined on a complete general probability space Ω equipped with a probability measure $\mathbb{P}$. Let $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \ldots$ be some positive constants that may differ from line to line.

3.1. Proof of Theorem 1, case $\alpha \in (0, 1)$

The proof of Theorem 1 in the case $\alpha \in (0, 1)$ is based on the following three results, whose proofs are given later in this section. In what follows, η is independent of $\{Z_\alpha(t), t \in \mathbb{R}\}$ and follows a standard exponential distribution.

Lemma 1. *For all $\alpha \in (0, 2)$,*

$$\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^\delta = \delta^{-1} \mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha(t) < 0,\ \sup_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha\left(t - \tfrac{\delta}{2}\cdot \operatorname{sgn}(t)\right) + \eta < 0\right\}.$$

As a side note, we remark that the representation in Lemma 1 yields a straightforward lower bound

$$\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^\delta \geq \delta^{-1} \mathbb{P}\left\{\sup_{t\in(\delta/2)\mathbb{Z}\setminus\{0\}} Z_\alpha(t) + \eta < 0\right\}$$

for all $\alpha \in (0, 2)$, $\delta > 0$.

Lemma 2. *For all $\alpha \in (0, 1)$ and $\delta > 0$,*

$$\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^\delta \leq \delta^{-1} \mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha(t) + \eta < 0\right\}.$$

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Proposition 2. *For any $\alpha \in (0, 2)$ and $\varepsilon > 0$, it holds that*

$$\mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha(t) + \eta < 0\right\} \leq \frac{\mathcal{H}_\alpha\sqrt{\pi}}{\sqrt{4-2^\alpha}}(1+\varepsilon)\cdot\delta^{1+\alpha/2}$$

for all $\delta > 0$ small enough.

Proof of Theorem 1, $\alpha \in (0, 1)$. Using the fact that $\mathcal{H}_\alpha^\delta \to \mathcal{H}_\alpha$ as $\delta \downarrow 0$, we may represent the discretization error $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$ as a telescoping series; that is,

$$\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta = \sum_{k=0}^{\infty} \mathcal{H}_\alpha^{2^{-(k+1)}\delta} - \mathcal{H}_\alpha^{2^{-k}\delta}.$$

Combining Lemma 2 and Proposition 2, we find that, with $\mathcal{C}$ denoting the constant from Proposition 2,

$$\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \leq \mathcal{C}\sum_{k=0}^{\infty} 2^{-k(1+\alpha/2)}\cdot\delta^{\alpha/2} = \frac{\mathcal{H}_\alpha\sqrt{\pi}(1+\varepsilon)}{(1-2^{-1-\alpha/2})\sqrt{4-2^\alpha}}\cdot\delta^{\alpha/2}$$

for all δ small enough. This completes the proof. □

Remark 1. If the upper bound in Lemma 2 holds also for $\alpha \in (1, 2)$, then the upper bound in Theorem 1(i) holds for all $\alpha \in (0, 2)$.

The remainder of this section is devoted to proving Lemma 1, Lemma 2, and Proposition 2.

In what follows, for any $\alpha \in (0, 2)$, let $\{X_\alpha(t), t \in \mathbb{R}\}$ be a centered, stationary Gaussian process with $\text{var}\{X_\alpha(t)\} = 1$, whose covariance function satisfies

$$\text{cov}(X_\alpha(t), X_\alpha(0)) = 1 - |t|^\alpha + o(|t|^\alpha), \quad t \to 0. \tag{5}$$

Before we give the proof of Lemma 1, we introduce the following result.

Lemma 3. *The finite-dimensional distributions of $\{u(X_\alpha(u^{-2/\alpha}t) - u) \mid X_\alpha(0) > u, t \in \mathbb{R}\}$ converge weakly to the finite-dimensional distributions of $\{Z_\alpha(t) + \eta, t \in \mathbb{R}\}$, where η is a random variable independent of $\{Z_\alpha(t), t \in \mathbb{R}\}$ following a standard exponential distribution.*

The result in Lemma 3 is well known; see, e.g., [1, Lemma 2], where the convergence of finite-dimensional distributions is established on $t \in \mathbb{R}_+$. The extension to $t \in \mathbb{R}$ is straightforward.

Proof of Lemma 1. The following proof is very similar in flavor to the proof of [4, Lemma 3.1]. From [34, Lemma 9.2.2] and the classical definition of the Pickands constant it follows that for any $\alpha \in (0, 2)$ and $\delta \geq 0$,

$$\mathcal{H}_\alpha^\delta = \lim_{T\to\infty}\lim_{u\to\infty}\frac{\mathbb{P}\left\{\sup_{t\in[0,T]_\delta} X_\alpha(u^{-2/\alpha}t) > u\right\}}{T\Psi(u)},$$

where $\Psi(u)$ is the complementary CDF (tail) of the standard normal distribution and $\{X_\alpha, t \in \mathbb{R}\}$ is the process introduced above Equation (5). Therefore,

$$\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^\delta = \lim_{T\to\infty}\lim_{u\to\infty}\frac{\mathbb{P}\left\{\max_{t\in[0,T]_{\delta/2}} X_\alpha(u^{-2/\alpha}t) > u, \max_{t\in[0,T]_\delta} X_\alpha(u^{-2/\alpha}t) < u\right\}}{T\Psi(u)}.$$

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Now, notice that we can decompose the event in the numerator above into a sum of disjoint events:

$$\frac{1}{\Psi(u)}\mathbb{P}\left\{\max_{t\in[0,T]_{\delta/2}} X_\alpha(u^{-2/\alpha}t) > u,\ \max_{t\in[0,T]_\delta} X_\alpha(u^{-2/\alpha}t) < u\right\}$$

$$= \sum_{\tau\in[0,T]_{\delta/2}} \mathbb{P}\left\{\max_{t\in[0,T]_{\delta/2}} X_\alpha(u^{-2/\alpha}t) \le X_\alpha(u^{-2/\alpha}\tau),\ \max_{t\in[0,T]_\delta} X_\alpha(u^{-2/\alpha}t) \le u \mid X_\alpha(u^{-2/\alpha}\tau) > u\right\}.$$

Using the stationarity of the process X_α, the above is equal to

$$\sum_{\tau\in[0,T]_{\delta/2}} \mathbb{P}\left\{\max_{t\in[0,T]_{\delta/2}} X_\alpha(u^{-2/\alpha}(t-\tau)) \le X_\alpha(0),\ \max_{t\in[0,T]_\delta} X_\alpha(u^{-2/\alpha}(t-\tau)) \le u \mid X_\alpha(0) > u\right\}.$$

Applying Lemma 3 to each element of the sum above, we find that the sum converges to $\sum_{\tau\in[0,T]_{\delta/2}} U(\tau, T)$ as $u \to \infty$, where

$$U(\tau, T) := \mathbb{P}\left\{\max_{t\in[0,T]_{\delta/2}} Z_\alpha(t-\tau) \le 0,\ \max_{t\in\delta\mathbb{Z}\cap[0,T]} Z_\alpha(t-\tau) + \eta \le 0\right\}.$$

We have now established that $\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta} = \lim_{T\to\infty} \frac{1}{T} \sum_{\tau\in[0,T]_{\delta/2}} U(\tau, T)$. Clearly,

$$U(\tau, \infty) = U(0, \infty) = \mathbb{P}\left\{\max_{t\in(\delta/2)\mathbb{Z}\setminus\{0\}} Z_\alpha(t) < 0,\ \max_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha\left(t - \tfrac{\delta}{2}\cdot \operatorname{sgn}(t)\right) + \eta < 0\right\}.$$

We will now show that $\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta}$ is lower-bounded and upper-bounded by $\delta^{-1}U(0, \infty)$, which will complete the proof. For the lower bound note that

$$\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta} \ge \lim_{T\to\infty} \frac{1}{T} \sum_{\tau\in[0,T]_{\delta/2}} U(\tau, \infty),$$

where the limit is equal to $\delta^{-1}U(0, \infty)$, because the sum above has $[T(\delta/2)^{-1}]$ elements, of which half are equal to 0 and the other half are equal to $U(0, \infty)$. In order to show the upper bound, consider $\varepsilon > 0$. For any $\tau \in (\varepsilon T, (1-\varepsilon)T)_{\delta/2}$ we have

$$U(\tau, T) \le \overline{U}(T, \varepsilon) := \mathbb{P}\left\{\max_{t\in(-\varepsilon T,\varepsilon T)_{\delta/2}} Z_\alpha(t) \le 0,\ \max_{t\in(-\varepsilon T,\varepsilon T)_\delta} Z_\alpha(t) + \eta \le 0\right\}.$$

Furthermore, we have the following decomposition:

$$\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta} = \lim_{T\to\infty} \frac{1}{T}\left(\sum_{\tau\in(\delta/2)\mathbb{Z}\cap I_-} U(\tau, T) + \sum_{\tau\in(\delta/2)\mathbb{Z}\cap I_0} U(\tau, T) + \sum_{\tau\in(\delta/2)\mathbb{Z}\cap I_+} U(\tau, T)\right),$$

where $I_- := [0, \varepsilon T]$, $I_0 := (\varepsilon T, (1-\varepsilon)T)$, $I_+ := [(1-\varepsilon), T]$. The first and last sums can be bounded by their number of elements, $[\varepsilon T(\delta/2)^{-1}]$, because $U(\tau, T) \le 1$. The middle sum can

