3

C^*-algebras of discrete groups

Group representations are one of the main sources of examples of C^*-algebras. The universal representation of a group G gives rise to the full or maximal C^*-algebra $C^*(G)$, while the left regular representation leads to the reduced C^*-algebra $C^*_\lambda(G)$. In this chapter we review some of their main properties when G is a discrete group.

3.1 Full (=Maximal) group C^*-algebras

We first recall some classical notation from noncommutative Abstract Harmonic Analysis on an arbitrary discrete group G.

We denote by e (and sometimes by e_G) the unit element. Let $\pi : G \to B(\mathcal{H})$ be a unitary representation of G. We denote by $C^*_\pi(G)$ the C^*-algebra generated by the range of π.

Equivalently, $C^*_\pi(G)$ is the closed linear span of $\pi(G)$.

In particular, this applies to the so-called universal representation of G, a notion that we now recall. Let $(\pi_j)_{j\in I}$ be a family of unitary representations of G, say

$$\pi_j : G \to B(H_j)$$

in which every equivalence class of a cyclic unitary representation of G has an equivalent copy. Now one can define the "universal" representation $U_G : G \to B(\mathcal{H})$ of G by setting

$$U_G = \oplus_{j\in I}\pi_j \quad \text{on} \quad \mathcal{H} = \oplus_{j\in I} H_j.$$

Then the associated C^*-algebra $C^*_{U_G}(G)$ is simply denoted by $C^*(G)$ and is called the "full" (or the "maximal") C^*-algebra of the group G, to distinguish it from the "reduced" one that is described in the sequel. Note that

$$C^*(G) = \overline{\text{span}}\{U_G(t) \mid t \in G\}.$$

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Let π be any unitary representation of G. By a classical argument, π is unitarily equivalent to a direct sum of cyclic representations, hence for any finitely supported function $x : G \to \mathbb{C}$ we have

$$\left\|\sum x(t)\pi(t)\right\| \leq \left\|\sum x(t)U_G(t)\right\|. \tag{3.1}$$

In particular, if π is the trivial representation

$$\left|\sum x(t)\right| \leq \left\|\sum x(t)U_G(t)\right\|. \tag{3.2}$$

Equivalently (3.1) means

$$\left\|\sum x(t)U_G(t)\right\|_{B(\mathcal{H})} = \sup\left\{\left\|\sum x(t)\pi(t)\right\|_{B(H_\pi)}\right\}$$

where the supremum runs over all possible unitary representations $\pi : G \to B(H_\pi)$ on an arbitrary Hilbert space H_π. More generally, for any Hilbert space K and any finitely supported function $x : G \to B(K)$ we have

$$\left\|\sum x(t) \otimes U_G(t)\right\|_{B(K\otimes_2\mathcal{H})} = \sup\left\{\left\|\sum x(t) \otimes \pi(t)\right\|_{B(K\otimes_2 H_\pi)}\right\}$$

where the sup is the same as before.

There is an equivalent description in terms of the group algebra $\mathbb{C}[G]$, the elements of which are simply the formal linear combinations of the elements of G, equipped with the obvious natural $*$-algebra structure. One equips $\mathbb{C}[G]$ with the norm (actually a C^*-norm)

$$\sum\nolimits_{t\in G} x(t)t \mapsto \sup\left\{\left\|\sum x(t)\pi(t)\right\|\right\}$$

where the supremum runs over all possible unitary representations π of G. One can then define $C^*(G)$ as the completion of $\mathbb{C}[G]$ with respect to the latter norm.

These formulae show that the norm of $C^*(G)$ is the largest possible C^*-norm on $\mathbb{C}[G]$. Whence the term "maximal" C^*-algebra of G.

Remark 3.1 (A recapitulation) By (3.1) there is a $1-1$ correspondence between the unitary representations $\pi : G \to B(H)$ and the $*$-homomorphisms $\psi : C^*(G) \to B(H)$. More precisely, for any π there is a unique $\psi : C^*(G) \to B(H)$ such that $\forall g \in G \quad \psi(U_G(g)) = \pi(g)$, or if we view G as a subset of $\mathbb{C}[G] \subset C^*(G)$ (which means we identify g and $U_G(g)$), we have

$$\forall g \in G \quad \pi(g) = \psi(g).$$

Remark 3.2 (c.b. and c.p. maps on $C^*(G)$) A linear map $u : C^*(G) \to B(K)$ is c.b. if and only if there exists a unitary group representation $\pi : G \to B(H_\pi)$ and operators $V, W : K \to H_\pi$ such that

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$$\forall t \in G \quad u(U_G(t)) = W^*\pi(t)V.$$

Moreover, we have $\|u\|_{cb} = \inf\{\|W\|\|V\|\}$ and the infimum is attained. Indeed, in view of the preceding remark this follows immediately from Theorem 1.50. The c.p. case is characterized similarly but with $V = W$. When $K = \mathbb{C}$ and hence $B(K) = \mathbb{C}$, this gives us a description of the dual of $C^*(G)$, as well as a characterization of states on $C^*(G)$.

The next result (in which we illustrate the preceding remark in the case of multipliers) is classical, and fairly easy to check.

Proposition 3.3 (Multipliers on $C^*(G)$) *Let $\varphi : G \to \mathbb{C}$. Consider the associated linear operator M_φ (a so-called multiplier, see §3.4) defined on span$\{U_G(t) \mid t \in G\}$ by $M_\varphi\big(\sum x(t)U_G(t)\big) = \sum x(t)\varphi(t)U_G(t)$. Then M_φ extends to a bounded operator on $C^*(G)$ if and only if there are a unitary representation $\pi : G \to B(H_\pi)$ and ξ, η in H_π such that*

$$\forall\, t \in G \quad \varphi(t) = \langle \eta, \pi(t)\xi \rangle. \tag{3.3}$$

Moreover we have for the resulting bounded operator (still denoted by M_φ)

$$\|M_\varphi\| = \|M_\varphi\|_{cb} = \inf\{\|\xi\|\|\eta\|\} \tag{3.4}$$

where the infimum (which is attained) runs over all possible π, ξ, η for which this holds. Lastly, if M_φ is positive (3.3) holds with $\xi = \eta$, and then M_φ is completely positive on $C^*(G)$.

Proof If $\|M_\varphi : C^*(G) \to C^*(G)\| \le 1$, let $f(x) = \sum_{t\in G} \varphi(t)x(t)$. Then by (3.2) $f \in C^*(G)^*$ with $\|f\| \le 1$. Note $f(U_G(t)) = \varphi(t)$. By Remark 1.54 there are π, ξ, and η with $\|\xi\|\|\eta\| \le \|f\| \le 1$ such that (3.3) holds. If M_φ (and hence f) is positive we find this with $\xi = \eta$. For the converse, since (like any unitary group representation) the mapping $U_G(t) \mapsto U_G(t) \otimes \pi(t)$ extends to a continuous $*$-homomorphism $\sigma : C^*(G) \to B(\mathcal{H} \otimes_2 H_\pi)$, we have $M_\varphi(\cdot) = V_2^*\sigma(\cdot)V_1$, with $V_1 h = h \otimes \xi$ and $V_2 h = h \otimes \eta$ ($h \in \mathcal{H}$) from which we deduce by (1.30) $\|M_\varphi\|_{cb} \le \|\xi\|\|\eta\|$. If $\xi = \eta$ then $V_1 = V_2$ and hence M_φ is c.p. on $C^*(G)$. □

Remark 3.4 By Remark 3.2 and (3.4) the space of bounded multipliers on $C^*(G)$ can be identified isometrically with $C^*(G)^*$. If f_φ is the linear form on $C^*(G)$ taking $U_G(t)$ to $\varphi(t)$ ($t \in G$) we have

$$\|M_\varphi\| = \|f_\varphi\|_{C^*(G)^*}.$$

Proposition 3.5 *Let G be a discrete group and let $\Gamma \subset G$ be a subgroup. Then the correspondence $U_\Gamma(t) \to U_G(t)$, ($t \in \Gamma$) extends to an isometric*

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(C^*-algebraic) embedding J of $C^*(\Gamma)$ into $C^*(G)$. Moreover, there is a completely contractive and completely positive projection P from $C^*(G)$ onto the range of this embedding, defined by $P(U_G(t)) = U_G(t)$ for any $t \in \Gamma$ and $P(U_G(t)) = 0$ otherwise.

Proof By the universal property of $C^*(\Gamma)$ the unitary representation $\Gamma \supset \gamma \mapsto U_G(\gamma)$ extends to a $*$-homomorphism $J : C^*(\Gamma) \to C^*(G)$ with $\|J\| = 1$. Let $\varphi = 1_\Gamma$. The projection P described in Proposition 3.5 coincides with the multiplier M_φ acting on $C^*(G)$. Thus, by Proposition 3.3 it suffices to show that there is a unitary representation $\pi : G \to B(H_\pi)$ of G and a unit vector $\xi \in H_\pi$ such that $\varphi(t) = \langle \xi, \pi(t)\xi \rangle$. Let $G = \bigcup_{s \in G/\Gamma} s\Gamma$ be the disjoint partition of G into left cosets. For any $t \in G$ the mapping $s\Gamma \mapsto ts\Gamma$ defines a permutation $\sigma(t)$ of the set G/Γ, and $t \mapsto \sigma(t)$ is a homomorphism. Let $H_\pi = \ell_2(G/\Gamma)$ and let $\pi : G \to B(H_\pi)$ be the unitary representation defined on the unit vector basis by $\pi(t)(\delta_s) = \delta_{\sigma(t)(s)}$ for any $s \in G/\Gamma$. Let $[[\Gamma]] \in G/\Gamma$ denote the coset Γ (i.e. $s\Gamma$ for $s = 1_G$) and let $\xi = \delta_{[[\Gamma]]}$. Then it is immediate that $\varphi(t) = \langle \xi, \pi(t)\xi \rangle$ for any $t \in G$. □

Remark 3.6 Let G be a discrete group and let $E \subset C^*(G)$ be any separable subspace. We claim that there is a countable subgroup $\Gamma \subset G$ such that with the notation of Proposition 3.5 we have $E \subset J(C^*(\Gamma))$. Indeed, since $C^*(G) \subset \overline{\mathrm{span}}[U_G(t) \mid t \in G]$ for any fixed $x \in C^*(G)$ there is clearly a countable subgroup $\Gamma_x \subset G$ and an analogous J_x such that $x \subset J_x(C^*(\Gamma_x))$. Arguing like this for each x in a dense countable sequence in E and taking the group generated by all the resulting Γ_x's gives us the claim.

