

**ON FANO MANIFOLDS, WHICH ARE  $P^k$ -BUNDLES OVER  $P^2$**

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In our earlier paper [8] we discussed Fano manifolds  $X$  that are of the form  $X = \mathbb{P}(\mathcal{E})$  with a rank-2 vector bundle  $\mathcal{E}$  on a surface  $S$ . Here we study a more general situation of Fano manifolds, ruled over the complex projective plane  $P^2$  as  $P^{r-1}$ -bundles, i.e., being of the form  $\mathbb{P}(\mathcal{E})$  with  $\mathcal{E}$ -a bundle of rank  $r \geq 3$  on  $P^2$ . We say that  $\mathcal{E}$  is a Fano bundle if  $\mathbb{P}(\mathcal{E})$  is a Fano manifold.

**THEOREM.** *All manifolds  $X = \mathbb{P}(\mathcal{E})$  with  $\mathcal{E}$  a vector bundle of rank  $r \geq 2$  over  $P^2$  are listed below:*

$c_1$	$c_2$	$\mathcal{E}$	$X = \mathbb{P}(\mathcal{E})$
0	0	$\mathcal{O}^r$	$P^2 \times P^{r-1}$
1	0	$\mathcal{O}(1) \oplus \mathcal{O}^{r-1}$	blow-up of $P^{r+1}$ along $P^{r-1}$
1	1	$T(-1) \oplus \mathcal{O}^{r-2}$	a divisor of degree (1, 1) in $P^2 \times P^r$
2	0	$\mathcal{O}(2) \oplus \mathcal{O}^{r-1}$	blow-up of the cone in $P^{r+4}$ over veronese ( $P^2$ ) $\subset P^5$ along its vertex ( $= P^{r-2}$ )
2	1	$\mathcal{O}(1)^2 \oplus \mathcal{O}^{r-2}$ ( $r \geq 3$ )	blow-up of the cone in $P^{r+3}$ over $P^1 \times P^2 \subset P^5$ along its vertex ( $= P^{r-3}$ )
2	2	$r = 2$ : a bundle $\mathcal{E}_2$ in $0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}_2(-1) \rightarrow J_x \rightarrow 0$ $r \geq 3$ : special: $\mathcal{O}^{r-1} \oplus \mathcal{E}_2$ general: $T(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}^{r-3}$	blow-up of $Q_3$ along a line; blow-up of a cone over a smooth quadric $Q_3$ (resp. $Q_4$ ) along a linear subspace $P^{r-1}$ containing the vertex $\cong P^{r-3}$ (resp. $P^{r-4}$ )
2	3	All bundles fitting into $0 \rightarrow \mathcal{O} \rightarrow (-1)^2 \rightarrow \mathcal{O}^{r+2} \rightarrow \mathcal{E} \rightarrow 0$ $r = 3$ : special $\mathcal{E}_3^s = \mathcal{O} \oplus \mathcal{E}_2$	blow-up of $P^3$ along a twisted cubic curve, if $r = 2$ blow-up of the cone over a twisted cubic curve in $P^4$ , i.e. the rational normal scroll $S(0, 3)$

Received January 13, 1989.