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be bounded by $\frac{1}{2} \cdot [(1-2\varepsilon)T(\delta/2)^{-1}]\overline{U}(T, \varepsilon)$, because half of its elements are equal to 0 and the other half can be upper-bounded by $\overline{U}(T, \varepsilon)$. Letting $T \to \infty$, we obtain

$$\mathcal{H}_\alpha^{\delta/2} - \mathcal{H}_\alpha^{\delta} \le 4\varepsilon\delta^{-1} + (1-2\varepsilon)\delta^{-1}U(0, \infty),$$

because $\overline{U}(T, \varepsilon) \to U(0, \infty)$ as $T \to \infty$. Finally, letting $\varepsilon \to 0$ yields the desired result. $\square$

Proof of Lemma 2. In light of Lemma 1, it suffices to show that

$$\mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha\left(t - \tfrac{\delta}{2}\cdot \operatorname{sgn}(t)\right) + \eta < 0\right\} \le \mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha(t) + \eta < 0\right\}.$$

The left-hand side of the above equals

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} \sqrt{2}B_\alpha\left(t - \tfrac{\delta}{2}\cdot \operatorname{sgn}(t)\right) - |t - \tfrac{\delta}{2}\cdot \operatorname{sgn}(t)|^\alpha + \eta < 0\right\} \\
&\quad = \mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} \frac{\sqrt{2}B_\alpha\left(t - \frac{\delta}{2}\cdot \operatorname{sgn}(t)\right)}{|t - \frac{\delta}{2}\cdot \operatorname{sgn}(t)|^{\alpha/2}} - |t - \tfrac{\delta}{2}\cdot \operatorname{sgn}(t)|^{\alpha/2} + \frac{\eta}{|t - \frac{\delta}{2}\cdot \operatorname{sgn}(t)|^{\alpha/2}} < 0\right\} \\
&\quad \le \mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} \frac{\sqrt{2}B_\alpha\left(t - \frac{\delta}{2}\cdot \operatorname{sgn}(t)\right)}{|t - \frac{\delta}{2}\cdot \operatorname{sgn}(t)|^{\alpha/2}} - |t|^{\alpha/2} + \frac{\eta}{|t|^{\alpha/2}} < 0\right\}.
\end{aligned}$$

Observe that for all $t, s \in \delta\mathbb{Z} \setminus \{0\}$ it holds that

$$\operatorname{cov}\left(\frac{\sqrt{2}B_\alpha\left(t - \frac{\delta}{2}\cdot \operatorname{sgn}(t)\right)}{|t - \frac{\delta}{2}\cdot \operatorname{sgn}(t)|^{\alpha/2}}, \frac{\sqrt{2}B_\alpha(s - \frac{\delta}{2}\cdot \operatorname{sgn}(s))}{|s - \frac{\delta}{2}\cdot \operatorname{sgn}(s)|^{\alpha/2}}\right) \le \operatorname{cov}\left(\frac{\sqrt{2}B_\alpha(t)}{|t|^{\alpha/2}}, \frac{\sqrt{2}B_\alpha(s)}{|s|^{\alpha/2}}\right); \tag{6}$$

the proof of this technical inequality is given in the appendix. Since in the case $t = s$ the covariances in Equation (6) are equal, we may apply the Slepian lemma [34, Lemma 2.1.1] and obtain

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} \frac{\sqrt{2}B_\alpha\left(t - \frac{\delta}{2}\cdot \operatorname{sgn}(t)\right)}{|t - \frac{\delta}{2}\cdot \operatorname{sgn}(t)|^{\alpha/2}} - |t|^{\alpha/2} + \frac{\eta}{|t|^{\alpha/2}} < 0\right\} \\
&\quad \le \mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} \frac{\sqrt{2}B_\alpha(t)}{|t|^{\alpha/2}} - |t|^{\alpha/2} + \frac{\eta}{|t|^{\alpha/2}} < 0\right\},
\end{aligned}$$

from which the claim follows. $\square$

We will now lay out the preliminaries necessary to prove Proposition 2. First, let us introduce some notation that will be used until the end of this section. For any $\delta > 0$, $\lambda > 0$ let

$$\begin{aligned}
&p(\delta) := \mathbb{P}\{A(\delta)\}, \quad \text{with} \quad A(\delta) := \{Z_\alpha(-\delta) < 0, Z_\alpha(\delta) < 0\}, \text{ and} \\
&q(\delta, \lambda) := \mathbb{P}\{A(\delta, \lambda)\}, \quad \text{with} \quad A(\delta, \lambda) := \{Z_\alpha(-\delta) + \lambda^{-1}\eta < 0, Z_\alpha(\delta) + \lambda^{-1}\eta < 0\}.
\end{aligned} \tag{7}$$

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For any $\delta > 0$ and $\lambda > 0$ we define the densities of the two-dimensional vectors $(Z_\alpha(-\delta), Z_\alpha(\delta))$ and $(Z_\alpha(-\delta) + \lambda^{-1}\eta, Z_\alpha(\delta) + \lambda^{-1}\eta)$ respectively, with $\mathbf{x} := \binom{x_1}{x_2} \in \mathbb{R}^2$, as follows:

$$\begin{aligned} f(\mathbf{x}; \delta) &:= \frac{\mathbb{P}\{Z_\alpha(-\delta) \in \mathrm{d}x_1, Z_\alpha(\delta) \in \mathrm{d}x_2\}}{\mathrm{d}x_1 \mathrm{d}x_2}, \\ g(\mathbf{x}; \delta, \lambda) &:= \frac{\mathbb{P}\left\{Z_\alpha(-\delta) + \lambda^{-1}\eta \in \mathrm{d}x_1, Z_\alpha(\delta) + \lambda^{-1}\eta \in \mathrm{d}x_2\right\}}{\mathrm{d}x_1 \mathrm{d}x_2}. \end{aligned} \tag{8}$$

We also define the densities of these random vectors conditioned to take negative values on both coordinates:

$$\begin{aligned} f^-(\mathbf{x}; \delta) &:= \frac{\mathbb{P}\{Z_\alpha(-\delta) \in \mathrm{d}x_1, Z_\alpha(\delta) \in \mathrm{d}x_2 \mid A(\delta)\}}{\mathrm{d}x_1 \mathrm{d}x_2}, \\ g^-(\mathbf{x}; \delta, \lambda) &:= \frac{\mathbb{P}\left\{Z_\alpha(-\delta) + \lambda^{-1}\eta \in \mathrm{d}x_1, Z_\alpha(\delta) + \lambda^{-1}\eta \in \mathrm{d}x_2 \mid A(\delta, \lambda)\right\}}{\mathrm{d}x_1 \mathrm{d}x_2}. \end{aligned} \tag{9}$$

Notice that both f^- and g^- are nonzero in the same domain $\mathbf{x} \le \mathbf{0}$. Now, let Σ be the covariance matrix of $(Z_\alpha(-1), Z_\alpha(1))$, that is,

$$\Sigma := \begin{pmatrix} \operatorname{cov}(Z_\alpha(-1), Z_\alpha(-1)) & \operatorname{cov}(Z_\alpha(-1), Z_\alpha(1)) \\ \operatorname{cov}(Z_\alpha(1), Z_\alpha(-1)) & \operatorname{cov}(Z_\alpha(1), Z_\alpha(1)) \end{pmatrix} = \begin{pmatrix} 2 & 2 - 2^\alpha \\ 2 - 2^\alpha & 2 \end{pmatrix}. \tag{10}$$

By the self-similarity property of fBm, the covariance matrix $\Sigma(\delta)$ of $(Z_\alpha(-\delta), Z_\alpha(\delta))$ equals $\Sigma(\delta) = \delta^\alpha \Sigma$. With $\mathbf{1}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we define

$$a(\mathbf{x}) := \mathbf{x}^\top \Sigma^{-1} \mathbf{x}, \quad b(\mathbf{x}) := \mathbf{x}^\top \Sigma^{-1} \mathbf{1}_2, \quad c := \mathbf{1}_2^\top \Sigma^{-1} \mathbf{1}_2 = \frac{2}{4 - 2^\alpha}, \tag{11}$$

so that, with $|\Sigma|$ denoting the determinant of matrix Σ, we have

$$\begin{aligned} f(\mathbf{x}; \delta) &= \frac{1}{2\pi |\Sigma|^{1/2} \delta^\alpha} \exp\left\{-\frac{(\mathbf{x} + \mathbf{1}_2 \delta^\alpha)^\top \Sigma(\delta)^{-1} (\mathbf{x} + \mathbf{1}_2 \delta^\alpha)}{2\delta^\alpha}\right\} \\ &= \frac{1}{2\pi |\Sigma|^{1/2} \delta^\alpha} \exp\left\{-\frac{a(\mathbf{x}) + 2b(\mathbf{x})\delta^\alpha + c\delta^{2\alpha}}{2\delta^\alpha}\right\}. \end{aligned}$$

The proofs of the following three lemmas are given in the appendix.

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Lemma 4. *For any $\lambda > 0$ there exist $\mathcal{C}_0, \mathcal{C}_1 > 0$ such that*

$$\mathcal{C}_0 \delta^{\alpha/2} \le q(\delta, \lambda) \le \mathcal{C}_1 \delta^{\alpha/2}$$

for all $\delta > 0$ sufficiently small.

In the following lemma, we establish the formulas for f^- and g^- and show that g^- is upper-bounded by f^- uniformly in δ, up to a positive constant.