By Proposition 3.5 this shows that there is a separable C^*-subalgebra $C \subset C^*(G)$ with $E \subset C$ for which there is a c.p. projection $P : C^*(G) \to C$.

Remark 3.7 Let G be any discrete group, let $A = C^*(G)$. Then $\bar{A} \simeq A$. Indeed, since for any unitary representation π on G, the complex conjugate $\bar{\pi}$ (as in Remark 2.14) is also a unitary representation, the correspondence $\pi \mapsto \bar{\pi}$ is a bijection on the set of unitary representations, from which the $\mathbb{C}$-linear isomorphism $\Phi : C^*(G) \to \overline{C^*(G)}$ follows immediately. Denoting by U_G the universal representation of G, this isomorphism takes $U_G(t)$ to $\overline{U_G(t)}$. Note that $\bar{A} \simeq A$ is in general not true (see [60]).

3.2 Full C^*-algebras for free groups

In this section, we start by comparing the C^*-algebras of free groups of different cardinals. Our goal is to make clear that we can restrict to $\mathscr{C} = C^*(\mathbb{F}_\infty)$ (or if we wish to $C^*(\mathbb{F}_2)$) for the various properties of interest to us in the

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sequel. Then we describe the operator space structure of the span of the free generators in $C^*(\mathbb{F})$ when $\mathbb{F}$ is any free group. The following simple lemma will be often invoked when we wish to replace $C^*(\mathbb{F})$ by $C^*(\mathbb{F}_\infty)$.

Lemma 3.8 *Let $\mathbb{F}$ be a free group with generators $(g_i)_{i\in I}$. Let $E \subset C^*(\mathbb{F})$ be any separable subspace. Then the inclusion $E \subset C^*(\mathbb{F})$ admits an extension $T_E : C^*(\mathbb{F}) \to C^*(\mathbb{F})$ that can be factorized as*

$$T_E : C^*(\mathbb{F}) \xrightarrow{w} C^*(\mathbb{F}_\infty) \xrightarrow{v} C^*(\mathbb{F})$$

where v, w are contractive c.p. maps.

For any C^*-algebra D and any $x \in D \otimes E$ we have

$$\|x\|_{D\otimes_{\max} C^*(\mathbb{F})} = \|(Id_D \otimes w)(x)\|_{D\otimes_{\max} C^*(\mathbb{F}_\infty)}. \tag{3.5}$$

In particular, $E \subset C^*(\mathbb{F})$ is completely isometric to $w(E) \subset C^*(\mathbb{F}_\infty)$.

Proof For any $x \in C^*(\mathbb{F})$ there is clearly a *countable* subgroup $\Gamma_x \subset \mathbb{F}$ such that

$$x \in \overline{\mathrm{span}}[U_{\mathbb{F}}(t) \mid t \in \Gamma_x].$$

By the separability of E, we can find a countable subgroup Γ such that $E \subset \overline{\mathrm{span}}[U_{\mathbb{F}}(t) \mid t \in \Gamma]$. Since any element of $t \in \Gamma$ can be written using only finitely many "letters" in $\{g_i \mid i \in I\}$, we may assume that Γ is the free subgroup generated by $(g_i)_{i\in I'}$ for some countable subset $I' \subset I$. Then, identifying $\overline{\mathrm{span}}[U_{\mathbb{F}}(t) \mid t \in \Gamma]$ with $C^*(\Gamma)$, Proposition 3.5 yields a mapping $T = JP : C^*(\mathbb{F}) \to C^*(\mathbb{F})$ with the required factorization through $C^*(\Gamma) = C^*(\mathbb{F}_{I'})$ that is the identity when restricted to E. If I' is infinite the proof is complete: since $C^*(\mathbb{F}_{I'}) = C^*(\mathbb{F}_\infty)$ we may take $T_E = T$.

Otherwise, we note that $\mathbb{F}_{I'} \subset \mathbb{F}_\infty$ as a subgroup and hence by Proposition 3.5 again we have a factorization of the same type $C^*(\mathbb{F}_{I'}) \xrightarrow{J'} C^*(\mathbb{F}_\infty) \xrightarrow{P'} C^*(\mathbb{F}_{I'})$ from which it is easy to conclude.

Note

$$\|x\|_{D\otimes_{\max} C^*(\mathbb{F})} = \|(Id_D \otimes T_E)(x)\|_{D\otimes_{\max} C^*(\mathbb{F})} = \|(Id_D \otimes vw)(x)\|_{D\otimes_{\max} C^*(\mathbb{F})}.$$

By Corollary 4.18 since v, w are c.p. contractions we have

$$\|x\|_{D\otimes_{\max} C^*(\mathbb{F})} \le \|(Id_D \otimes w)(x)\|_{D\otimes_{\max} C^*(\mathbb{F}_\infty)} \quad \text{and}$$
$$\|(Id_D \otimes w)(x)\|_{D\otimes_{\max} C^*(\mathbb{F}_\infty)} \le \|x\|_{D\otimes_{\max} C^*(\mathbb{F})},$$

from which (3.5) follows. □

Let $\mathbb{F}$ be a free group with generators $(g_i)_{i\in I}$. We start with a basic property of the span of the free generators in $C^*(\mathbb{F})$.

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Lemma 3.9 *Let $\mathbb{F}$ be a free group with generators $(g_i)_{i\in I}$. Let $U_i = U_{\mathbb{F}}(g_i) \in C^*(\mathbb{F})$. Let $E = \mathrm{span}[(U_i)_{i\in I}, 1] \subset C^*(\mathbb{F})$ and $E_I = \mathrm{span}[(U_i)_{i\in I}] \subset C^*(\mathbb{F})$. Then for any linear map $u : E \to B(H)$ and any $v : E_I \to B(H)$ we have*

$$\begin{aligned} \|u\|_{cb} &= \|u\| = \max\{\sup_{i\in I} \|u(U_i)\|, \|u(1)\|\} \quad \textit{and} \\ \|v\|_{cb} &= \|v\| = \max\{\sup_{i\in I} \|v(U_i)\|\}. \end{aligned} \tag{3.6}$$

Proof It clearly suffices to show that $\max\{\sup_{i\in I} \|u(U_i)\|, \|u(1)\|\} \le 1$ implies $\|u\|_{cb} \le 1$. When $u(1) = 1$ and all $u(U_i)$ are unitaries this is easy: indeed there is a (unique) group representation $\sigma : \mathbb{F} \to B(H)$ such that $\sigma(g_i) = u(U_i)$ and the associated linear extension $u_\sigma : C^*(\mathbb{F}) \to B(H)$ is a $*$-homomorphism automatically satisfying $\|u_\sigma\|_{cb} = 1$, and hence $\|u\|_{cb} = 1$. This same argument works if we merely assume that $u(1)$ is unitary. Indeed, we may replace u by $x \mapsto u(1)^{-1}u$, which takes us back to the previous easy case. Since the general case is easy to reduce to that of a finite set, we assume that I is finite. Then the Russo–Dye Theorem A.18 shows us that any u such that $\max\{\sup_{i\in I} \|u(U_i)\|, \|u(1)\|\} \le 1$ lies in the closed convex hull of u's for which $u(1)$ and all the $u(U_i)$s are unitaries, and hence $\|u\|_{cb} \le 1$ in that case also. □

The first part of the next result is based on the classical observation that a unitary representation $\pi : \mathbb{F} \to B(H)$ is entirely determined by its values $u_i = \pi(g_i)$ on the generators, and if we let π run over all possible unitary representations, then we obtain all possible families (u_i) of unitary operators. The second part is also well known.

Lemma 3.10 *Let $A \subset B(H)$ be a C^*-algebra. Let $\mathbb{F}$ be a free group with generators $(g_i)_{i\in I}$. Let $U_i = U_{\mathbb{F}}(g_i) \in C^*(\mathbb{F})$. Let $(x_i)_{i\in I}$ be a family in A with only finitely many nonzero terms. Consider the linear map $T : \ell_\infty(I) \to A$ defined by $T((\alpha_i)_{i\in I}) = \sum_{i\in I} \alpha_i x_i$. Then we have*

$$\left\|\sum\nolimits_{i\in I} U_i \otimes x_i\right\|_{C^*(\mathbb{F})\otimes_{\min} A} = \|T\|_{cb} = \sup\left\{\left\|\sum u_i \otimes x_i\right\|_{\min}\right\} \tag{3.7}$$

where the sup runs over all possible Hilbert spaces K and all families (u_i) of unitaries on K. Actually, the latter supremum remains the same if we restrict it to finite-dimensional Hilbert spaces K. Moreover, in the case when $A = B(H)$ with $\dim(H) = \infty$, we have

$$\left\|\sum\nolimits_{i\in I} U_i \otimes x_i\right\|_{C^*(\mathbb{F})\otimes_{\min} B(H)} = \inf\left\{\left\|\sum y_i y_i^*\right\|^{1/2} \left\|\sum z_i^* z_i\right\|^{1/2}\right\} \tag{3.8}$$

where the infimum, which runs over all possible factorizations $x_i = y_i z_i$ with y_i, z_i in $B(H)$, is actually attained.