<p><i>general: non-splitting</i></p> $0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}_3^{\otimes r} \rightarrow \mathcal{E}_2 \rightarrow 0$ <p><math>r \geq 4</math>: <i>special: <math>\mathcal{E} = \mathcal{O}^{r-2} \oplus \mathcal{E}_2</math></i></p> <p><i>special: <math>\mathcal{E} = \mathcal{O}^{r-3} \oplus \mathcal{E}_3^{\otimes r}</math></i></p> <p><i>general: sums of <math>\mathcal{O}^{r-4}</math> and a nontrivial extension of <math>\mathcal{E}_3^{\otimes r}</math> by <math>\mathcal{O}</math>.</i></p>	<p><i>blow-up of <math>\mathbf{P}^{r+1}</math> along the rational normal scroll <math>\mathbf{S}(1, 2)</math></i></p> <p><i>blow-up of <math>\mathbf{P}^{r+1}</math> along <math>\mathbf{S}(0, \dots, 0, 3)</math></i></p> <p><i>blow-up of <math>\mathbf{P}^{r+1}</math> along <math>\mathbf{S}(0, \dots, 0, 1, 2)</math></i></p> <p><i>blow-up of <math>\mathbf{P}^{r+1}</math> along <math>\mathbf{S}(0, \dots, 1, 1, 1)</math>.</i></p>
<p>2 4 <math>r \geq 2</math> a bundle <math>\mathcal{E}</math> in</p> $\mathcal{O} \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}^{r+1} \rightarrow \mathcal{E} \rightarrow 0;$ <p>such <math>\mathcal{E}</math> is a direct sum of <math>\mathcal{O}^j</math> and some non-splitting <math>\mathcal{E}</math> of rank <math>\leq 5</math>.</p>	<p>divisor of degree <math>(2, 1)</math> in <math>\mathbf{P}^2 \times \mathbf{P}^r</math>;</p>

In the above table  $\mathbf{S}(a_1, \dots, a_n)$  ( $0 \leq a_1 \leq \dots \leq a_n > 0$ ) denotes a rational normal scroll, that is the image of the scroll  $\mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbf{P}^1}(a_n)$  under the map associated to the complete system of sections of the bundle  $\mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbf{P}^1}(a_n)$ , see [3]. Also  $T(-1)$  denotes the tangent bundle to  $\mathbf{P}^2$  twisted by the line bundle  $\mathcal{O}_{\mathbf{P}^2}(-1)$  and  $\mathbf{J}_x$  denotes the sheaf of ideals of a point  $x \in \mathbf{P}^2$ .

The paper is organized as follows. Section 1 is introductory, In section 2 we show first that if  $0 \leq c_1(\mathcal{E}) \leq r - 1$ , then  $\xi_r$ , i.e., the relative very ample sheaf on  $\mathbb{P}(\mathcal{E})$ , is numerically effective and then we prove that  $\xi_r$  is spanned. Here we use methods of Mori Theory. In section 3 we obtain, by classical vector bundle methods, the description of the bundles considered, as stated in the 3<sup>rd</sup> column of the Theorem. Then we study a more geometric picture, i.e., the image of  $\mathbb{P}(\mathcal{E})$  under the contraction  $\phi$  defined by the linear system  $|\xi_r|$  on  $\mathbb{P}(\mathcal{E})$ . Note that the case  $r = 2$  was discussed in [8], and also in [2]; see [6] for a general classification of Fano threefolds.

**§ 1. Preliminaries**

All our manifolds are defined over the complex number field  $\mathbb{C}$ . The symbol  $\mathbb{P}(\mathcal{E})$  denotes the projectivization of a vector bundle  $\mathcal{E}$  on  $\mathbf{P}^2$  in the sense of Grothendieck, i.e., the fibre over  $x \in \mathbf{P}^2$  of the projection morphism from  $\mathbb{P}(\mathcal{E})$  onto  $\mathbf{P}^2$  consists of hyperplanes in the affine fibre  $\mathcal{E}_x$ . The rest of our notation is standard. Let us recall basic facts and formulas that we need in the sequel.

(1.1) The Riemann-Roch formula for twisted rank- $r$  vector bundles on  $\mathbf{P}^2$  is

$$\chi(\mathcal{E}(m)) = \frac{1}{2}(c_1^2(\mathcal{E}) + (2m + 3)c_1(\mathcal{E}) + rm(m + 3)) + r - c_2(\mathcal{E}).$$

As in [8], in a standard way we obtain the following formula for the anticanonical divisor of  $\mathbb{P}(\mathcal{E})$ :

$$(1.2) \quad -K_{\mathbb{P}(\mathcal{E})} = r\xi_{\mathcal{E}} + (3 - c_1(\mathcal{E}))H,$$

where  $H$  is the pullback of  $\mathcal{O}(1)$  from  $\mathbf{P}^2$  and  $\xi_{\mathcal{E}}$ -the relative ample sheaf  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  on  $\mathbb{P}(\mathcal{E})$ . (We will write simply  $\xi$  instead of  $\xi_{\mathcal{E}}$  if no confusion is likely.) We will frequently use the isomorphism of cohomology  $H^i(\mathcal{O}(\xi_{\mathcal{E}} + kH)) \cong H^i(\mathcal{E}(k))$ .