Lemma 5. *For any $\lambda > 0$,*

(i) $f^-(\boldsymbol{x}; \delta) = p(\delta)^{-1} f(\boldsymbol{x}; \delta) \mathbb{1}\{\boldsymbol{x} \le 0\}$;

(ii) $g^-(\boldsymbol{x}; \delta, \lambda) = q(\delta, \lambda)^{-1} f(\boldsymbol{x}; \delta) \int_0^\infty \lambda \exp\left\{ -\frac{cz^2 + 2z((\lambda - c)\delta^\alpha - b(\boldsymbol{x}))}{2\delta^\alpha} \right\} \mathrm{d}z \cdot \mathbb{1}\{\mathbf{x} \le 0\}$;

(iii) there exists $C > 0$, depending only on λ, such that for all δ small enough, $g^-(\boldsymbol{x}; \delta, \lambda) \le C f^-(\boldsymbol{x}; \delta)$ for all $\boldsymbol{x} \le 0$.

Recall the definition of Σ in Equation (10). In what follows, for $k \in \mathbb{Z}$ we define

$$\begin{aligned} \begin{pmatrix} c^-(k) \\ c^+(k) \end{pmatrix} &:= \Sigma^{-1} \cdot \begin{pmatrix} \mathrm{cov}(Z_\alpha(k), Z_\alpha(-1)) \\ \mathrm{cov}(Z_\alpha(k), Z_\alpha(1)) \end{pmatrix} \\ &= \frac{1}{2^\alpha(4 - 2^\alpha)} \cdot \begin{pmatrix} 2 & 2^\alpha - 2 \\ 2^\alpha - 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} k^\alpha + 1 - (k + \mathrm{sgn}(k))^\alpha \\ k^\alpha + 1 - (k - \mathrm{sgn}(k))^\alpha \end{pmatrix}. \end{aligned} \tag{12}$$

Lemma 6. *For $k \in \mathbb{Z} \setminus \{0\}$,*

(i) $(2 - 2^{\alpha-1})^{-1} < c^-(k) + c^+(k) \le 1$ *when* $\alpha \in (0, 1)$;

(ii) $1 \le c^-(k) + c^+(k) \le (2 - 2^{\alpha-1})^{-1}$ *when* $\alpha \in (1, 2)$.

We are now ready to prove Proposition 2. In what follows, for any $\delta > 0$ and $t \in \mathbb{R}$ let

$$\begin{aligned} Y_\alpha^\delta(t) &:= Z_\alpha(t) - \mathbb{E}\{Z_\alpha(t) \mid (Z_\alpha(-\delta), Z_\alpha(\delta))\} \\ &= Z_\alpha(t) - \big(c^-(k) Z_\alpha(-\delta) + c^+(k) Z_\alpha(\delta)\big). \end{aligned} \tag{13}$$

It is a well-known fact that $\{Y_\alpha^\delta(t), t \in \mathbb{R}\}$ is independent of $(Z_\alpha(-\delta), Z_\alpha(\delta))$.

Proof of Proposition 2. Recall the definition of the events $A(\delta, \lambda)$ and $A(\delta)$ in (7). We have

$$\begin{aligned} &\mathbb{P}\left\{ \sup\nolimits_{t \in \delta\mathbb{Z} \setminus \{0\}} Z_\alpha(t) + \eta \le 0 \right\} \\ &\qquad = \mathbb{P}\left\{ \sup\nolimits_{k \in \mathbb{Z} \setminus \{-1,0,1\}} Y_\alpha^\delta(\delta k) + \Big(c^-(k) Z_\alpha(-\delta) + c^+(k) Z_\alpha(\delta) \Big) + \eta < 0; A(\delta, 1) \right\}, \end{aligned}$$

with $Y_\alpha(t)$ as defined in (13). Let $\lambda^* := 1$ when $\alpha \in (0, 1]$, and $\lambda^* := (2 - 2^{\alpha-1})^{-1}$ when $\alpha \in [1, 2)$. By Lemma 6 we have $\lambda^* \ge 1$ and $c^-(k) + c^+(k) \le \lambda^*$; thus $A(\delta, 1) \subseteq A(\delta, \lambda^*)$, and the

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display above is upper-bounded by

$$\mathbb{P}\left\{\sup_{k\in\mathbb{Z}\setminus\{-1,0,1\}} Y_\alpha^\delta(\delta k) + \left(c^-(k)\left(Z_\alpha(-\delta)+\tfrac{\eta}{\lambda^*}\right) + c^+(k)\left(Z_\alpha(\delta)+\tfrac{\eta}{\lambda^*}\right)\right) < 0; A(\delta,\lambda^*)\right\}$$

$$= q(\delta,\lambda^*)\cdot\mathbb{P}\left\{\sup_{k\in\mathbb{Z}\setminus\{-1,0,1\}} Y_\alpha^\delta(\delta k) + \left(c^-(k)\left(Z_\alpha(-\delta)+\tfrac{\eta}{\lambda^*}\right) + c^+(k)\left(Z_\alpha(\delta)+\tfrac{\eta}{\lambda^*}\right)\right) < 0 \,\middle|\, A(\delta,\lambda^*)\right\}$$

$$= q(\delta,\lambda^*)\cdot\int_{\mathbf{x}\le\mathbf{0}} \mathbb{P}\left\{\sup_{k\in\mathbb{Z}\setminus\{-1,0,1\}} Y_\alpha^\delta(\delta k) + \left(c^-(k)x_1 + c^+(k)x_2\right) < 0\right\} g^-(\mathbf{x};\delta,\lambda^*)\mathrm{d}\mathbf{x},$$

where g^- is as defined in (9). By using Lemma 5(iii), in particular Equation (25), we know that for every $\varepsilon > 0$ and all $\delta > 0$ small enough, with

$$\mathcal{C}(\delta,\varepsilon) := \tfrac{1}{2}\lambda^*\sqrt{\pi(4-2^\alpha)}(1+\varepsilon)\delta^{\alpha/2}q(\delta,\lambda^*)^{-1}p(\delta),$$

the expression above is upper-bounded by

$$q(\delta,\lambda^*)\cdot\int_{\mathbf{x}\le\mathbf{0}} \mathbb{P}\left\{\sup_{k\in\mathbb{Z}\setminus\{-1,0,1\}} Y_\alpha^\delta(\delta k) + \left(c^-(k)x_1 + c^+(k)x_2\right) < 0\right\} \mathcal{C}(\delta,\varepsilon) f^-(\mathbf{x};\delta)\mathrm{d}\mathbf{x}$$

$$= \mathcal{C}(\delta,\varepsilon)q(\delta,\lambda^*)\mathbb{P}\left\{\sup_{k\in\mathbb{Z}\setminus\{-1,0,1\}} Y_\alpha^\delta(\delta k) + \left(c^-(k)Z_\alpha(-\delta) + c^+(k)Z_\alpha(\delta)\right) < 0 \,\middle|\, A(\delta)\right\}$$

$$= \mathcal{C}(\delta,\varepsilon)q(\delta,\lambda^*)p(\delta)^{-1}\mathbb{P}\left\{A(\delta),\ \sup_{k\in\mathbb{Z}\setminus\{-1,0,1\}} Y_\alpha^\delta(\delta k) + \left(c^-(k)Z_\alpha(-\delta) + c^+(k)Z_\alpha(\delta)\right) < 0\right\}$$

$$= \mathcal{C}(\delta,\varepsilon)q(\delta,\lambda^*)p(\delta)^{-1}\mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha(t) \le 0\right\}$$

$$= \tfrac{1}{2}\lambda^*\sqrt{\pi(4-2^\alpha)}(1+\varepsilon)\delta^{\alpha/2}\mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha(t) \le 0\right\}.$$

Finally, from [23, Proposition 4] we know that $\mathbb{P}\left\{\sup_{t\in\delta\mathbb{Z}\setminus\{0\}} Z_\alpha(t) \le 0\right\} \sim \delta\mathcal{H}_\alpha$; therefore, after substituting for λ^*, we find that the above is upper-bounded by

$$\frac{\mathcal{H}_\alpha\sqrt{\pi}}{\sqrt{4-2^\alpha}}(1+\varepsilon)\cdot\delta^{1+\alpha/2}$$

for all $\delta > 0$ sufficiently small. □

3.2. Proof of Theorem 1, case $\alpha \in (1, 2)$

The following lemma provides a crucial bound for $\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta$.

Lemma 7. *For sufficiently small $\delta > 0$ it holds that*

$$\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta \le 2\mathbb{E}\left\{\sup_{t\in[0,1]} e^{Z_\alpha(t)} - \sup_{t\in[0,1]_\delta} e^{Z_\alpha(t)}\right\}.$$

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Proof of Lemma 7. As follows from the proof of [17, Theorem 1], in particular the first equation on p. 12 with $c_\delta := [1/\delta]\delta$, where $[\cdot]$ is the integer part of a real number, it holds that

$$\begin{aligned}\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta &\le c_\delta^{-1}\mathbb{E}\left\{\sup_{t\in[0,c_\delta]}e^{Z_\alpha(t)} - \sup_{t\in[0,c_\delta]_\delta}e^{Z_\alpha(t)}\right\}\\ &\le 2\mathbb{E}\left\{\sup_{t\in[0,c_\delta]}e^{Z_\alpha(t)} - \sup_{t\in[0,c_\delta]_\delta}e^{Z_\alpha(t)}\right\}\\ &\le 2\mathbb{E}\left\{\sup_{t\in[0,1]}e^{Z_\alpha(t)} - \sup_{t\in[0,1]_\delta}e^{Z_\alpha(t)}\right\}.\end{aligned}$$

This completes the proof. □

Now we are ready to prove Theorem 1(ii).