Moreover, all this remains true if we enlarge the family $(U_i)_{i\in I}$ by including the unit element of $C^*(\mathbb{F})$.

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Proof It is easy to check going back to the definitions that on one hand

$$\left\|\sum U_i \otimes x_i\right\|_{\min} = \sup\left\{\left\|\sum u_i \otimes x_i\right\|_{\min}\right\},$$

where the sup runs over all possible families of unitaries (u_i), and on the other hand that

$$\|T\|_{cb} = \sup\left\{\left\|\sum t_i \otimes x_i\right\|_{\min}\right\},$$

where the sup runs over all possible families of contractions (t_i). By the Russo–Dye Theorem A.18, any contraction is a norm limit of convex combinations of unitaries, so (3.7) follows by convexity. Actually, the preceding sup obviously remains unchanged if we let it run only over all possible families of contractions (t_i) on a *finite-dimensional* Hilbert space. Thus it remains unchanged when restricted to families of *finite-dimensional* unitaries (u_i).

Now assume $\|T\|_{cb} = 1$. By the factorization of $c.b.$ maps we can write $T(\alpha) = V^*\pi(\alpha)W$ where $\pi : \ell_\infty(I) \to B(\widehat{H})$ is a representation and where V, W are in $B(H, \widehat{H})$ with $\|V\|\,\|W\| = \|T\|_{cb}$. Since we assume $\dim(H) = \infty$ and may assume I finite (because $i \mapsto x_i$ is finitely supported), by Remark 1.51 we may as well take $\widehat{H} = H$. Let $(e_i)_{i\in I}$ be the canonical basis of $\ell_\infty(I)$, we set

$$y_i = V^*\pi(e_i) \quad \text{and} \quad z_i = \pi(e_i)W.$$

It is then easy to check $\left\|\sum y_i y_i^*\right\|^{1/2} \left\|\sum z_i^* z_i\right\|^{1/2} \leq \|V\|\,\|W\| = \|T\|_{cb}$. Thus we obtain one direction of (3.8). The converse follows from (2.2) (easy consequence of Cauchy–Schwarz) applied to $a_i = U_i \otimes y_i$ and $b_i = 1 \otimes z_i$. Finally, the last assertion follows from the forthcoming Remark 3.12. □

Remark 3.11 (Russo–Dye) The Russo–Dye Theorem A.18 shows that the sup of any continuous convex function on the unit ball of a unital C^*-algebra coincides with its sup over all its unitary elements.

Remark 3.12 Let $\{0\}$ be a singleton disjoint from the set I and let $\dot{I} = \{0\} \cup I$. Then for any finitely supported family $\{x_j \mid j \in \dot{I}\}$ in $B(H)$ (H arbitrary) we have

$$\left\|I \otimes x_0 + \sum\nolimits_{i\in I} U_i \otimes x_i\right\|_{\min} = \sup\left\{\left\|\sum\nolimits_{j\in\dot{I}} u_j \otimes x_j\right\|_{\min}\right\} \tag{3.9}$$

where the supremum runs over all possible families $(u_j)_{j\in\dot{I}}$ of unitaries.

Indeed, since

$$\left\|\sum\nolimits_{j\in\dot{I}} u_j \otimes x_j\right\|_{\min} = \left\|I \otimes x_0 + \sum\nolimits_{i\in I} u_0^{-1}u_i \otimes x_i\right\|_{\min},$$

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the right-hand side of (3.9) is the same as the supremum of

$$\left\| I \otimes x_0 + \sum\nolimits_{i \in I} u_i \otimes x_i \right\|_{\min} \tag{3.10}$$

over all possible families of unitaries $(u_i)_{i \in I}$. Therefore (recalling $U(g_i) = U_i$) we find

$$\left\| I \otimes x_0 + \sum\nolimits_{i \in I} U_i \otimes x_i \right\|_{\min} = \sup \left\{ \left\| I \otimes x_0 + \sum\nolimits_{i \in I} u_i \otimes x_i \right\|_{\min} \middle| u_i \text{ unitary} \right\},$$

where the sup runs over all Hilbert spaces $\mathcal{H}$ and all families (u_i) of unitaries in $B(\mathcal{H})$.

Moreover, by the same argument we used for Lemma 3.10, we can restrict to finite-dimensional $\mathcal{H}$'s:

$$\left\| I \otimes x_0 + \sum\nolimits_{i \in I} U_i \otimes x_i \right\|_{\min} = \sup\nolimits_{n \geq 1} \left\{ \left\| I \otimes x_0 + \sum\nolimits_{i \in I} u_i \otimes x_i \right\|_{\min} \middle| u_i \ n \times n \text{ unitaries} \right\} \tag{3.11}$$

so that the supremum on the right-hand side is restricted to families of *finite-dimensional* unitaries. Indeed, by Russo–Dye (Remark 3.11) the suprema of (3.10) taken over u_i's in the unit ball of $B(H)$ and over unitary u_i's are the same. Replacing u_i by $P_E u_{i|E}$ with $E \subset H$, $\dim(E) < \infty$ shows that the supremum of (3.10) is the same if we restrict it to u_i's in the unit ball of $B(E)$ with $\dim(E) < \infty$. Then, invoking Russo–Dye (Remark 3.11) again, we obtain (3.11).

Remark 3.13 Using (3.11) when I is a singleton and the fact that a single unitary generates a commutative unital C^*-algebra, it is easy to check that $\|T\| = \|T\|_{cb}$ for any $T : \ell_\infty^2 \to B(H)$.

Remark 3.14 ($\ell_1(I)$ as operator space) In the particular case $A = \mathbb{C}$, (3.7) becomes

$$\left\| \sum\nolimits_{i \in I} U_i x_i \right\|_{C^*(\mathbb{F})} = \sum\nolimits_{i \in I} |x_i|, \tag{3.12}$$

which shows that $E_I = \overline{\operatorname{span}}[U_i, i \in I] \simeq \ell_1(I)$ isometrically.

Note that (3.8) generalizes the classical fact that $B_{\ell_1} = B_{\ell_2} B_{\ell_2}$ for the pointwise product.

More generally, Lemma 3.9 shows that the dual operator space E_I^* can be identified with the von Neumann algebra $\ell_\infty(I)$ equipped with its natural operator space structure as a C^*-algebra, i.e. the one such that we have $M_n(\ell_\infty(I)) = \ell_\infty(I; M_n)$ isometrically for all n. Lemma 3.10 describes the

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dual operator space of the operator space (actually a C^*-subalgebra) $c_0(I) \subset \ell_\infty(I)$ that is the closed span of the canonical basis in $\ell_\infty(I)$. We obtain $c_0(I)^* = E_I$ completely isometrically, which is the operator space analogue of the isometric identity $c_0(I)^* = \ell_1(I)$. Indeed, together with Lemma 3.9, (3.7) tells us that $CB(c_0(I), M_n) = M_n(E_I)$ isometrically for all n.

3.3 Reduced group C^*-algebras: Fell's absorption principle

We denote by $C^*_\lambda(G)$ (resp. $C^*_\rho(G)$) the so-called reduced C^*-algebra generated in $B(\ell_2(G))$ by λ_G (resp. ρ_G). Equivalently, $C^*_\lambda(G) = \overline{\text{span}}\{\lambda_G(t) | \, t \in G\}$ and $C^*_\rho(G) = \overline{\text{span}}\{\rho_G(t) \mid t \in G\}$. Note that $\lambda_G(t)$ and $\rho_G(s)$ commute for all t, s in G.

We denote λ_G and ρ_G simply by λ and ρ (and U_G by U) when there is no ambiguity.

The following very useful result is known as Fell's "absorption principle."

Proposition 3.15 *For any unitary representation $\pi : G \to B(H)$, we have*

$$\lambda_G \otimes \pi \simeq \lambda_G \otimes I \quad \textit{(unitary equivalence).}$$

Here I stands for the trivial representation of G in $B(H)$ (i.e. $I(t) = Id_H$ $\forall t \in G$). In particular, for any finitely supported functions $a : G \to \mathbb{C}$ and $b : G \to B(\ell_2)$, we have

$$\left\| \sum a(t)\lambda_G(t) \otimes \pi(t) \right\|_{C^*_\lambda(G) \otimes_{\min} B(H)} = \left\| \sum a(t)\lambda_G(t) \right\|, \tag{3.13}$$

$$\left\| \sum b(t) \otimes \lambda_G(t) \otimes \pi(t) \right\|_{B(\ell_2) \otimes_{\min} C^*_\lambda(G) \otimes_{\min} B(H)} = \left\| \sum b(t) \otimes \lambda_G(t) \right\|_{B(\ell_2) \otimes_{\min} C^*_\lambda(G)}.$$

Proof Note that $\lambda_G \otimes \pi$ acts on the Hilbert space $K = \ell_2(G) \otimes_2 H \simeq \ell_2(G; H)$. Let $V : K \to K$ be the unitary operator taking $x = (x(t))_{t \in G}$ to $(\pi(t^{-1})x(t))_{t \in G}$. A simple calculation shows that

$$V^{-1}(\lambda_G(t) \otimes Id_H)V = \lambda_G(t) \otimes \pi(t).$$

□

We will often use the following immediate consequence:

Corollary 3.16 *For any unitary representation $\pi : G \to B(H)$, the linear map*

$$\sigma_\pi : \text{span}[\lambda_G(G)] \to B(\ell_2(G) \otimes_2 H) \textit{ defined by}$$
$$\sigma_\pi(\lambda_G(g)) = \lambda_G(t) \otimes \pi(t) \ (\forall t \in G)$$

extends to a (contractive) $*$-homomorphism from $C^*_\lambda(G)$ to $B(\ell_2(G) \otimes_2 H)$.