For computing the intersection of divisors on  $\mathbb{P}(\mathcal{E})$  let us note that  $\xi^{r-1}H^2 = 1$  and  $H^3 \equiv 0$ . Then, by the theorem of Leray and Hirsch we get

$$(1.3) \quad \begin{aligned} \xi^r - c_1(\mathcal{E})\xi^{r-1}H + c_2(\mathcal{E})\xi^{r-2}H^2 &= 0; \\ \xi^{r+1} = c_1^2(\mathcal{E}) - c_2(\mathcal{E}), \quad \xi^r H &= c_1(\mathcal{E}); \\ (-K)^{r+1} = r^r \left( r(c_1^2(\mathcal{E}) - c_2(\mathcal{E})) \right. \\ &\quad \left. + (r + 1)((3 - c_1(\mathcal{E}))c_1(\mathcal{E}) + \frac{1}{2}(3 - c_1(\mathcal{E}))^2) \right) \end{aligned}$$

Let us also collect necessary facts from Mori theory, see [5] for a complete exposition of the theory. Whenever we use the term “contraction”, we mean it in the sense of the following

(1.4) **KAWAMATA-SHOKUROV CONTRACTION THEOREM.** *For any extremal ray  $R$  on a manifold  $X$  there exists a normal projective variety  $Y$  and a surjective morphism  $\text{contr}_R: X \rightarrow Y$  with connected fibres that contracts precisely the curves from  $R$ , i.e., for any integral curve  $C$  on  $X$ ,  $\dim(\text{contr}_R(C)) = 0$  if and only if  $[C] \in R$ .*

(1.5) **COROLLARY.** *If  $D$  is a numerically effective divisor and  $aD - K$  is ample for some  $a > 0$ , then  $D$  is semiample, i.e., some tensor power of  $D$  is spanned. In particular, on Fano manifolds  $\text{nef} \Rightarrow \text{semiample}$ .*

A contraction  $\phi$  is called of fibre type if  $\dim Y < \dim X$  or, equivalently, if the extremal ray  $R$  is numerically effective. A divisorial contraction is that with  $\dim Y = \dim X$  and the smallest  $A \subset X$ ,  $A \neq X$  being such that  $\phi: X \setminus A \xrightarrow{\cong} Y \setminus \phi(A)$  is of dimension  $\dim X - 1$ .

We now give some estimates for  $c_1(\mathcal{E})$  and  $c_2(\mathcal{E})$ . Assume  $\mathcal{E}$  is a rank- $r$  vector bundle on  $\mathbf{P}^2$ , with  $r \geq 3$  and  $0 \leq c_1(\mathcal{E}) \leq r - 1$ . Then

(1.6) If  $\mathbb{P}(\mathcal{E})$  is Fano, then  $c_1(\mathcal{E}) \leq 2$ .

*Proof.* Suppose  $c_1 \geq 3$ . Then the ampleness of the anticanonical divisor  $-K_{\mathbb{P}(\mathcal{E})} = r\xi + (3 - c_1(\mathcal{E}))H$  would yield the ampleness of  $\xi_r$  i.e.,  $\mathcal{E}$  would be ample. Because  $c_1(\mathcal{E}) \leq r - 1$ , on any line  $L \subset \mathbf{P}^2$  there is  $E|L = \mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_r)$  with at least one number  $a_i \leq 0$ . Such bundle cannot be ample.

(1.7) If  $\mathcal{E}$  with  $c_1(\mathcal{E}) \geq 0$  is Fano, then  $\mathcal{E}(1)$  is ample.