Proof of Theorem 1, $\alpha \in (1, 2)$. Note that for any $y \le x$ it holds that $e^x - e^y \le (x - y)e^x$. Implementing this inequality, we find that for $s, t \in [0, 1]$,

$$\begin{aligned}\left|e^{Z_\alpha(t)} - e^{Z_\alpha(s)}\right| &\le e^{\max(Z_\alpha(t),Z_\alpha(s))}|Z_\alpha(t) - Z_\alpha(s)|\\ &\le e^{\sqrt{2}\max\limits_{w\in[0,1]}B_\alpha(w)}\left|\sqrt{2}(B_\alpha(t) - B_\alpha(s)) - (t^\alpha - s^\alpha)\right|.\end{aligned}$$

Next, by Lemma 7 we have

$$\begin{aligned}\mathcal{H}_\alpha - \mathcal{H}_\alpha^\delta &\le 2\mathbb{E}\left\{\sup_{t,s\in[0,1],|t-s|\le\delta}\left|e^{Z_\alpha(t)} - e^{Z_\alpha(s)}\right|\right\}\\ &\le 2\sqrt{2}\mathbb{E}\left\{e^{\sqrt{2}\max\limits_{w\in[0,1]}B_\alpha(w)}\sup_{t,s\in[0,1],|t-s|\le\delta}|B_\alpha(t) - B_\alpha(s)|\right\}\\ &\quad + 2\mathbb{E}\left\{e^{\sqrt{2}\max\limits_{w\in[0,1]}B_\alpha(w)}\sup_{t,s\in[0,1],|t-s|\le\delta}|t^\alpha - s^\alpha|\right\}.\end{aligned}$$

Clearly, the second term is upper-bounded by $\mathcal{C}_1\delta$ for all δ small enough. Using the Hölder inequality, the first term can be bounded by

$$2\sqrt{2}\mathbb{E}\left\{e^{2\sqrt{2}\max\limits_{w\in[0,1]}B_\alpha(w)}\right\}^{1/2}\mathbb{E}\left\{\left(\sup_{t,s\in[0,1],|t-s|\le\delta}(B_\alpha(t) - B_\alpha(s))\right)^2\right\}^{1/2}.$$

The first expectation is finite. The random variable inside the second expectation is called the *uniform modulus of continuity*. From [10, Theorem 4.2, p. 164] it follows that there exists $\mathcal{C} > 0$ such that

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$$\mathbb{E}\left\{\left(\sup_{t,s\in[0,1],|t-s|\le\delta}(B_\alpha(t)-B_\alpha(s))\right)^2\right\}^{1/2}\le\mathcal{C}\delta^{\alpha/2}|\log(\delta)|^{1/2}.$$

This concludes the proof. □

Remark 2. Note that the proofs of Lemma 7 and Theorem 1 also work in the case $\alpha\in(0,1]$.

3.3. Proofs of Theorems 2 and 3

Let $\delta\ge 0$. Define a measure μ_δ such that for real numbers $a\le b$,

$$\mu_\delta([a,b])=\begin{cases}\delta\cdot\#\{[a,b]_\delta\}, & \delta>0,\\ b-a, & \delta=0.\end{cases}$$

Proof of Theorem 2. For any $T>0$, $x\ge 1$, and $\delta\ge 0$, we have

$$\begin{aligned}\mathbb{P}\left\{\xi_\alpha^\delta>x\right\}&\le\mathbb{P}\left\{\frac{e^{\sup_{t\in\mathbb{R}}Z_\alpha(t)}}{\int_{\mathbb{R}}e^{Z_\alpha(t)}\mathrm{d}\mu_\delta}>x\right\}\\&=\mathbb{P}\left\{\frac{e^{\sup_{t\in[-T,T]}Z_\alpha(t)}}{\int_{\mathbb{R}}e^{Z_\alpha(t)}\mathrm{d}\mu_\delta}>x\text{ and }Z_\alpha(t)\text{ achieves its maximum at }t\in[-T,T]\right\}\\&\quad+\mathbb{P}\left\{\frac{e^{\sup_{t\in\mathbb{R}\setminus[-T,T]}Z_\alpha(t)}}{\int_{\mathbb{R}}e^{Z_\alpha(t)}\mathrm{d}\mu_\delta}>x\text{ and }Z_\alpha(t)\text{ achieves its maximum at }t\in\mathbb{R}\setminus[-T,T]\right\}\\&\le\mathbb{P}\left\{\frac{e^{\sup_{t\in[-T,T]}Z_\alpha(t)}}{\int_{\mathbb{R}}e^{Z_\alpha(t)}\mathrm{d}\mu_\delta}>x\right\}+\mathbb{P}\left\{\exists t\in\mathbb{R}\setminus[-T,T]:Z_\alpha(t)>0\right\}\\&\le\mathbb{P}\left\{\frac{e^{\sup_{t\in[-T,T]}Z_\alpha(t)}}{\int_{[-T,T)}e^{Z_\alpha(t)}\mathrm{d}\mu_\delta}>x\right\}+2\mathbb{P}\left\{\exists t\ge T:Z_\alpha(t)>0\right\}\\&=:p_1(T,x)+2p_2(T).\end{aligned}\tag{14}$$

Estimation of $p_2(T)$. By the self-similarity of fBm we have

$$\begin{aligned}p_2(T)&\le\sum_{k=1}^{\infty}\mathbb{P}\left\{\exists t\in[kT,(k+1)T]:\sqrt{2}B_\alpha(t)-t^\alpha>0\right\}\\&=\sum_{k=1}^{\infty}\mathbb{P}\left\{\exists t\in[1,1+\frac{1}{k}]:\sqrt{2}B_\alpha(t)(kT)^{\alpha/2}>(kT)^\alpha t^\alpha\right\}\\&\le\sum_{k=1}^{\infty}\mathbb{P}\left\{\exists t\in[1,2]:\sqrt{2}B_\alpha(t)>(kT)^{\alpha/2}\right\}.\end{aligned}$$

Thus, using the Borell–TIS inequality, we find that for all $T\ge 1$,

$$p_2(T)\le\sum_{k=1}^{\infty}\mathcal{C}e^{-\frac{(kT)^\alpha}{10}}\le\mathcal{C}e^{-\frac{T^\alpha}{10}}.\tag{15}$$

Estimation of $p_1(T, x)$. Observe that for $T, x \geq 1$ and $\delta \in [0, 1]$,

$$p_1(T, x) \leq \mathbb{P}\left\{\frac{\sum_{k=-T}^{T-1} e^{\sup_{t\in[k,k+1]} Z_\alpha(t)}}{\sum_{k=-T}^{T-1} \int_{[k,k+1)} e^{Z_\alpha(t)} \mathrm{d}\mu_\delta} > x\right\} =: \mathbb{P}\left\{\frac{\sum_{k=-T}^{T-1} a_k(\omega)}{\sum_{k=-T}^{T-1} b_k(\omega)} > x\right\}.$$

Since the event $\{\sum_{k=-T}^{T-1} a_k(\omega)/\sum_{k=-T}^{T-1} b_k(\omega) > x\}$ implies $\{a_k(\omega)/b_k(\omega) > x,$ for some $k \in [-T, T-1]_1\}$, we have

$$\mathbb{P}\left\{\frac{\sum_{k=-T}^{T-1} a_k(\omega)}{\sum_{k=-T}^{T-1} b_k(\omega)} > x\right\} \leq \sum_{k=-T}^{T-1} \mathbb{P}\left\{\frac{a_k(\omega)}{b_k(\omega)} > x\right\} \leq 2T \sup_{k\in[-T,T]} \mathbb{P}\left\{\frac{\sup_{t\in[k,k+1]} e^{Z_\alpha(t)}}{\int_{[k,k+1)} e^{Z_\alpha(t)} \mathrm{d}\mu_\delta} > x\right\}.$$

Therefore, we obtain that for $x, T \geq 1$

$$p_1(T, x) \leq 2T \sup_{k\in[-T,T]} \mathbb{P}\left\{\frac{\sup_{t\in[k,k+1]} e^{Z_\alpha(t)}}{\int_{[k,k+1)} e^{Z_\alpha(t)} \mathrm{d}\mu_\delta} > x\right\}.$$

Next, by the stationarity of the increments of fBm, for $x, T \geq 1$ we have

$$\begin{aligned}
\mathbb{P}\left\{\frac{\sup_{t\in[k,k+1]} e^{Z_\alpha(t)}}{\int_{[k,k+1)} e^{Z_\alpha(t)} \mathrm{d}\mu_\delta} > x\right\} &\leq \mathbb{P}\left\{\frac{\sup_{t\in[k,k+1]} e^{Z_\alpha(t)}}{\mu_\delta[k, k+1] \inf_{[k,k+1]} e^{Z_\alpha(t)}} > x\right\} \\
&\leq \mathbb{P}\left\{\exists t, s \in [k, k+1] : Z_\alpha(t) - Z_\alpha(s) > \log(\tfrac{x}{2})\right\} \\
&\leq \mathbb{P}\left\{\exists t, s \in [k, k+1] : B_\alpha(t) - B_\alpha(s) > \frac{\log(\frac{x}{2}) - \sup_{t,s\in[k,k+1]}(|t|^\alpha - |s|^\alpha)}{\sqrt{2}}\right\} \\
&\leq \mathbb{P}\left\{\exists t \in [0, 1] : B_\alpha(t) > \frac{\log x - \mathcal{C} \max(1, T^{\alpha-1})}{\sqrt{2}}\right\},
\end{aligned}$$

where in the second line we used that $\mu_\delta[k, k+1] \geq 1/2$ for $\delta \in [0, 1]$. Thus, for $T, x \geq 1$,

$$p_1(T, x) \leq 2T\mathbb{P}\left\{\exists t \in [0, 1] : B_\alpha(t) > \frac{\log x - \mathcal{C} \max(1, T^{\alpha-1})}{\sqrt{2}}\right\}. \tag{16}$$