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Remark 3.17 Let $\mathbb{F}$ be a free group with free generators (g_j). Then for any finitely supported sequence of scalars (a_j), for any H and for any family (u_j) of unitary operators in $B(H)$ we have

$$\left\|\sum a_j\lambda(g_j)\otimes u_j\right\|_{\min} = \left\|\sum a_j\lambda(g_j)\right\|.$$

Indeed, this follows from (3.13) applied to the function a defined by $a(g_j) = a_j$ and $= 0$ elsewhere, and to the unique unitary representation π of F such that $\pi(g_j) = u_j$.

Proposition 3.18 *Let G be a discrete group and let $\Gamma \subset G$ be a subgroup. Then the correspondence $\lambda_\Gamma(t) \to \lambda_G(t)$, $(t \in \Gamma)$ extends to an isometric (C^*-algebraic) embedding $J_\lambda : C^*_\lambda(\Gamma) \to C^*_\lambda(G)$. Moreover there is a completely contractive and completely positive projection P_λ from $C^*_\lambda(G)$ onto the range of this embedding, taking $\lambda_G(t)$ to 0 for any $t \notin \Gamma$.*

Proof Let $Q = G/\Gamma$ and let $G = \bigcup_{q\in Q} \Gamma g_q$ be the partition of G into (disjoint) right cosets. For convenience, let us denote by 1 the equivalence class of the unit element of G. Since $G \simeq \Gamma \times Q$, we have an identification

$$\ell_2(G) \simeq \ell_2(\Gamma) \otimes_2 \ell_2(Q)$$

such that

$$\forall\, t \in \Gamma \quad \lambda_G(t) = \lambda_\Gamma(t) \otimes I.$$

This shows of course that J_λ is an isometric embedding. Moreover, we have a natural (linear) isometric embedding $V : \ell_2(\Gamma) \to \ell_2(G)$ (note that the range of V coincides with $\ell_2(\Gamma) \otimes \delta_1$ in the preceding identification), such that $\lambda_\Gamma(t) = V^*\lambda_G(t)V$ for all $t \in \Gamma$. Let $u(x) = V^*xV$. Clearly for any $t \in G$ we have $u(\lambda_G(t)) = \lambda_\Gamma(t)$ if $t \in \Gamma$ and $u(\lambda_G(t)) = 0$ if $t \notin \Gamma$. Therefore $P_\lambda = J_\lambda u$ is the announced completely positive and completely contractive projection from $C^*_\lambda(G)$ onto $J_\lambda C^*_\lambda(\Gamma)$. $\square$

As an immediate application, we state for further use the following particular case:

Corollary 3.19 (The diagonal subgroup in $G \times G$) *Let $\Delta = \{(g, g) \mid g \in G\} \subset G \times G$ be the diagonal subgroup. There are:*

- *a complete isometry $J^\Delta : C^*_\lambda(G) \to C^*_\lambda(G) \otimes_{\min} C^*_\lambda(G)$ such that $J^\Delta(\lambda_G(t)) = \lambda_G(t) \otimes \lambda_G(t)$, and*
- *a c.p. map $Q^\Delta : C^*_\lambda(G) \otimes_{\min} C^*_\lambda(G) \to C^*_\lambda(G)$ with $\|Q^\Delta\| = 1$ such that $Q^\Delta(\lambda_G(t) \otimes \lambda_G(s)) = 0$ whenever $s \neq t$ and $Q^\Delta(\lambda_G(t) \otimes \lambda_G(t)) = \lambda_G(t)$.*

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Proof We apply Proposition 3.18 to the subgroup Δ and we use the identification

$$C^*_\lambda(G) \otimes_{\min} C^*_\lambda(G) \simeq C^*_\lambda(G \times G)$$

which follows easily from the definition of both sides (see §4.3 for more such identifications). □

The projection P_λ in the preceding proposition is an example of mapping associated to a "multiplier."

3.4 Multipliers

Let $\varphi : G \to \mathbb{C}$ be a (bounded) function and let π be a unitary representation of G. Let M_φ be the linear mapping defined on the linear span of $\{\pi(t) \mid t \in G\}$ by

$$\forall t \in G \quad M_\varphi(\pi(t)) = \varphi(t)\pi(t).$$

As anticipated in Proposition 3.3, we say that φ is a bounded (resp. c.b. rresp. c.p.) multiplier on $C^*_\pi(G)$ if M_φ extends to a bounded (resp. c.b. rresp. c.p.) linear map on $C^*_\pi(G)$.

We will be mainly interested in the cases when $\pi = \lambda_G$ or $\pi = U_G$.

In the commutative case (or when G is amenable) the bounded or c.b. multipliers of $C^*_\lambda(G)$ coincide with the linear combinations of positive definite functions, and the latter, as we explain next, are the c.p. multipliers. However, in general the situation is more complicated. The next statement characterizes the c.b. case. We may even include $B(H)$-valued multipliers.

Theorem 3.20 ([35, 136]) *Let G be a discrete group, H a Hilbert space. The following properties of a function $\varphi : G \to B(H)$ are equivalent:*

(i) *The linear mapping defined on* $\mathrm{span}[\lambda(t) \mid t \in G]$ *by*

$$M_\varphi(\lambda(t)) = \lambda(t) \otimes \varphi(t)$$

extends to a c.b. map
$M_\varphi : C^*_\lambda(G) \to C^*_\lambda(G) \otimes_{\min} B(H) \subset B(\ell_2(G) \otimes_2 H)$ *with* $\|M_\varphi\|_{cb} \le 1$.

(ii) *There is a Hilbert space $\widehat{H}$ and bounded functions $x : G \to B(H, \widehat{H})$ and $y : G \to B(H, \widehat{H})$ with* $\sup_{t \in G} \|x(t)\| \le 1$ *and* $\sup_{s \in G} \|y(s)\| \le 1$ *such that*

$$\varphi(s^{-1}t) = y(s)^* x(t). \qquad \forall\, s, t \in G$$

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Proof Assume (i). Then by Theorem 1.50 there are a Hilbert space $\widehat{H}$, a representation $\pi : C^*_\lambda(G) \to B(\widehat{H})$ and operators $V_j : \ell_2(G) \otimes_2 H \to \widehat{H}$ $(j = 1,2)$ with $\|V_1\|\|V_2\| \le 1$ such that

$$\forall\, \theta \in G \quad \lambda(\theta) \otimes \varphi(\theta) = M_\varphi(\lambda(\theta)) = V_2^* \pi(\lambda(\theta)) V_1. \tag{3.14}$$

We will use this for $\theta = s^{-1}t$, in which case we have $\langle \delta_{s^{-1}}, \lambda(\theta)\delta_{t^{-1}} \rangle = 1$. We define $x(t) \in B(H,\widehat{H})$ and $y(s) \in B(H,\widehat{H})$ by $x(t)h = \pi(\lambda(t))\ V_1(\delta_{t^{-1}} \otimes h)$ and $y(s)k = \pi(\lambda(s))V_2(\delta_{s^{-1}} \otimes k)$. Note that when $\theta = s^{-1}t$

$$\langle \delta_{s^{-1}} \otimes k, (\lambda(\theta) \otimes \varphi(\theta))(\delta_{t^{-1}} \otimes h) \rangle = \langle k, \varphi(s^{-1}t)h \rangle,$$

and hence (3.14) implies

$$\langle k, \varphi(s^{-1}t)h \rangle = \langle k, y(s)^* x(t)h \rangle,$$

and we obtain (ii).

Conversely assume (ii). Define $\pi : C^*_\lambda(G) \to B(\ell_2(G) \otimes_2 \widehat{H})$ by $\pi(x) = x \otimes Id_{\widehat{H}}$. Let

$$V_j : \ell_2(G) \otimes_2 H \to \ell_2(G) \otimes_2 \widehat{H}$$

be defined by $V_1(\delta_t \otimes h) = \delta_t \otimes x(t)h$ and $V_2(\delta_s \otimes k) = \delta_s \otimes y(s)k$. Note that $\|V_1\| = \sup_{t\in G} \|x(t)\| \le 1$ and $\|V_2\| = \sup_{s \in G} \|y(s)\| \le 1$. Then for any θ, t, s, h, k we have

$$\begin{aligned}\langle \delta_s \otimes k, V_2^* \pi(\lambda(\theta)) V_1(\delta_t \otimes h) \rangle &= \langle \delta_s, \lambda(\theta)\delta_t \rangle \langle k, y(s)^* x(t)h \rangle \\ &= \langle \delta_s \otimes k, (\lambda(\theta) \otimes \varphi(\theta))(\delta_t \otimes h) \rangle,\end{aligned}$$

equivalently $V_2^* \pi(\lambda(\theta)) V_1 = M_\varphi(\lambda(\theta))$, so the converse part of Theorem 1.50 yields (ii) $\Rightarrow$ (i). □

In the particular case $\mathbb{C} = B(H)$ the preceding result yields:

Corollary 3.21 (Characterization of c.b. multipliers on $C^*_\lambda(G)$) *Consider a complex-valued function* $\varphi : G \to \mathbb{C}$*. Then* $\|M_\varphi : C^*_\lambda(G) \to C^*_\lambda(G)\|_{cb} \le 1$ *if and only if there are Hilbert space valued functions* x, y *with* $\sup_t \|x(t)\| \le 1$ *and* $\sup_s \|y(s)\| \le 1$ *such that*

$$\varphi(s^{-1}t) = \langle y(s), x(t) \rangle. \qquad \forall\, s,t \in G$$

Remark 3.22 (On positive definiteness) A function $\varphi : G \to \mathbb{C}$ is called positive definite if for any n and any $t_1, \ldots, t_n \in G$ the $n \times n$-matrix $[\varphi(t_i^{-1}t_j)]$ is positive (semi)definite, i.e. we have

$$\forall x \in \mathbb{C}^n \quad \sum \overline{x_i} x_j \varphi(t_i^{-1}t_j) \ge 0.$$

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Equivalently

$$\forall x \in \mathbb{C}[G] \quad \sum \overline{x(s)}x(t)\varphi(s^{-1}t) \geq 0.$$

Using the scalar product defined by the latter condition, we find, after passing to the quotient and completing in the usual way, a Hilbert space H_φ and a mapping $\mathbb{C}[G] \to H_\varphi$ denoted by $x \mapsto \dot{x}$ with dense range (so that $\|\dot{x}\|^2_{H_\varphi} = \sum \overline{x(s)}x(t)\varphi(s^{-1}t)$ for all $x \in \mathbb{C}[G]$) and a unitary representation π_φ of G extending left translation on $\mathbb{C}[G]$. Let $\delta_e \in \mathbb{C}[G]$ denote the indicator function of the unit element of G. We have

$$\langle \dot{\delta}_e, \pi_\varphi(g)\dot{\delta}_e \rangle_{H_\varphi} = \langle \dot{\delta}_e, \dot{\delta}_g \rangle_{H_\varphi} = \varphi(g). \tag{3.15}$$

Thus φ is a (diagonal) matrix coefficient of π.