*Proof.* The divisor  $r\xi_{\mathcal{E}(1)} = r\xi_{\mathcal{E}} + rH = (r\xi_{\mathcal{E}} + (3 - c_1(\mathcal{E})H)) + (r - 3 + c_1(\mathcal{E}))H$  can be represented as a sum of an ample divisor and a nef one.

(1.8) The splitting type of a Fano bundle with  $c_1(\mathcal{E}) = 0, 1, 2$  can only be

for  $c_1(\mathcal{E}) = 0$ :  $(0, \dots, 0, 0)$ ,

for  $c_1(\mathcal{E}) = 1$ :  $(0, \dots, 0, 1)$ ,

for  $c_1(\mathcal{E}) = 2$ :  $(0, \dots, 1, 1)$  or  $(0, \dots, 0, 2)$ .

(1.9) If  $c_1(\mathcal{E}) = 0$  then  $\mathcal{E}$  is trivial. If  $c_1(\mathcal{E}) = 1$  and  $r = 3$ , then either  $\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$  or  $\mathcal{E} = \mathbf{T}(-1) \oplus \mathcal{O}$ .

*Proof.* From (1.8) we obtain immediately that a bundle as in (1.9) is uniform, hence the conclusion of the proposition follows from the classification of uniform bundles, see [7].

(1.10) If  $0 \leq c_1(\mathcal{E}) \leq 2$  and  $\mathcal{E}$  is spanned, then  $\mathcal{E}$  is Fano. In particular all bundles listed in the theorem are Fano.

*Proof.* The divisor  $\xi_r$  is nef, also  $\xi_r$  and  $H$  belong to different rays of the cone of numerically effective divisors in  $\text{Pic}(\mathbb{P}(\mathcal{E})) = \mathbb{Z}\xi + \mathbb{Z}H$ . Because  $c_1 = 0, 1, 2$ , then  $r\xi + (3 - c_1(\mathcal{E}))H$  is ample. The second statement of the lemma is clear, for rank-2 bundles see [8].

(1.11) For Fano bundles the following estimates hold: if  $c_1(\mathcal{E}) = 1$ , then  $c_2(\mathcal{E}) < (5r + 4)/r$  and  $c_1(\mathcal{E}) = 2$ , then  $c_2(\mathcal{E}) < (13r + 5)/2r$ .

*Proof.* This follows from the evaluation of the left-hand side of the inequality  $(-K_{\mathbb{P}(\mathcal{E})})^{r+1} > 0$ , see (1.3).

(1.12) Let  $\mathcal{F}$  be a Fano bundle of rank 2 on  $\mathbf{P}^2$  such that  $1 \leq c_1(\mathcal{F}) \leq 2$ . Then

a)  $\mathcal{F}$  is spanned;

- b)  $\dim H^1(\mathcal{F}^*) = \begin{cases} c_2(\mathcal{F}) - 1 & \text{if } c_1(\mathcal{F}) = 2, c_2(\mathcal{F}) = 2, 3, 4, \\ 0 & \text{otherwise;} \end{cases}$
- c)  $\mathcal{O}^{r-2} \oplus \mathcal{F}$  is a Fano  $r$ -bundle.

*Proof.* The properties a) and b) follow from the presentation of all Fano 2-bundles on  $P^2$ , see [8]; c) then follows from (1.10).

**§ 2. Spannedness of Fano bundles**

The aim of this section is to prove

(2.1) PROPOSITION. *Let  $\mathcal{E}$  be a rank- $r$  vector bundle on  $P^2$  with  $r \geq 3$  and  $0 \leq c_1(\mathcal{E}) \leq r - 1$ . Then  $\mathcal{E}$  is Fano if and only if  $\mathcal{E}$  is spanned.*

Because of (1.10), it remains to prove that if  $\mathbb{P}(\mathcal{E})$  is a Fano manifold, then  $\mathcal{E}$  is spanned. The case  $c_1(\mathcal{E}) = 0$  has already been discussed in (1.9) and hence in the sequel we assume that  $c_1(\mathcal{E}) = 1$  or  $c_1(\mathcal{E}) = 2$ . The proof will be divided into two major steps:

- 1) we shall show that  $\xi_s$  is nef and then
- 2) assuming  $\xi_s$  nef, we prove that  $\mathcal{E}$  is spanned.