Combining the statement above with (14) and (15), for $x, T \geq 1$ we have

$$\mathbb{P}\left\{\frac{\sup_{t\in\mathbb{R}} e^{Z_\alpha(t)}}{\int_{\mathbb{R}} e^{Z_\alpha(t)} \mathrm{d}\mu_\delta} > x\right\} \leq \widetilde{\mathcal{C}} e^{-\frac{T^\alpha}{10}} + 2T\mathbb{P}\left\{\exists t \in [0, 1] : B_\alpha(t) > \frac{\log x - \mathcal{C} \max(1, T^{\alpha-1})}{\sqrt{2}}\right\}. \tag{17}$$

Assume that $\alpha \leq 1$. Then, choosing $T = x$ in the line above, by the Borell–TIS inequality we have for any fixed $\varepsilon > 0$ and sufficiently large x that

$$\mathbb{P}\left\{\frac{\sup_{t\in\mathbb{R}} e^{Z_\alpha(t)}}{\int_{\mathbb{R}} e^{Z_\alpha(t)} \mathrm{d}\mu_\delta} > x\right\} \leq e^{-\frac{log^2 x}{4+\varepsilon}}.$$

Assume that $\alpha > 1$. Taking $T = \mathcal{C}'(\log x)^{\frac{1}{\alpha-1}}$ with sufficiently small $\mathcal{C}' > 0$, we obtain by the Borell–TIS inequality that for any fixed $\varepsilon > 0$ and sufficiently large x,

$$\mathbb{P}\left\{\frac{\sup_{t\in\mathbb{R}} e^{Z_\alpha(t)}}{\int_{\mathbb{R}} e^{Z_\alpha(t)} \mathrm{d}\mu_\delta} > x\right\} \leq \widetilde{\mathcal{C}} e^{-\mathcal{C}''(\log x)^{\frac{\alpha}{\alpha-1}}} + e^{-\frac{\log^2 x}{4+\varepsilon}}.$$

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The first claim now follows since $\mathbb{P}\left\{\xi_\alpha^\delta(T) > x\right\} = p_1(T, x)$ and $\frac{\alpha}{\alpha-1} > 2$. The second claims follows in the same way by (17). □

Proof of Theorem 3. Observe that $|x^p - y^p| \le p|x-y|(x^{p-1} + y^{p-1})$ for all $x, y \ge 0$ and $p \ge 1$; this can be shown straightforwardly by differentiation. Hence we have

$$\begin{aligned}
&\left|\mathbb{E}\left\{(\xi_\alpha^\delta(T))^p\right\} - \mathbb{E}\left\{(\xi_\alpha^\delta)^p\right\}\right| \le \mathbb{E}\left\{\left|(\xi_\alpha^\delta(T))^p - (\xi_\alpha^\delta)^p\right|\right\}\\
&\le p\mathbb{E}\left\{\left|\xi_\alpha^\delta(T) - \xi_\alpha^\delta\right| \cdot \left((\xi_\alpha^\delta(T))^{p-1} + (\xi_\alpha^\delta)^{p-1}\right)\right\}\\
&\le p\left(\mathbb{E}\left\{\left(\xi_\alpha^\delta(T) - \xi_\alpha^\delta\right)^2\right\}\right)^{1/2} \cdot \left(\mathbb{E}\left\{\left((\xi_\alpha^\delta(T))^{p-1} + (\xi_\alpha^\delta)^{p-1}\right)^2\right\}\right)^{1/2}\\
&\le \sqrt{2}p\left(\mathbb{E}\left\{\left(\xi_\alpha^\delta(T) - \xi_\alpha^\delta\right)^2\right\}\right)^{1/2} \cdot \left(\mathbb{E}\left\{(\xi_\alpha^\delta(T))^{2p-2} + (\xi_\alpha^\delta)^{2p-2}\right\}\right)^{1/2}\\
&=: \sqrt{2}p\left(\mathbb{E}\left\{\beta^2\right\}\right)^{1/2} \cdot \left(\mathbb{E}\left\{\kappa_p\right\}\right)^{1/2}.
\end{aligned}$$

We have

$$\beta := \frac{\sup_{t\in\delta\mathbb{Z}} e^{Z_\alpha(t)}}{\int_{\mathbb{R}} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta} - \frac{\sup_{t\in[-T,T]_\delta} e^{Z_\alpha(t)}}{\int_{[-T,T]} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta} = \beta_1 - \beta_2\beta_3,$$

where

$$\begin{aligned}
\beta_1 &= \frac{\sup\limits_{t\in\delta\mathbb{Z}} e^{Z_\alpha(t)} - \sup\limits_{t\in[-T,T]_\delta} e^{Z_\alpha(t)}}{\int_{\mathbb{R}} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta} \ge 0,\\
\beta_2 &= \frac{\int_{\mathbb{R}\setminus[-T,T]} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta}{\int_{\mathbb{R}} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta} > 0,\\
\beta_3 &= \frac{\sup\limits_{t\in[-T,T]_\delta} e^{Z_\alpha(t)}}{\int_{[-T,T]} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta} > 0.
\end{aligned}$$

Applying the Hölder inequality, we obtain

$$\mathbb{E}\left\{\beta^2\right\} \le 2\mathbb{E}\left\{\beta_1^2\right\} + 2\sqrt{\mathbb{E}\left\{\beta_2^4\right\}\mathbb{E}\left\{\beta_3^4\right\}}. \tag{18}$$

We have by (16) that for $x, T \ge 1$,

$$\mathbb{P}\left\{\beta_3 > x\right\} \le 2T\mathbb{P}\left\{\exists t \in [0, 1] : B_\alpha(t) > \frac{\log x - \mathcal{C}\max(1, T^{\alpha-1})}{\sqrt{2}}\right\},$$

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which implies that for $T \geq 1$,

$$\begin{aligned}\mathbb{E}\left\{\beta_3^4\right\} &= \int_0^\infty \mathbb{P}\left\{\beta_2 > x^{1/4}\right\} \mathrm{d}x \\ &\leq 2T \int_0^\infty \mathbb{P}\left\{\exists t \in [0, 1] : B_\alpha(t) > \frac{\frac{1}{4}\log x - \mathcal{C}\max(1, T^{\alpha-1})}{\sqrt{2}}\right\} \mathrm{d}x \qquad (19) \\ &\leq 2T\left(\int_0^{\exp(5\mathcal{C}\max(1,T^{\alpha-1}))} 1\mathrm{d}x + \int_{\exp(5\mathcal{C}\max(1,T^{\alpha-1}))}^\infty \mathbb{P}\left\{\exists t \in [0, 1] : B_\alpha(t) > \mathcal{C}_3 \log x\right\} \mathrm{d}x\right) \\ &\leq \mathcal{C}_1 e^{\mathcal{C}\max(1,T^{\alpha-1})}.\end{aligned}$$

Finally for $\alpha \in (0, 2)$ and $T \geq 1$ we have

$$\mathbb{E}\left\{\beta_3^4\right\} \leq \mathcal{C}_1 e^{\mathcal{C}\max(1,T^{\alpha-1})}. \qquad (20)$$

Next, we focus on the properties of β_2. For $k > 0$ and sufficiently large T we have

$$\begin{aligned}\mathbb{P}\left\{\int_{[kT,(k+1)T)} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta > e^{-\frac{1}{2}T^\alpha k^\alpha}\right\} &\leq \mathbb{P}\left\{(T+1)\sup_{t\in[kT,(k+1)T]} e^{Z_\alpha(t)} > e^{-\frac{1}{2}T^\alpha k^\alpha}\right\} \\ &= \mathbb{P}\left\{\log(T+1) + \sup_{t\in[kT,(k+1)T]} Z_\alpha(t) > -\frac{1}{2}T^\alpha k^\alpha\right\} \\ &= \mathbb{P}\left\{\exists t \in [kT, (k+1)T] : \frac{B_\alpha(t)}{t^{\alpha/2}} > \frac{t^\alpha - \frac{1}{2}T^\alpha k^\alpha - \log(T+1)}{\sqrt{2}t^{\alpha/2}}\right\} \\ &\leq \mathbb{P}\left\{\exists t \in [kT, (k+1)T] : \frac{B_\alpha(t)}{t^{\alpha/2}} > (Tk)^{\alpha/2}/3\right\},\end{aligned}$$

which, by the Borell–TIS inequality, is upper-bounded by $e^{-k^\alpha T^\alpha/19}$. By the lines above we obtain that with probability at least $1 - \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{-|k|^\alpha T^\alpha/19} \geq 1 - e^{-T^\alpha/20}$, for large T,

$$\int_{\mathbb{R}\setminus[-T,T]} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta \leq \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{-\frac{1}{2}T^\alpha|k|^\alpha} \leq e^{-T^\alpha/3}.$$