Conversely, any φ of the form $\varphi(g) = \langle \xi, \pi(g)\xi \rangle$ (with π unitary and $\xi \in H_\pi$) is positive definite.

Proposition 3.23 *Let $\varphi : G \to \mathbb{C}$. The following are equivalent:*

(i) *φ is a c.p. multiplier of $C^*_\lambda(G)$.*
(ii) *φ is positive definite.*

Moreover, in that case we have $\|M_\varphi\| = \|M_\varphi\|_{cb} = \varphi(e)$ where e is the unit of G.

Proof Assume (i). Let $t_1, \ldots, t_n \in G$. Consider the matrix a defined by $a_{ij} = \lambda_G(t_i)^{-1}\lambda_G(t_j)$. Clearly $a \in M_n(C^*_\lambda(G))_+$. Then $(Id_{M_n} \otimes M_\varphi)(a) = [\varphi(t_i^{-1}t_j)a_{ij}] \in M_n(C^*_\lambda(G))_+$. Therefore, for any $x_1, \ldots, x_n \in \ell_2(G)$ we have $\sum \varphi(t_i^{-1}t_j)\langle x_i, a_{ij}x_j \rangle \geq 0$. Choosing $x_j = \lambda_j \delta_{t_j^{-1}}$ ($\lambda_j \in \mathbb{C}$) we find $\langle x_i, a_{ij}x_j \rangle = \overline{\lambda_i}\lambda_j$ for all i, j, and we conclude that φ is positive definite.

Assume (ii). By (3.15) we have for any $g \in G$

$$M_\varphi(\lambda_G(g)) = \varphi(g)\lambda_G(g) = V^*([\lambda_G \otimes \pi_\varphi](g))V$$

where $V : \ell_2(G) \to \ell_2(G) \otimes_2 H_\varphi$ is defined by $V(h) = h \otimes \dot{\delta}_e$. By Corollary 3.16 we have $M_\varphi(\cdot) = V^*(\sigma_\pi(\cdot))V$, and hence M_φ is c.p. on $C^*_\lambda(G)$. Moreover $M_\varphi(1) = \varphi(e)1$, so $\|M_\varphi(1)\| = \varphi(e)$. $\square$

Remark 3.24 The reader can easily check that the preceding statement remains valid for $B(H)$-valued functions, in analogy with Theorem 3.20, for the natural extension of positive definiteness, defined by requesting that $[\varphi(t_i^{-1}t_j)] \in M_n(B(H))_+$ for all n. Such functions are sometimes called *completely* positive definite.

In the preceding construction, we associated a linear mapping M_φ to a function φ. We now go conversely. We will associate to a c.b. mapping a

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multiplier. In other words, we will describe a linear projection from the set of c.b. maps to the subspace formed by those associated to multipliers.

Proposition 3.25 (Haagerup) *Let $u : C_\lambda^*(G) \to C_\lambda^*(G)$ be a c.b. map. Then the function φ_u defined by (recall e is the unit of G)*

$$\varphi_u(t) = \langle \delta_t, u(\lambda_G(t))\delta_e \rangle$$

is a c.b. multiplier on $C_\lambda^*(G)$ with $\|M_{\varphi_u}\|_{cb} \le \|u\|_{cb}$. If u is c.p. then the multiplier is also c.p.

If u has finite rank then $\varphi_u \in \ell_2(G)$.

Moreover, if $u = M_\varphi$ then $\varphi_u = \varphi$.

Proof We have

$$\|Id_{C_\lambda^*(G)} \otimes u : C_\lambda^*(G) \otimes_{\min} C_\lambda^*(G) \to C_\lambda^*(G) \otimes_{\min} C_\lambda^*(G)\| \le \|u\|_{cb}.$$

It is easy to see that $C_\lambda^*(G) \otimes_{\min} C_\lambda^*(G)$ can be identified with $C_\lambda^*(G \times G)$. With this identification, we have, for the mappings J^Δ, Q^Δ in Corollary 3.19, for any $t \in G$

$$\varphi_u(t)\lambda_G(t) = Q^\Delta[Id_{C_\lambda^*(G)} \otimes u]J^\Delta(\lambda_G(t)).$$

In other words, $M_{\varphi_u} = Q^\Delta[Id_{C_\lambda^*(G)} \otimes u]J^\Delta$. All the assertions are now evident. We just note that if u has rank 1, say $u(x) = f(x)y$ with $f \in C_\lambda^*(G)^*$ and $y \in C_\lambda^*(G)$, then $\varphi_u(t) = f(\lambda_G(t))y(t)$, and $t \mapsto f(\lambda_G(t))$ is bounded while $y(t) = \langle \delta_t, y\delta_e \rangle$ is in $\ell_2(G)$; this shows $\varphi_u \in \ell_2(G)$. □

Remark 3.26 With the notation of the next section we have

$$\varphi_u(t) = \tau_G(\lambda_G(t)^* u(\lambda_G(t))),$$

while with that of §11.2 it becomes $\varphi_u(t) = \langle \lambda_G(t), u(\lambda_G(t)) \rangle_{L_2(\tau_G)}$.

The preceding two statements combined show that if u is decomposable as a linear combination of c.p. maps on $C_\lambda^*(G)$ (as in Chapter 6) then φ_u is a linear combination of positive definite functions. In particular:

Corollary 3.27 *Let $\varphi : G \to \mathbb{C}$. The associated mapping M_φ is decomposable on $C_\lambda^*(G)$ if and only if φ is a linear combination of positive definite functions.*

We will now complete the description started in Proposition 3.3 of multipliers on the full algebra $C^*(G)$. In this case the picture is simpler.

Proposition 3.28 *Let $\varphi : G \to \mathbb{C}$. The following are equivalent:*

(i) *φ is a bounded multiplier on $C^*(G)$.*
(ii) *φ is a linear combination of positive definite functions.*
(iii) *φ is c.b. multiplier on $C^*(G)$.*

Moreover, φ is positive definite if and only if M_φ is c.p. on $C^*(G)$.

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Proof We already know (i) $\Leftrightarrow$ (iii) from Proposition 3.3. Assume (i). Then by Proposition 3.3 φ satisfies (3.3) for some π, η, ξ. By the polarization formula, we can rewrite φ as a linear combination of four functions of the form $t \mapsto \langle \xi, \pi(t)\xi \rangle$ with $\eta = \xi$. But the latter are clearly positive definite. This shows (i) $\Rightarrow$ (ii). Assume φ positive definite. By (3.15) and by the case $\xi = \eta$ in Proposition 3.3 M_φ is c.p. and hence a fortiori c.b. Now (ii) $\Rightarrow$ (iii) is clear. □

3.5 Group von Neumann Algebra

We denote by $M_G \subset B(\ell_2(G))$ the von Neumann algebra generated by λ_G. This means that $M_G = \lambda_G(G)''$. Equivalently M_G is the weak* closure of the linear span of $\lambda_G(G)$, and also the weak* closure of $C_\lambda(G)$. See §A.16 for some background on von Neumann algebras (in particular on the bicommutant Theorem A.46).

Let $f \in \ell_2(G)$. Note that a priori, the operator of left convolution by f, $T_f : x \mapsto f * x$ is only bounded from $\ell_2(G)$ to $\ell_\infty(G)$. An operator $T \in B(\ell_2(G))$ belongs to M_G if and only if there is a (uniquely determined by $f = T_f(\delta_e)$) function $f \in \ell_2(G)$ such that $x \mapsto f * x$ defines a bounded operator on $\ell_2(G)$ such that $T = T_f$.

We have

$$M_G' = \lambda_G(G)' = \rho_G(G)'' \text{ and } \rho_G(G)' = M_G.$$

Let $\Gamma \subset G$ be a subgroup. Since the embedding $J_\lambda : C_\lambda^*(\Gamma) \to C_\lambda^*(G)$ in Proposition 3.18 is clearly bicontinuous with respect to the weak* topologies of $B(\ell_2(\Gamma))$ and of $B(\ell_2(G))$, it extends to an embedding

$$M_\Gamma \subset M_G,$$

with which we may identify M_Γ to a von Neumann subalgebra of M_G.