Step 1 can be formulated as

(2.2) PROPOSITION. *If  $\mathcal{E}$  is Fano and satisfies assumptions of (2.1) then  $\xi_s$  is nef.*

For a proof of (2.2), let us first recall two results on contractions of extremal rays on Fano manifolds, see [1] and [9].

(2.3) PROPOSITION. *Let  $X$  be a Fano manifold and  $\phi: X \rightarrow Y$  be a contraction of an extremal ray on  $X$ . Assume that all fibres of  $\phi$  are of dimension  $\leq 1$ . Then  $Y$  is smooth and one of the following is true*

- i)  $\phi: X \rightarrow Y$  makes  $X$  a conic bundle over  $T$ ,
- ii)  $\phi: X \rightarrow Y$  makes  $X$  a blow-up of  $Y$  along a smooth submanifold  $Z \subset Y$  of codimension 2.

(2.4) PROPOSITION. *Let  $X$  be a Fano manifold and  $D$  an irreducible and reduced effective divisor on  $X$ . Assume that  $D$  is either ample or nef (in particular semiample) and that the locus of no extremal ray of  $X$  is contracted to a 0-dimensional set by the map associated to the system  $|mD|$ ,  $m \gg 0$ . Let  $\mathcal{L}$  be a line bundle on  $X$  and  $\mathcal{L}_D$  its restriction to  $D$ . Then  $\mathcal{L}$  is ample (respectively, nef) if and only if  $\mathcal{L}_D$  is, unless there exists a contraction with the properties i) or ii) of (2.3). In any of these cases  $D$*

does not contain any curve contracted by  $\phi$  and  $\mathcal{L}$  has non-positive (resp. negative) intersection with any curve contracted by  $\phi$ .

Let us now take a line  $L$  on  $\mathbf{P}^2$  and let  $D = p^{-1}(L)$ ,  $\mathcal{L} = \mathcal{O}(\xi_s)$ . Then  $\mathcal{L}_D$  is nef. Hence, the divisor  $\xi_s$  is always nef, unless the cases i) and ii) of (2.3) occur. Let us assume that is not nef and discuss these remaining cases. Since the splitting type of  $\mathcal{E}$  is as in (1.8), then  $H^2(\mathcal{E}) = H^0(\mathcal{E}^*(-3)) = 0$ . Therefore

$$(2.5) \quad \dim H^0(\mathcal{E}) \geq \chi(\mathcal{E}) = \frac{1}{2}(3c_1(\mathcal{E}) + c_1^2(\mathcal{E})) + r - c_2(\mathcal{E})$$

and then (1.6) and (1.11) give  $\dim H^0(\mathcal{E}) \geq 1$  (we bear in mind (1.9)) with the equality possible only for

$$(2.6) \quad \begin{cases} r = 3, & c_1(\mathcal{E}) = 2, & c_2(\mathcal{E}) = 7, & \text{or} \\ r = 4, & c_1(\mathcal{E}) = 1, & c_2(\mathcal{E}) = 5. \end{cases}$$

This makes the fibre contraction impossible, since  $\dim H_0(\mathcal{E}) > 0$  implies that  $\xi_s$  is an effective divisor, hence has a non-negative intersection with the numerically effective ray whose contraction is being considered. On the other hand, in the divisorial case ii) the effective but not numerically effective divisor, which intersects the contracted ray negatively, is uniquely determined, so ii) cannot happen if  $\dim H^0(\mathcal{E}) > 1$ . Therefore we shall be done if we show that a divisorial contraction cannot occur in the cases listed in (2.6). However, in a divisorial contraction,  $\xi_s$  is the exceptional divisor of the blow-down  $\phi$ . Let  $A$  be the pullback of the ample generator of  $\text{Pic } Y \cong \mathbb{Z}$ . Since the pair  $(\xi_s, A)$  forms a basis of  $\text{Pic } X \cong \mathbb{Z}\xi + \mathbb{Z}H$ , it follows that  $A \equiv k\xi_s + H$  with a positive integer  $k$  (recall that  $A$  and  $H$  generate the cone of nef divisors and  $\xi_s$  is assumed to be not nef thus  $A \not\equiv k\xi_s - H$ ). As  $\xi_s$  is blown down to a codimension-2 variety in  $Y$  we should have