Putting everything together, we find that for sufficiently large T,

$$\mathbb{P}\left\{\int_{\mathbb{R}\setminus[-T,T]} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta > e^{-T^\alpha/3}\right\} \leq e^{-T^\alpha/20}. \qquad (21)$$

Next we notice that for $T \geq 1$,

$$\mathbb{P}\left\{\int_{[-T,T]} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta < e^{-\frac{T^\alpha}{4}}\right\} \leq \mathbb{P}\left\{\int_{[0,1]} e^{Z_\alpha(t)}\mathrm{d}t\mu_\delta < e^{-\frac{T^\alpha}{4}}\right\} \leq \mathbb{P}\left\{\sup_{t\in[0,1]} Z_\alpha(t) < -\frac{T^\alpha}{4}\right\},$$

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so, by the Borell–TIS inequality, the above is bounded by $e^{-\frac{T^{2\alpha}}{65}}$ for all sufficiently large T. This result in combination with (21) gives us $\mathbb{P}\left\{\beta_2 > e^{-T^\alpha/12}\right\} \le e^{-T^\alpha/21}$ for all sufficiently large T. Thus, since $\beta_2 \in [0, 1]$, from the line above we immediately obtain that

$$\mathbb{E}\left\{\beta_2^4\right\} \le \mathcal{C}_1 e^{-\mathcal{C}T^\alpha} \tag{22}$$

for $T \ge 1$. By (15) we observe that

$$\mathbb{P}\{\beta_1 > 0\} \le \mathbb{P}\{\exists t \notin [-T, T] : Z_\alpha(t) > 0\} \le 2p_2(T) \le e^{-\mathcal{C}T^\alpha}.$$

Next, by Theorem 2, for $x \ge 1$ we have

$$\mathbb{P}\{\beta_1 > x\} \le \mathbb{P}\left\{\frac{\sup_{t\in\delta\mathbb{Z}} e^{Z_\alpha(t)}}{\int_{\mathbb{R}} e^{Z_\alpha(t)}\mathrm{d}\mu_\delta} > x\right\} \le \mathcal{C}_1 e^{-\mathcal{C}_2 \log^2 x},$$

and thus $\mathbb{E}\left\{\beta_1^4\right\} < \mathcal{C}$ for a positive constant $\mathcal{C}$ that does not depend on T. With $A_T := \{\beta_1(\omega) > 0\}$, by the Hölder inequality, for large T we have

$$\mathbb{E}\left\{\beta_1^2\right\} = \mathbb{E}\left\{\beta_1^2 \cdot \mathbb{1}(\Omega_T)\right\} \le \sqrt{\mathbb{E}\left\{\beta_1^4\right\}}\sqrt{\mathbb{E}\left\{\mathbb{1}(\Omega_T)\right\}} \le \mathcal{C}_1 e^{-\mathcal{C}_2 T^\alpha}.$$

By the line above and (20), (22), and (18), for $T \ge 1$ we obtain

$$\mathbb{E}\left\{\beta^2\right\} \le \mathcal{C}_2 e^{-\mathcal{C}_1 T^\alpha}. \tag{23}$$

Our next aim is to estimate $\mathbb{E}\left\{\kappa_p\right\}$. We have for $T \ge 1$ that

$$\begin{aligned}\mathbb{E}\left\{(\xi_\alpha^\delta(T))^{2p-2}\right\} &= \int_0^\infty \mathbb{P}\left\{\xi_\alpha^\delta(T) > x^{\frac{1}{2p-2}}\right\}\mathrm{d}x = \int_0^\infty \mathbb{P}\left\{p_1(T, x^{\frac{1}{2p-2}})\right\}\mathrm{d}x \\ &\le \int_0^\infty \mathbb{P}\left\{\exists t \in [0, 1] : B_\alpha(t) > \frac{\log x^{\frac{1}{2p-2}} - \mathcal{C}\max(1, T^{\alpha-1})}{\sqrt{2}}\right\}\mathrm{d}x.\end{aligned}$$

By the same arguments as in Equation (19), the last integral above does not exceed $\mathcal{C}_1 e^{\max \mathcal{C}_2(T^{\alpha-1},1)}$, and since Theorem 2 implies that ξ_α^δ has all finite moments uniformly bounded for all $\delta \ge 0$, we obtain for $T \ge 1$ that

$$\kappa_p \le \mathcal{C}_1 e^{\max \mathcal{C}_2(T^{\alpha-1},1)}.$$

Combining the bound above with (23), we have $\sqrt{2}p\mathbb{E}\left\{\beta^2\right\}\mathbb{E}\left\{\kappa_p\right\} \le e^{-\mathcal{C}_p T^\alpha}$ for sufficiently large T, and the claim follows. □

3.4. Proofs of Corollaries 1 and 2

In the following, ϕ stands for the probability density function of a standard Gaussian random variable and

$$v(\eta) := \eta \exp\left(2\sum_{k=1}^\infty \frac{\Psi(\sqrt{\eta k/2})}{k}\right), \qquad \eta > 0,$$

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with (recall) Ψ the survival function of a standard Gaussian random variable. Before giving the proofs we introduce the following auxiliary lemma, whose proof is given in the appendix.

Lemma 8. *It holds that for any $\eta > 0$,*

$$v'(\eta) = \exp\left(2\sum_{k=1}^{\infty} \frac{\Psi\left(\sqrt{\frac{\eta k}{2}}\right)}{k}\right)\left(1 - \frac{\sqrt{\eta}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-\frac{\eta k}{4}}}{\sqrt{k}}\right).$$

Proof of Corollary 1, $\alpha = 1$. By Proposition 1(i), we find that

$$\mathcal{A} := \lim_{\eta\to 0}\frac{\mathcal{H}_0 - \mathcal{H}_\eta}{\sqrt{\eta}} = \lim_{\eta\to 0}\frac{1 - 1/v(\eta)}{\sqrt{\eta}}.$$

Since $\mathcal{H}_\eta = v(\eta)^{-1} \to 1$ as $\eta \to 0$ (see, e.g., [23]), we conclude that $\lim_{\eta\to 0} v(\eta) = 1$ and hence $\mathcal{A} = \lim_{\eta\to 0}\frac{v(\eta)-1}{\sqrt{\eta}}$. Implementing L'Hôpital's rule, we obtain by Lemma 8 that

$$\mathcal{A} = \lim_{\eta\to 0}\frac{v'(\eta)}{1/(2\sqrt{\eta})} = 2\lim_{\eta\to 0}\sqrt{\eta}\exp\left(2\sum_{k=1}^{\infty}\frac{\Psi\left(\sqrt{\frac{\eta k}{2}}\right)}{k}\right)\left(1 - \frac{\sqrt{\eta}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-\frac{\eta k}{4}}}{\sqrt{k}}\right).$$

Note that by the definition of $v(\eta)$, the observation that $\lim_{\eta\to 0} v(\eta) = 1$ implies

$$\sqrt{\eta}\exp\left(2\sum_{k=1}^{\infty}\frac{\Psi\left(\sqrt{\frac{\eta k}{2}}\right)}{k}\right) \sim \frac{1}{\sqrt{\eta}}, \qquad \eta \to 0,$$

and hence, with $x := \sqrt{\eta}/2$,

$$\mathcal{A} = \lim_{x\to 0}\frac{1}{x}\Big(1 - \frac{x}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-x^2 k}}{\sqrt{k}}\Big)\Big) = \lim_{x\to 0}\frac{1}{x}\Big(1 - \frac{x}{\sqrt{\pi}}\mathrm{Li}_{\frac{1}{2}}(e^{-x^2})\Big),$$

where $\mathrm{Li}_{\frac{1}{2}}$ is the polylogarithm function; see, e.g., [9]. As follows from [35, Equation (9.3)],

$$\begin{aligned}
&\lim_{x\to 0}\frac{1}{x}\Big(1 - \frac{x}{\sqrt{\pi}}\mathrm{Li}_{\frac{1}{2}}(e^{-x^2})\Big)\\
&= \lim_{x\to 0}\frac{1}{x}\Big(1 - \frac{x}{\sqrt{\pi}}\Big(\Gamma(1/2)(x^2)^{-1/2} + \zeta(1/2) + \sum_{k=1}^{\infty}\zeta(1/2-k)\frac{(-x^2)^k}{k!}\Big)\Big)\\
&= \frac{\zeta(1/2)}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}\lim_{x\to 0}\Big(\sum_{k=1}^{\infty}\zeta(1/2-k)\frac{x^{2k}(-1)^k}{k!}\Big).
\end{aligned}$$

Thus, to prove the claim it is enough to show that

$$\lim_{x\to 0}\sum_{k=1}^{\infty}\zeta(1/2-k)\frac{x^{2k}(-1)^k}{k!} = 0. \tag{24}$$

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By the Riemann functional equation (see [25, Equation (2.3)]) and the observation that $\zeta(s)$ is strictly decreasing for real $s > 1$, we have for any natural number k that

$$|\zeta(1/2-k)| \le 2^{1/2-k}\pi^{-1/2-k}\Gamma(1/2+k)\zeta(1/2+k) \le 2^{-k}\Gamma(k+1)\zeta(3/2) = \frac{\zeta(3/2)k!}{2^k}.$$

Thus, for $|x| < 1$ we have

$$\Big|\sum_{k=1}^{\infty}\zeta(1/2-k)\frac{x^{2k}(-1)^k}{k!}\Big| \le x^2\sum_{k=1}^{\infty}\frac{|\zeta(1/2-k)|}{k!} \le x^2\zeta(3/2)\sum_{k=1}^{\infty}2^{-k} = \zeta(3/2)x^2,$$

and (24) follows, which completes the proof of the first statement.