Let $\{\delta_t \mid t \in G\}$ denote the canonical basis of $\ell_2(G)$. There is a distinguished tracial state τ_G defined on M_G by

$$\tau_G(T) = \langle \delta_e, T(\delta_e) \rangle.$$

Of course this makes sense on the whole of $B(\ell_2(G))$, but it is tracial only if we restrict to M_G:

$$\forall S, T \in M_G \quad \tau_G(TS) = \tau_G(ST).$$

Clearly τ_G is "normal" (meaning continuous for the weak* topology of $B(\ell_2(G))$) and faithful (meaning $\tau_G(T^*T) = 0 \Rightarrow T = 0$) and $\tau_G(1) = 1$. Thus (M_G, τ_G) is the basic example of a "tracial (or noncommutative) probability space” that we will consider in Chapter 12 when we discuss the Connes embedding problem.

Remark 3.29 Let φ be as in Corollary 3.21. Let $\Phi(s,t) = \varphi(s^{-1}t)$. Then the Schur multiplier $u_\Phi : B(\ell_2(G)) \to B(\ell_2(G))$ associated to Φ according to (iii) in Theorem 1.57 is completely contractive on $B(\ell_2(G))$ if and only if φ satisfies the equivalent conditions in Corollary 3.21. Moreover, the latter Schur multiplier is weak* continuous, meaning continuous from $B(\ell_2(G))$ to $B(\ell_2(G))$ when both spaces are equipped with the weak* topology. Therefore, if we restrict to M_G we obtain a weak* continuous (also called normal) complete contraction from M_G to M_G that extends the multiplier $M_\varphi : C^*_\lambda(G) \to C^*_\lambda(G)$. We will call the resulting maps weak* continuous multipliers on M_G.

A similar argument, based on Proposition 3.23, shows that φ is positive definite if and only if M_φ extends to a weak* continuous c.p. multiplier on M_G.

Lastly, the conclusion of Proposition 3.25 holds with the same proof for any c.b. map $u : M_G \to M_G$. The resulting multiplier M_{φ_u} is weak* continuous on M_G, with $\|M_{\varphi_u}\|_{cb} \le \|u\|_{cb}$. Moreover, if u is c.p. on M_G, so is M_{φ_u}.

3.6 Amenable groups

We review some basic facts on amenability.

A discrete group G is called amenable if it admits an invariant mean, i.e. a functional φ in $\ell_\infty(G)^*_+$ with $\varphi(1) = 1$ such that $\varphi(\delta_t * f) = \varphi(f)$ for any f in $\ell_\infty(G)$ and any t in G.

Theorem 3.30 *The following are equivalent:*

- (i) *G is amenable.*
- (i)’ *There is a net (h_i) in the unit sphere of $\ell_2(G)$ that is approximately translation invariant, i.e. such that $\|\lambda_G(t)h_i - h_i\|_2 \to 0$ for any $t \in G$.*
- (ii) $C^*(G) = C^*_\lambda(G)$.
- (iii) *For any finitely supported function $f : G \to \mathbb{C}$ we have* $\left|\sum f(t)\right| \le \left\|\sum f(t)\lambda_G(t)\right\|$.
- (iii)’ *For any finite subset $E \subset G$, we have $|E| = \left\|\sum_{t\in E} \lambda_G(t)\right\|$.*
- (iv) *There is a generating subset $S \subset G$ with $e \in S$ such that, for any finite subset $E \subset S$, we have $|E| = \left\|\sum_{t\in E} \lambda_G(t)\right\|$.*
- (v) *M_G is injective.*

Proof Assume (i). Let φ be the invariant mean. Note that φ is in the unit ball of $\ell_1(G)^{**}_+$. Therefore, there is a net (φ_i) in the unit ball of $\ell_1(G)_+$ tending in the sense of $\sigma(\ell_1(G)^{**}, \ell_1(G)^*)$ to φ. Let $\mathbf{1}$ be the constant function equal to 1

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on G. Since $\varphi_i(\mathbf{1}) \to 1$, we may assume after renormalization that $\varphi_i(\mathbf{1}) = \|\varphi_i\|_{\ell_1(G)} = 1$. Fix $t \in G$. Since $\delta_t * \varphi = \varphi$, we have $\delta_t * \varphi_i - \varphi_i \to 0$ when $i \to \infty$. But since $\delta_t * \varphi_i - \varphi_i$ lies in $\ell_1(G)$ this means that $\lim_{i\to\infty}(\delta_t * \varphi_i - \varphi_i) = 0$ for the weak topology of $\ell_1(G)$. By (Mazur's) Theorem A.9, passing to convex combinations of elements of a subnet (here we leave some details to the reader, see Remark A.10) we may assume that $\lim_{i\to\infty} \|\delta_t * \varphi_i - \varphi_i\|_{\ell_1(G)} = 0$. A priori, this was obtained for each fixed t, but, by suitably refining the argument (here again we skip some details), we can obtain the same for each finite subset $T \subset G$. Let $h_i = \sqrt{\varphi_i}$. We claim that $\|\delta_t * h_i - h_i\|_2 \to 0$ for any $t \in T$. This claim clearly implies (i)'. To check the claim, using $|x^{1/2} - y^{1/2}| \le |x - y|^{1/2}$ for any $x, y \in \mathbb{R}_+$, we observe that $|\delta_t * h_i(s) - h_i(s)| \le |\delta_t * \varphi_i(s) - \varphi_i(s)|^{1/2}$ and hence $\|\delta_t * h_i - h_i\|_2 \to 0$ for any $t \in T$. This shows (i) $\Rightarrow$ (i)'.

Assume (i)'. Let $x = \sum x(t)\lambda_G(t) \in \text{span}[\lambda_G(t) \mid t \in G]$. Let $\pi : G \to B(H)$ be any unitary representation. By the absorption principle (3.13) $\|\sum x(t)\lambda_G(t)\| = \|\sum x(t)\pi(t) \otimes \lambda_G(t)\|$. We claim that $\|\sum x(t)\pi(t) \otimes \lambda_G(t)\| \ge \|\sum x(t)\pi(t)\|$. Indeed, let f_i be the state on $B(\ell_2(G))$ defined by $f_i(T) = \langle h_i, T h_i\rangle$. Then we have clearly

$$\left\|[Id \otimes f_i]\left(\sum x(t)\pi(t) \otimes \lambda_G(t)\right)\right\| \le \left\|\sum x(t)\pi(t) \otimes \lambda_G(t)\right\|$$

but

$$[Id \otimes f_i]\left(\sum x(t)\pi(t) \otimes \lambda_G(t)\right) = \sum x(t)\pi(t) f_i(\lambda_G(t)) \to \sum x(t)\pi(t),$$

where at the last step we use $f_i(\lambda_G(t)) = \langle h_i, \delta_t * h_i\rangle \to 1$. This implies the claim and hence $\left\|\sum x(t)\lambda_G(t)\right\| \ge \left\|\sum x(t)\pi(t)\right\|$. Taking the sup over the πs we obtain (by "maximality" of U_G) $\left\|\sum x(t)\lambda_G(t)\right\| = \left\|\sum x(t)U_G(t)\right\|$. This shows (i)' $\Rightarrow$ (ii).

Assume (ii). Then (iii) holds by (3.2), and (iii) $\Rightarrow$ (iii)' $\Rightarrow$ (iv) are trivial.

Assume (iv). We will show (i)'. Fix E as in (iv). Let $M_E = |E|^{-1}\sum_{t\in E}\lambda_G(t)$ so that $\|M_E\| = 1$. There is a net (x_i) in the unit sphere of $\ell_2(G)$ such that $\|M_E(x_i)\| \to 1$. By the uniform convexity of $\ell_2(G)$ (see §A.3), this implies $\delta_t * x_i - x_i \to 0$ in $\ell_2(G)$ for any $t \in E$. Rearranging the net (here again we leave the details to the reader) we find a net (h_i) in the unit sphere of $\ell_2(G)$ such that the same holds for any $t \in S$, and since S generates G, still the same for any $t \in G$. This shows (iv) $\Rightarrow$ (i)'.

Assume (i)'. We will show (i). Let $\varphi \in \ell_\infty(G)^*$ be defined by

$$\forall x \in \ell_\infty(G) \quad \varphi(x) = \lim_{\mathcal{U}} \sum x(t)|h_i(t)|^2,$$

where $\mathcal{U}$ is an ultrafilter refining the net (see Remark A.6). Let $D_x \in B(\ell_2(G))$ be the diagonal operator associated to x. Note that $\varphi(x) = \lim_{\mathcal{U}}\langle h_i, D_x h_i\rangle$, and also

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$$\lambda_G(t)D_x\lambda_G(t)^{-1} = D_{\delta_t * x}. \tag{3.16}$$

Therefore

$$\varphi(\delta_t * x) = \lim_{\mathcal{U}}\langle h_i, D_{\delta_t * x}h_i\rangle = \lim_{\mathcal{U}}\langle \lambda_G(t)h_i, D_x\lambda_G(t)h_i\rangle$$
$$= \lim_{\mathcal{U}}\langle h_i, D_x h_i\rangle = \varphi(x).$$

Thus φ is an invariant mean, so (i) holds. This proves the equivalence of (i)–(iv), (i)', and (iii)'. It remains to show that (i) and (v) are equivalent.