$$(2.7) \quad 0 = \xi_s A^r = k^{r-2} \left( k^2(c_1^2(\mathcal{E}) - c_2(\mathcal{E})) + krc_1(\mathcal{E}) + \frac{r(r-1)}{2} \right).$$

One checks easily that if  $r$ ,  $c_1(\mathcal{E})$  and  $c_2(\mathcal{E})$  are as in (2.6), then the above equality holds for no positive integer  $k$ . This concludes the proof of (2.2).

We shall now make Step 2 and prove that  $\xi_s$  is not only nef, but also globally generated. The fact that  $\xi_s$  is nef, has the following direct implications:

(2.8) Assume  $\mathcal{E}$  is Fano. If  $c_1(\mathcal{E}) = 1$ , then  $c_2(\mathcal{E}) \leq 1$ . If  $c_2(\mathcal{E}) \leq 2$ , then  $c_2(\mathcal{E}) \leq 4$ ,

*Proof.* Indeed, in such a situation  $0 \leq \xi^{r+1} = c_1^2(\mathcal{E}) - c_2(\mathcal{E})$ .

(2.9) If  $c_1(\mathcal{E}) = 1$ , then  $-K_{\mathbb{P}(\mathcal{E})} + \xi_{\mathcal{E}}$  and  $-K_{\mathbb{P}(\mathcal{E})} - H$  are ample. If  $c_1(\mathcal{E}) = 2$ , then  $-K_{\mathbb{P}(\mathcal{E})} + \xi_{\mathcal{E}}$  is ample, also if  $c_2(\mathcal{E}) < c_1^2(\mathcal{E})$ , then  $-K_{\mathbb{P}(\mathcal{E})} + \xi_{\mathcal{E}} - H$  is nef and big.

*Proof.* It follows from (1.2) and (1.3) by simple calculation.

(2.10) Let  $\mathcal{E}$  be as above, i.e. of rank  $r \geq 3$ ,  $c_1(\mathcal{E}) = 1, 2$  and with  $\mathbb{P}(\mathcal{E})$  being Fano. Then  $H^1(\mathcal{E}) = H^2(\mathcal{E}) = 0$  and

$$\dim H^0(\mathcal{E}) = \begin{cases} r + 2 + c_2(\mathcal{E}) & \text{if } c_1(\mathcal{E}) = 1 \\ r + 5 - c_2(\mathcal{E}) & \text{if } c_1(\mathcal{E}) = 2, \end{cases}$$

moreover  $H^1(\mathcal{E}(-1)) = H^2(\mathcal{E}(-1)) = 0$  if  $(c_1(\mathcal{E}), c_2(\mathcal{E})) \neq (2, 4)$  and

$$\dim H^0(\mathcal{E}(-1)) = \begin{cases} 1 - c_2(\mathcal{E}) & \text{if } c_1(\mathcal{E}) = 1 \\ 3 - c_2(\mathcal{E}) & \text{if } c_1(\mathcal{E}) = 2, c_2(\mathcal{E}) \leq 3. \\ 0 & \text{if } c_1(\mathcal{E}) = 2, c_2(\mathcal{E}) = 4. \end{cases}$$