For the statement (ii), by Proposition 1(ii) we have, as $\delta \to 0$,

$$\begin{aligned}\mathcal{H}_2 - \mathcal{H}_2^{\delta} &= \frac{1}{\sqrt{\pi}} - \frac{2}{\delta}\left(\Phi(\frac{\delta}{\sqrt{2}}) - \frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}\left(1 - \frac{\sqrt{2}}{\delta}\int_0^{\delta/\sqrt{2}} e^{-x^2/2}\mathrm{d}x\right)\\ &= \frac{1}{\sqrt{\pi}}\left(1 - \frac{\sqrt{2}}{\delta}\int_0^{\delta/\sqrt{2}}\left(1 - \frac{x^2}{2}\right)\mathrm{d}x + O(\delta^4)\right) = \frac{1}{12\sqrt{\pi}},\end{aligned}$$

and the claim follows. □

Proof of Corollary 2. Case $\alpha = 1$. First we show that $v(\eta)$ is an increasing function for $\eta > 0$, which is equivalent to the fact that $v'(\eta) > 0$ for $\eta > 0$. In light of Lemma 8 it is sufficient to show that

$$\frac{\sqrt{\eta}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-\frac{\eta k}{4}}}{\sqrt{k}} < 1, \qquad \eta > 0.$$

We have

$$\begin{aligned}\frac{\sqrt{\eta}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-\frac{\eta k}{4}}}{\sqrt{k}} &< \frac{1}{\sqrt{\pi}}\sqrt{\frac{\eta}{4}}\int_0^{\infty} e^{-\frac{\eta z}{4}}z^{-1/2}dz = \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-\frac{\eta z}{4}}(\frac{\eta z}{4})^{-1/2}d(\frac{\eta z}{4})\\ &= \frac{1}{\sqrt{\pi}}\Gamma(1/2) = 1,\end{aligned}$$

and hence $\mathcal{H}_1^{\eta} = 1/v(\eta)$ is decreasing for $\eta > 0$. Since, by the classical definition, $\mathcal{H}_1^0 > \mathcal{H}_1^{\eta}$ for any $\eta > 0$, we obtain the claim.

Case $\alpha = 2$. By Proposition 1(ii) we have

$$\mathcal{H}_2^{\delta} = \frac{2}{\delta}\left(\Phi(\delta/\sqrt{2}) - \frac{1}{2}\right) = \frac{2}{\delta\sqrt{2\pi}}\int_0^{\delta/\sqrt{2}} e^{-x^2/2}\mathrm{d}x = \frac{1}{\eta\sqrt{\pi}}\int_0^{\eta} e^{-x^2/2}\mathrm{d}x,$$

where $\eta = \delta/\sqrt{2}$. The derivative of the last integral above with respect to η equals

$$\frac{1}{\sqrt{\pi}}\left(-\frac{1}{\eta^2}\int_0^{\eta} e^{-x^2/2}\mathrm{d}x + \frac{1}{\eta}e^{-\eta^2/2}\right) = \frac{1}{\sqrt{\pi}\eta^2}\left(\int_0^{\eta}(e^{-\eta^2/2} - e^{-x^2/2})\mathrm{d}x\right) < 0,$$

and the claim follows. □

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Appendix

Proof of Equation (6). Let $t, s \in \mathbb{R}$ be fixed, and let

$$c(\delta; t, s) := \operatorname{cov}\left(\frac{B_\alpha(t+\delta\cdot \operatorname{sgn}(t))}{|t+\delta\cdot\operatorname{sgn}(t)|^{\alpha/2}}, \frac{B_\alpha(s+\delta\cdot\operatorname{sgn}(s))}{|s+\delta\cdot\operatorname{sgn}(s)|^{\alpha/2}}\right).$$

We will show that $\delta \mapsto c(\delta; t, s)$ is a nondecreasing function, which will conclude the proof. We have

$$c(\delta; t, s) = \frac{|t+\delta\cdot\operatorname{sgn}(t)|^\alpha + |s+\delta\cdot\operatorname{sgn}(s)|^\alpha - |t-s+\delta\cdot(\operatorname{sgn}(t)-\operatorname{sgn}(s))|^\alpha}{2|s+\delta\cdot\operatorname{sgn}(s)|^{\alpha/2}\cdot|t+\delta\cdot\operatorname{sgn}(t)|^{\alpha/2}}.$$

We consider two cases: (i) $t, s > 0$, and (ii) $s < 0 < t$. Consider case (i) first. Without loss of generality we assume that $t \geq s$; then

$$c(\delta; t, s) = \frac{(t+\delta)^\alpha + (s+\delta)^\alpha - (t-s)^\alpha}{2((s+\delta)(t+\delta))^{\alpha/2}}.$$

It suffices to show that the first derivative of $\delta \mapsto c(\delta; t, s)$ is nonnegative. We have

$$\frac{\partial}{\partial\delta} c(\delta, t, s) = \frac{\alpha(1-x)(t+\delta)^{\alpha/2}/4}{(s+\delta)^{1+\alpha/2}}\cdot\Big((1-x)^{\alpha-1}(1+x) - x^\alpha - 1\Big),$$

where $x := \frac{s+\delta}{t+\delta} \in (0, 1]$. The derivative above is nonnegative if and only if $G_1(x, \alpha) := (1-x)^{\alpha-1}(1+x) + x^\alpha - 1 \geq 0$ for all $x \in (0, 1]$. It is easy to see that, for any fixed $x \in (0, 1]$, $\alpha \mapsto G_1(x, \alpha)$ is a nondecreasing function; this observation combined with the fact that $G(x, 2) = 0$ completes the proof of case (i). In case (ii) we need to show that

$$c(\delta; t, -s) = \frac{(t+\delta)^\alpha + (s+\delta)^\alpha - (t+s+2\delta)^\alpha}{2((s+\delta)(t+\delta))^{\alpha/2}}$$

is a nondecreasing function of x for any $s, t > 0$. Without loss of generality let $0 < s \leq t$. Again, we take the first derivative of the above and see that

$$\frac{\partial}{\partial\delta} c(\delta, t, s) = \frac{\alpha(1-x)(t+\delta)^{\alpha/2}/4}{(s+\delta)^{1+\alpha/2}}\cdot\Big((1-x)(1+x)^{\alpha-1} + x^\alpha - 1\Big),$$

where $x := \frac{s+\delta}{t+\delta} \in (0, 1]$. The derivative above is nonnegative if and only if $G_2(x, \alpha) := (1-x)(1+x)^{\alpha-1} + x^\alpha - 1 \geq 0$. Notice that $G_2(x, 1) = 0$. We will now show that $\frac{\partial}{\partial\alpha} G_2(x, \alpha) \leq 0$ for all $\alpha \in [0, 1]$ and $x \in (0, 1]$, which will conclude the proof. We have

$$\begin{aligned}
\frac{\partial}{\partial\alpha} G_2(x, \alpha) &= (1-x)(1+x)^{\alpha-1}\log(x+1) + x^\alpha \log x \\
&\leq (1-x)x^{\alpha-1}\log(x+1) + x^\alpha \log x \\
&= x^{\alpha-1}((1-x)\log(x+1) + x\log x) \\
&\leq x^{\alpha-1}((1-x)x + x(x-1)) = 0,
\end{aligned}$$

where in the last line we used the fact that $\log(1+x) \leq x$ for all $x > -1$. □

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Proof of Lemma 4. For the lower bound, observe that

$$\begin{aligned}
q(\delta, \lambda) &= \mathbb{P}\left\{\sqrt{2}B_\alpha(-\delta) - \delta^\alpha + \lambda^{-1}\eta < 0, \sqrt{2}B_\alpha(\delta) - \delta^\alpha + \lambda^{-1}\eta < 0)\right\} \\
&\geq \mathbb{P}\left\{\sqrt{2}B_\alpha(-\delta) + \lambda^{-1}\eta < 0, \sqrt{2}B_\alpha(\delta) + \lambda^{-1}\eta < 0, \lambda^{-1}\eta < \delta^{\alpha/2}\right\} \\
&\geq \mathbb{P}\left\{\sqrt{2}B_\alpha(-\delta) < -\delta^{\alpha/2}, \sqrt{2}B_\alpha(\delta) < -\delta^{\alpha/2}, \lambda^{-1}\eta < \delta^{\alpha/2}\right\} \\
&= \mathbb{P}\left\{\sqrt{2}B_\alpha(-1) > 1, \sqrt{2}B_\alpha(1) > 1\right\} \mathbb{P}\left\{\lambda^{-1}\eta < \delta^{\alpha/2}\right\} \\
&= \mathbb{P}\left\{\sqrt{2}B_\alpha(-1) > 1, \sqrt{2}B_\alpha(1) > 1\right\} \cdot \left(1 - \exp\left\{-\lambda\delta^{\alpha/2}\right\}\right),
\end{aligned}$$