Assume (i). We will show that there is a c.p. projection $P : B(\ell_2(G)) \to M_G$ with $\|P\| = 1$. Let $T \in B(\ell_2(G))$. We define $\Phi_T : G \to B(\ell_2(G))$ by $\Phi_T(g) = \rho_G(g)T\rho_G(g)^{-1}$. We will define $P(T)$ as the "integral" with respect to φ of the function Φ_T, but some care is needed since φ is not really a measure on G. Let $[T(s,t)]$ be the "matrix" associated to T defined by $T(s,t) = \langle\delta_s, T\delta_t\rangle$ $(s,t \in G)$. Observe that $g \mapsto \Phi_T(g)(s,t)$ is in $\ell_\infty(G)$. Then we set

$$P(T)(s,t) = \varphi(\Phi_T(\cdot)(s,t)).$$

This defines a matrix and it is easy to see that the associated linear operator on $\text{span}[\delta_t \mid t \in G]$ extends to a bounded one (still denoted by $P(T)$) on $\ell_2(G)$ such that $\|P(T)\| \le \|T\|$. We have $\Phi_T(g)(s,t) = T(sg,tg)$ and hence, by the left invariance of φ, $P(T)(s,t) = P(T)(st^{-1},e)$. This shows that $P(T)$ acts on $\ell_2(G)$ as a left convolution bounded operator, in other words $P(T) \in M_G$. Moreover, if $T \in M_G$ then T commutes with ρ_G so we have $P(T) = T$. This proves that $P : B(\ell_2(G)) \to M_G$ is a contractive projection. A simple verification left to the reader shows that it is c.p. (but this is automatic by Tomiyama's Theorem 1.45). This shows (i) $\Rightarrow$ (v).

Assume (v). Let $P : B(\ell_2(G)) \to M_G$ be a projection with $\|P\| = 1$. Invoking Theorem 1.45 again, we know that P is a c.p. conditional expectation. We define

$$\forall x \in \ell_\infty(G) \quad \varphi(x) = \tau_G(P(D_x)) = \langle\delta_e, P(D_x)\delta_e\rangle.$$

Clearly $\varphi \in \ell_\infty(G)^*_+$, $\varphi(1) = 1$ and by (3.16), (1.28) and the trace property of τ_G

$$\forall t \in G \quad \varphi(\delta_t * x) = \tau_G[P(\lambda_G(t)D_x\lambda_G(t)^{-1})] = \tau_G[\lambda_G(t)P(D_x)\lambda_G(t)^{-1}]$$
$$= \tau_G[P(D_x)] = \varphi(x).$$

Thus φ is an invariant mean on G. This shows (v) $\Rightarrow$ (i). □

Remark 3.31 If the generating set S is finite, the condition (iv) obviously reduces (by the triangle inequality) to

$$|S| = \left\|\sum\nolimits_{t \in S}\lambda_G(t)\right\|.$$

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Remark 3.32 The net (h_i) in (i)' is sometimes called asymptotically left invariant. By density (and after renormalization) when it exists, it can always be found in the group algebra $\mathbb{C}[G]$.

Remark 3.33 (On Følner sequences) It is well known (see e.g. [194]) that for any amenable discrete group G the net (h_i) appearing in (i)' in Theorem 3.30 can be chosen of the form $h_i = 1_{B_i}|B_i|^{-1/2}$ for some family (B_i) of finite subsets of G. For (h_i) of the latter form, (i)' boils down to the assertion that the symmetric differences $(tB_i)\Delta B_i$ satisfy

$$\forall t \in G \quad |tB_i \Delta B_i||B_i|^{-1} \to 0.$$

A net of finite subsets (B_i) satisfying this is called a Følner net, and a Følner sequence when the index set is $\mathbb{N}$. Thus a (resp. countable) group G is amenable if and only if it admits a Følner net (resp. sequence). For instance, for $G = \mathbb{Z}^d$ $(1 \le d < \infty)$, the sequence $B_n = [-n, n]^d$ is a Følner sequence.

This gives us the following special property of the *reduced* C^*-algebra, called the CPAP in the sequel (see Definition 4.8):

Lemma 3.34 *If G is amenable, there is a net of finite rank maps $u_i \in CP(C^*_\lambda(G), C^*_\lambda(G))$ (resp. $u_i \in CP(C^*(G), C^*(G))$) with $\|u_i\| = 1$ that tends pointwise to the identity on $C^*_\lambda(G)$ (resp. $C^*(G)$). Moreover, in both cases the u_i's are multiplier operators.*

Proof By Remark 3.32, there is a net (h_i) in $\mathbb{C}[G]$ in the unit sphere of $\ell_2(G)$ such that $\|\lambda_G(t)h_i - h_i\|_2 \to 0$ for any t in G. Let $h_i^*(t) = \overline{h_i(t^{-1})}$ $(t \in G)$. A simple verification show that $\varphi_i = h_i^* * h_i$ is a positive definite function on G such that $\varphi_i(e) = \|h_i\|_2^2 = 1$. Moreover, φ_i is finitely supported and tends pointwise to the constant function 1 on G. Let u_i be the associated multiplier operator on $C^*_\lambda(G)$ (resp. $C^*(G)$). Its rank being equal to the cardinality of the support of φ_i is finite. By Proposition 3.23 (resp. Proposition 3.28), u_i is c.p. and since $u_i(1) = \varphi_i(e)1 = 1$, we have $\|u_i\| = 1$ by (1.20). For any $x = \sum x(t)\lambda_G(t)$ (resp. $x = \sum x(t)U_G(t)$) with x finitely supported, $u_i(x)$ obviously tends to x in the norm of $C^*_\lambda(G)$ (resp. $C^*(G)$). Since such finite sums are dense in $C^*_\lambda(G)$ (resp. $C^*(G)$) and $\sup_i \|u_i\| < \infty$, we conclude that $u_i(x) \to x$ for any $x \in C^*_\lambda(G)$ (resp. $C^*(G)$). □

Remark 3.35 (Examples of amenable groups) All commutative groups are amenable. If G is commutative (and discrete), its dual $\widehat{G}$ is defined as the group formed of all homomorphisms $\gamma : G \to \mathbb{T}$, which is compact for the pointwise convergence topology. For any finitely supported function $f : G \to \mathbb{C}$ we define its "Fourier transform" by $\widehat{f}(\gamma) = \sum f(g)\overline{\gamma(g)}$. (This is the usual convention but we could remove the bar from $\overline{\gamma(g)}$ if we wished). As

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is entirely classical $f \mapsto \widehat{f}$ extends to an isometric isomorphism from $\ell_2(G)$ to $L_2(\widehat{G}, m)$, where m is the normalized Haar measure on $\widehat{G}$, and convolution of two functions on G is transformed into the pointwise product of their Fourier transforms. Using the latter fact one shows that the correspondence $f \mapsto \widehat{f}$ extends to an isometric isomorphism from $C^*_\lambda(G)$ to the C^*-algebra $C(\widehat{G})$ of all continuous functions on $\widehat{G}$. Thus in the commutative case we have

$$C^*(G) = C^*_\lambda(G) \simeq C(\widehat{G}). \tag{3.17}$$

All finitely generated groups of polynomial growth are amenable. The growth is defined using the length. If G is generated by a symmetric set S the smallest number of elements of S needed to write an element $g \in G$ (as a word in letters in S) is denoted by $\ell_S(g)$. The growth function is the function $\Phi(R) = |\{g \in G \mid \ell_S(g) \le R\}$. The group G is called of polynomial growth if $\Phi(R)$ grows less than a power of R when $R \to \infty$. For instance $G = \mathbb{Z}^n$ is of polynomial growth (but $\mathbb{F}_n$ is not whenever $n \ge 2$).

Remark 3.36 By Kesten's famous work on the spectral radius of random walks on the free group $\mathbb{F}_n$ with n generators, the set $S_1 \subset \mathbb{F}_n$ formed of the $2n$ elements of length 1 (i.e. these are either generators or their inverses), satisfies

$$\left\| \sum\nolimits_{s \in S_1} \lambda_{\mathbb{F}_n}(s) \right\| = 2\sqrt{2n-1}. \tag{3.18}$$

Kesten also observed that it is not difficult to deduce from this that for any group G and any symmetric subset $S \subset G$ with $|S| = k$ we have

$$\left\| \sum\nolimits_{s \in S} \lambda_G(s) \right\| \ge 2\sqrt{k-1}.$$

Akemann and Ostrand [2] proved that any $S \subset S_1$ in $\mathbb{F}_n$ with $|S| = k$ satisfies

$$\left\| \sum\nolimits_{s \in S} \lambda_{\mathbb{F}_n}(s) \right\| = 2\sqrt{k-1}. \tag{3.19}$$

In particular

$$\left\| \sum\nolimits_{j=1}^{n} \lambda_{\mathbb{F}_n}(g_j) \right\| = 2\sqrt{n-1}. \tag{3.20}$$

The subsets S of a discrete group for which (3.19) holds have been characterized by Franz Lehner in [166], as the translates of the union of a free set and the unit.

Let $S \subset \mathbb{F}_n$ be the set formed of the unit and the n free generators, so that $|S| = n + 1$. Then a variant of what precedes is that for $G = \mathbb{F}_n$

$$\left\| \sum\nolimits_{s \in S} \lambda_G(s) \right\| = 2\sqrt{n}. \tag{3.21}$$

When $n \ge 2$, this is $< n + 1$, and hence (iii) or (iv) in Theorem 3.30 fails. This shows that $\mathbb{F}_n$ is not amenable for $n \ge 2$.

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Since amenability passes to subgroups (by Proposition 3.18 and (i)⇔(iv) in Theorem 3.30), any group containing a copy of $\mathbb{F}_2$ as a subgroup is nonamenable. The converse, whether nonamenable groups must contain $\mathbb{F}_2$, remained a major open question for a long time but was disproved by A. Olshanskii, see [126] for details. See Monod's [178] for what seems to be currently the simplest construction of nonamenable groups not containing $\mathbb{F}_2$ as a subgroup.

3.7 Operator space spanned by the free generators in $C^*_\lambda(\mathbb{F}_n)$

The next statement gives us a description up to complete isomorphism of the span of the generators in $C^*_\lambda(\mathbb{F}_n)$ (and also implicitly in $C^*_\lambda(\mathbb{F}_\infty)$). See [168] for a more precise (completely isometric) description.