which behaves like $\lambda\delta^{\alpha/2}\mathbb{P}\left\{\sqrt{2}B_\alpha(-1) > 1, \sqrt{2}B_\alpha(1) > 1\right\}$ as $\delta \downarrow 0$. For the upper bound, observe that

$$\begin{aligned}
q(\delta, \lambda) &\leq \mathbb{P}\left\{Z_\alpha(\delta) + \lambda^{-1}\eta < 0\right\} = \mathbb{P}\left\{\delta^{\alpha/2}\sqrt{2}B_\alpha(1) - \delta^\alpha + \lambda^{-1}\eta < 0\right\} \\
&= \int_0^\infty \Phi\left(\tfrac{\delta^\alpha - z}{\sqrt{2}\delta^{\alpha/2}}\right) \lambda e^{-\lambda z} \mathrm{d}z \leq \sqrt{2}\delta^{\alpha/2}\lambda \int_0^\infty \Phi\left(\tfrac{\delta^{\alpha/2}}{\sqrt{2}} - z\right) \mathrm{d}z \\
&\leq \sqrt{2}\delta^{\alpha/2}\lambda \left(\tfrac{\delta^{\alpha/2}}{\sqrt{2}} + \int_0^\infty \Psi(z)\mathrm{d}z\right) = \delta^{\alpha/2}\lambda\left(\delta^{\alpha/2} + \frac{\mathbb{E}|\mathcal{N}(0, 1)|}{\sqrt{2}}\right),
\end{aligned}$$

where (recall) $\Phi(\cdot)$, $\Psi(\cdot)$ are the CDF and complementary CDF, respectively, of the standard normal distribution. This concludes the proof. □

Proof of Lemma 5. Part (i) follows directly from the definition. For part (ii), for $\mathbf{x} \leq 0$ we have

$$\begin{aligned}
g^-(\mathbf{x}; \delta, \lambda) &= q(\delta, \lambda)^{-1} \int_0^\infty f(\mathbf{x} - \mathbf{1}_2 z; \delta) \cdot \lambda \mathrm{e}^{-\lambda z} \mathrm{d}z \\
&= \int_0^\infty \frac{q(\delta, \lambda)^{-1}}{2\pi|\Sigma|\delta^\alpha} \exp\left\{-\frac{(\mathbf{x} + \mathbf{1}_2(\delta^\alpha - z))^\top \Sigma^{-1}(\mathbf{x} + \mathbf{1}_2(\delta^\alpha - z))}{2\delta^\alpha}\right\} \cdot \lambda \mathrm{e}^{-\lambda z}\mathrm{d}z \\
&= \int_0^\infty \frac{\lambda q(\delta, \lambda)^{-1}}{2\pi|\Sigma|\delta^\alpha} \exp\left\{-\frac{a(\mathbf{x}) + 2b(\mathbf{x})(\delta^\alpha - z) + c(\delta^{2\alpha} - 2\delta^\alpha z + z^2) + 2\lambda\delta^\alpha z}{2\delta^\alpha}\right\} \mathrm{d}z \\
&= q(\delta, \lambda)^{-1} f(\mathbf{x}; \delta) \int_0^\infty \lambda \exp\left\{-\frac{cz^2 + 2z((\lambda - c)\delta^\alpha - b(\mathbf{x}))}{2\delta^\alpha}\right\} \mathrm{d}z.
\end{aligned}$$

For part (iii), we have $\Sigma^{-1} = \left(2^\alpha(4 - 2^\alpha)\right)^{-1} \cdot \begin{pmatrix} 2 & 2^\alpha - 2 \\ 2^\alpha - 2 & 2 \end{pmatrix}$; thus $\Sigma^{-1}\mathbf{1}_2 \geq \mathbf{0}$ element-wise. It then follows that $b(\mathbf{x}) \leq \mathbf{0}$ for all $\mathbf{x} \leq \mathbf{0}$. Since g^- is nonzero only on $\mathbf{x} \leq \mathbf{0}$, this yields the following upper bound:

$$g^-(\mathbf{x}; \delta, \lambda) \leq q(\delta, \lambda)^{-1} f(\mathbf{x}; \delta) \int_0^\infty \lambda \exp\left\{-\frac{cz^2 + 2z(\lambda - c)\delta^\alpha}{2\delta^\alpha}\right\} \mathrm{d}z.$$

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Now, after applying the substitution $z := \delta^{\alpha/2}z$, we find that for any $\varepsilon > 0$ and all δ small enough,

$$\int_0^\infty \lambda \exp\left\{-\frac{cz^2 + 2z(\lambda - c)\delta^\alpha}{2\delta^\alpha}\right\} \mathrm{d}z = \int_0^\infty \lambda\delta^{\alpha/2} \exp\left\{-\frac{cz^2 + 2\delta^{\alpha/2}z(\lambda - c)}{2}\right\} \mathrm{d}z$$
$$< \frac{\lambda\sqrt{\pi}}{\sqrt{2c}}((1+\varepsilon)\cdot\delta^{\alpha/2}.$$

Hence, using part (i) and substituting $c = 2/(4 - 2^\alpha)$ as in Equation (11), we obtain

$$g^-(\mathbf{x}; \delta, \lambda) \le \tfrac{1}{2}\lambda\sqrt{\pi(4-2^\alpha)}(1+\varepsilon)\cdot\delta^{\alpha/2}q(\delta, \lambda)^{-1}p(\delta)f^-(\mathbf{x}; \delta) \tag{25}$$

for all $\delta > 0$ sufficiently small. The proof is concluded by noting that $p(\delta) \to \mathbb{P}\{B_\alpha(-1) < 0, B_\alpha(1) < 0\} > 0$ and $\delta^{\alpha/2}q(\delta, \lambda)^{-1} = O(1)$, by Lemma 4. □

Proof of Lemma 6. After some algebraic transformations, from (12) we find that

$$c^-(k) + c^+(k) = \frac{2 + 2|k|^\alpha - (|k|-1)^\alpha - (|k|+1)^\alpha}{4 - 2^\alpha}, \quad k \in \mathbb{Z}\setminus\{0\}.$$

Let $f(x) = 2x^\alpha - (x-1)^\alpha - (x+1)^\alpha$. For $x > 1$ we have

$$\begin{aligned} f'(x) &= \alpha x^{\alpha-1}\left(2 - (1-1/x)^{\alpha-1} - (1+1/x)^{\alpha-1}\right) \\ &= \alpha x^{\alpha-1}\left[2 - \sum_{n=0}^\infty (-1)^n \frac{(\alpha-1)\cdots(\alpha-n)}{n!}x^{-n} - \sum_{n=0}^\infty \frac{(\alpha-1)\cdots(\alpha-n)}{n!}x^{-n}\right] \\ &= \alpha x^{\alpha-3}(\alpha-1)(2-\alpha)\left[\frac{1}{2!} + \sum_{n=1}^\infty \frac{(\alpha-3)\cdots(\alpha-2(n+1))}{(2(n+1))!}x^{-2n}\right]. \end{aligned}$$

We see that each of the terms in the sum above is positive, so $\operatorname{sgn}(f'(x)) = \operatorname{sgn}(\alpha - 1)$. Thus, $f'(x)$ is negative for $\alpha \in (0, 1)$ and positive for $\alpha \in (1, 2)$. Finally, since

$$\lim_{x\to\infty} \frac{2 + 2x^\alpha - (x-1)^\alpha - (x+1)^\alpha}{4 - 2^\alpha} = \frac{2}{4 - 2^\alpha} = (2 - 2^{\alpha-1})^{-1}$$

and $c^-(1) + c^+(1) = 1$, the claim follows. □

Proof of Lemma 8. It is sufficient to show that for any $\eta > 0$,

$$\frac{\partial}{\partial\eta}\left(\sum_{k=1}^\infty \frac{\Psi\left(\sqrt{\frac{\eta k}{2}}\right)}{k}\right) = \sum_{k=1}^\infty \frac{\partial}{\partial\eta}\left(\frac{\Psi\left(\sqrt{\frac{\eta k}{2}}\right)}{k}\right).$$

Take $a, b > 0$ such that $\eta \in [a, b]$, $f(\eta) = \sum_{k=1}^\infty (\Psi(\sqrt{\eta k/2})/k)$ and $f_n(\eta) = \sum_{k=1}^n (\Psi(\sqrt{\eta k/2})/k)$, $n \in \mathbb{N}$. According to [30, paragraph 3.1, p. 385], to claim the line above it is enough to show that (1) there exists $\eta_0 \in [a, b]$ such that the sequence $\{f_n(\eta_0)\}_{n\in\mathbb{N}}$ converges to a finite limit, and (2) ${f'}_n(\eta)$, $\eta \in [a, b]$, converge uniformly to some function.

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The first condition holds since $\Psi(x) < e^{-x^2/2}$ for $x > 0$. For the second condition we need to prove that, uniformly for all $\eta \in [a, b]$, it holds that $\sum_{k=n+1}^{\infty} f'_k(\eta) \to 0$ as $n \to \infty$. We have

$$\sum_{k=n+1}^{\infty} f'_k(\eta) = \sum_{k=n+1}^{\infty} \frac{\phi(\sqrt{\eta k/2})}{2\sqrt{2k\eta}} = \sum_{k=n+1}^{\infty} \frac{e^{-\eta k/4}}{4\sqrt{\pi k\eta}} \leq \mathcal{C}e^{-\mathcal{C}_1 n} \to 0, \quad n \to \infty,$$

so the claim holds. □

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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