Theorem 3.37 *Let $(g_j)_{1\le j\le n}$ be the generators in $\mathbb{F}_n (n \ge 1)$. Then for any Hilbert space H and any $a_j \in B(H) (1 \le j \le n)$ we have*

$$\max\left\{\left\|\sum a_j^* a_j\right\|^{1/2}, \left\|\sum a_j a_j^*\right\|^{1/2}\right\} \le \left\|\sum a_j \otimes \lambda_{\mathbb{F}_n}(g_j)\right\|_{\min}$$
$$\le \left\|\sum a_j^* a_j\right\|^{1/2} + \left\|\sum a_j a_j^*\right\|^{1/2}. \tag{3.22}$$

In particular for any $\alpha_j \in \mathbb{C}$ we have

$$\left(\sum |\alpha_j|^2\right)^{1/2} \le \left\|\sum \alpha_j \lambda_{\mathbb{F}_n}(g_j)\right\| \le 2\left(\sum |\alpha_j|^2\right)^{1/2}.$$

Proof We will first prove the upper bound in (3.22). Let $C_i^+ \subset \mathbb{F}_n$ (resp. $C_i^- \subset \mathbb{F}_n$) be the subset formed by all the reduced words which start with g_i (resp. g_i^{-1}). Note: except for the empty word e, every element of G can be written as a reduced word in the generators admitting a well-defined "first" and "last" letter (where we read from left to right). Let P_i^+ (resp. P_i^-) be the orthogonal projection on $\ell_2(\mathbb{F}_n)$ with range $\overline{\text{span}}[\delta_t \mid t \in C_i^+]$ (resp. $\overline{\text{span}}[(\delta_t \mid t \in C_i^-])$. The $2n$ projections $\{P_i^+, P_i^- \mid 1 \le i \le n\}$ are mutually orthogonal. Then it is easy to check that

$$\begin{aligned}\lambda_{\mathbb{F}_n}(g_j) &= \lambda_{\mathbb{F}_n}(g_j)P_j^- + \lambda_{\mathbb{F}_n}(g_j)(1 - P_j^-)\\ &= \lambda_{\mathbb{F}_n}(g_j)P_j^- + P_j^+\lambda_{\mathbb{F}_n}(g_j)(1 - P_j^-)\\ &= \lambda_{\mathbb{F}_n}(g_j)P_j^- + P_j^+\lambda_{\mathbb{F}_n}(g_j)\end{aligned}$$

so that setting $\lambda_{\mathbb{F}_n}(g_j) = x_j + y_j$ with $x_j = \lambda_{\mathbb{F}_n}(g_j)P_j^-$ and $y_j = P_j^+\lambda_{\mathbb{F}_n}(g_j)$ we find

$$\left\|\sum x_j^* x_j\right\| = \left\|\sum P_j^-\right\| \le 1 \quad \text{and} \quad \left\|\sum y_j y_j^*\right\| = \left\|\sum P_j^+\right\| \le 1.$$

Therefore for any finite sequence (a_j) in $B(H)$ we have by (1.11) (note $a_j \otimes x_j = (a_j \otimes 1)(1 \otimes x_j)$ and similarly for $a_j \otimes y_j$)

$$\begin{aligned}\left\|\sum a_j \otimes \lambda_{\mathbb{F}_n}(g_j)\right\| &\le \left\|\sum a_j \otimes x_j\right\| + \left\|\sum a_j \otimes y_j\right\| \\ &\le \left\|\sum a_j a_j^*\right\|^{1/2} + \left\|\sum a_j^* a_j\right\|^{1/2}.\end{aligned}$$

The inverse inequality follows from a more general one valid for any discrete group G: for any finitely supported function $a : G \to B(H)$ we have

$$\max\left\{\left\|\sum a(t)^* a(t)\right\|^{1/2}, \left\|\sum a(t)a(t)^*\right\|^{1/2}\right\} \le \left\|\sum a(t) \otimes \lambda_G(t)\right\|_{\min}. \tag{3.23}$$

To check this, let $T = \sum a(t) \otimes \lambda_G(t)$. For any h in B_H we have $T(h \otimes \delta_e) = \sum a(t)h \otimes \delta_t$ so that $\|T(h \otimes \delta_e)\| = \left(\sum_t \|a(t)h\|^2\right)^{1/2}$ and hence

$$\left\|\sum a(t)^* a(t)\right\|^{1/2} = \sup_{h \in B_H}\left(\sum \|a(t)h\|^2\right)^{1/2} \le \|T\|.$$

Similarly since $T^* = \sum a(t^{-1})^* \otimes \lambda_G(t)$ we find

$$\left\|\sum a(t)a(t)^*\right\|^{1/2} \le \|T^*\| = \|T\|$$

and we obtain (3.23). In the case $G = \mathbb{F}_n$, (3.23) implies the left-hand side of (3.22). The second inequality follows by taking $a_j = \alpha_j 1$. □

Corollary 3.38 *For any $n, N \ge 1$ and any unitaries $a \in \mathbb{U}_n$, $x_1, \ldots, x_n \in \mathbb{U}_N$ we have*

$$\left\|\sum\nolimits_{i,j=1}^n a_{ij} x_i \otimes \lambda_{\mathbb{F}_n}(g_j)\right\| \le 2\sqrt{n}.$$

Proof Let $a_j = \sum_i a_{ij} x_i$. Since a is unitary a simple verification (using (A.13)) shows that we have $\|\sum a_j^* a_j\|^{1/2} \le \|\sum x_j^* x_j\|^{1/2} = \sqrt{n}$ and $\|\sum a_j a_j^*\|^{1/2} \le \|\sum x_j x_j^*\|^{1/2} = \sqrt{n}$. □

3.8 Free products of groups

Let $(G_i)_{i \in I}$ be a family of groups. The free product $G = *_{i \in I} G_i$ is a group containing each G_i as a subgroup and possessing the following universal

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property that characterizes it: for any group G' and any family of homomorphisms $f_i : G_i \to G'$, there is a *unique* homomorphism $f : G \to G'$ extending each f_i.

When $I = \{1,2\}$ we denote $G_1 * G_2$ the free product $*_{i\in I} G_i$.

When $I = \{1,\dots,n\}$ and $G_1 = \cdots = G_n = \mathbb{Z}$ it is easy to see that $G = *_{i\in I} G_i$ can be identified with $\mathbb{F}_n$.

More generally, any free group $\mathbb{F}$ that is generated by a family of free elements $(g_i)_{i\in I}$ can be identified with the free product $*_{i\in I} G_i$ relative to $G_i = \mathbb{Z}$ for all $i \in I$. We denote that group by $\mathbb{F}_I$.

It is well known that any group G is a quotient of some free group. Indeed, if G is generated by a family $(t_i)_{i\in I}$, let $f : \mathbb{F}_I \to G$ be the (unique) homomorphism such that $f(g_i) = t_i$ for all $i \in I$. Then f is onto G. Thus $G \simeq \mathbb{F}_I/\ker(f)$. The analogous fact for C^*-algebras is the next statement.

Proposition 3.39 *Any unital C^*-algebra A is a quotient of $C^*(\mathbb{F}_I)$ for some set I. If A is separable (resp. is generated by n unitaries) then we can take $I = \mathbb{N}$ (resp. $I = \{1,\dots,n\}$).*

Proof Let G be the unitary group of A. Let $f : \mathbb{F}_I \to G$ be a surjective homomorphism. Let $\pi : C^*(\mathbb{F}_I) \to A$ be the associated $*$-homomorphism, as in Remark 3.1. By the Russo–Dye Theorem A.18, the range of π is dense in A, but since it is closed (see §A.14), π must be surjective. Thus A is a quotient of $C^*(\mathbb{F}_I)$. If A is generated as a C^*-algebra by a family of unitaries $(u_i)_{i\in I}$, we can replace G in the preceding argument by the group generated by $(u_i)_{i\in I}$. This settles the remaining assertions. □

Remark 3.40 As we saw in Remark 3.36, $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ is not amenable. More generally it can be shown that $\mathbb{Z}_n * \mathbb{Z}_m$ is not amenable if $n \ge 2$ and $m \ge 3$, and in fact contains a subgroup isomorphic to $\mathbb{F}_\infty$. The group $\mathbb{Z}_2 * \mathbb{Z}_2$ is a slightly surprising exception, it is amenable because it happens to have polynomial growth (an exercise left to the reader).

3.9 Notes and remarks

The main results of this section are by now well known, and sometimes for general locally compact groups (for instance Proposition 3.5 is proved in greater generality in [225]), but we choose to focus on the discrete ones. Section 3.2 on free groups is just a reformulation of operator space duality illustrated on the pair (ℓ_1, ℓ_∞). Lemmas 3.9 and 3.10 are elementary facts from operator space theory (see [80, 208]). The classical reference that exploited C^*-algebra theory in noncommutative harmonic analysis is Eymard's thesis [85]. The name of Fell is attached to the notions of weak containment

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and weak equivalence of group representations, which apparently led him to the principle enunciated in Theorem 3.15. Concerning multipliers, those considered in Theorem 3.20 are sometimes called Herz–Schur multipliers (in honor of Carl Herz). The characterization in Theorem 3.20 and its Corollary is due to Jolissaint [136], but the simple proof we give is due to Bożejko and Fendler [35]. Our treatment is inspired by Haagerup's unpublished (but widely circulated) notes on multipliers, where in particular he proves Proposition 3.25. There are many known characterizations of amenability, the main one going back to Kesten, with variants due to Hulanicki and many authors. We refer the reader to [194] (or [199]) for details and references. Theorem 3.37 appears in [118]. In [168] Lehner gives an exact computation of the norm of $\sum a_j \otimes \lambda_{\mathbb{F}_n}(g_j)$ when the coefficients a_j are matricial or equivalently when $\dim(H) < \infty$.

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