*Proof.* In view of (2.9), the vanishing of higher cohomology follows by Kodaira-Kawamata-Vieweg vanishing theorem. Then the equality for  $\dim H^0(\mathcal{E})$  and  $\dim H^0(\mathcal{E}(-1))$  follows from the Riemann-Roch formula (1.1) except for  $\dim H^0(\mathcal{E}(-1))$  in the case  $c_1(\mathcal{E}) = 2, c_2(\mathcal{E}) = 4$ . In this case  $-K_{\mathbb{P}(\mathcal{E})} + \xi_{\mathcal{E}(-1)}$  is not big. However,  $\xi_{\mathcal{E}}^{r+1} = 0$  and hence  $\xi_{\mathcal{E}}^r(\xi_{\mathcal{E}} - H) = 0 - c_1(\mathcal{E}) = -2$ . The divisor  $\xi_{\mathcal{E}} - H$  cannot then be effective, i.e.,  $H^0(\mathcal{E}(-1)) = H^0(\xi_{\mathcal{E}} - H) = 0$ .

(2.11) PROPOSITION. Let  $\mathcal{E}$  be as above, i.e.,  $\mathbb{P}(\mathcal{E})$  is a Fano manifold, ruled over  $\mathbb{P}^2$ ,  $c_1(\mathcal{E}) = 1, 2$ ,  $c_1^2(\mathcal{E}) \geq c_2(\mathcal{E})$  and  $\xi_{\mathcal{E}}$  is nef. Then  $\mathcal{E}$  is spanned.

*Proof.* We shall study the cases  $(c_1(\mathcal{E}), c_2(\mathcal{E})) \neq (2, 4)$  and its opposite separately. Let  $x$  be a point of  $\mathbb{P}^2$  and  $L$  a line through  $x$ . Consider a diagram with an exact row and column:

$$(2.12) \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & H^0(\mathcal{E} \otimes J_L) = H^0(\mathcal{E}(-1)) & & & \\ & & & \downarrow & & & \\ 0 & \longrightarrow & H^0(\mathcal{E} \otimes J_x) & \longrightarrow & H^0(\mathcal{E}) & \xrightarrow{\psi} & \mathcal{E}_x \\ & & \downarrow & & & & \\ & & H^0(\mathcal{E} \otimes L \otimes J_x) & & & & \end{array}$$

where  $\psi$  is the evaluation morphism and the arrows are defined in a natural way. On  $L$ , the bundle  $\mathcal{E}$  decomposes as stated in (1.8). In all cases where  $c_1(\mathcal{E}) = 1$  or  $2$ , we have  $\dim H^0(\mathcal{E}(-1)|L) = c_1(\mathcal{E})$ . By an obvious diagram chasing in (2.12) we obtain  $\dim H^0(\mathcal{E} \otimes \mathcal{J}_x) \leq 2 - c_2(\mathcal{E})$  if  $c_1(\mathcal{E}) = 1$  and  $\dim H^0(\mathcal{E} \otimes \mathcal{J}_x) \leq 5 - c_2(\mathcal{E})$  if  $c_1(\mathcal{E}) = 2$  and then  $\dim(\text{Im}(\psi)) \geq (r + 2 - c_2) - (2 - c_2) = r$  in case  $c_1(\mathcal{E}) = 1$  and also  $\dim(\text{Im}(\psi)) \geq (r + 5 - c_2) - (5 - c_2) = r$  when  $c_1(\mathcal{E}) = 2$ . This proves (2.11), unless  $c_1(\mathcal{E}) = 2$ ,  $c_2(\mathcal{E}) = 4$ .

(2.13) LEMMA. *Let  $X$  be  $\mathbb{P}(\mathcal{E})$  with a Fano vector bundle  $\mathcal{E}$  over  $\mathbb{P}^2$ ,  $\phi: X \rightarrow Y$  be a contraction of  $X$  (with connected fibres) defined by a multiple of the relative ample divisor  $\xi_{\mathcal{E}}$  on  $X$ . If  $\phi$  has a fibre  $F$  of dimension  $\geq 2$ , then the bundle  $\mathcal{E}$  decomposes as  $\mathcal{E} = \mathcal{E}' \oplus \mathcal{O}$ .*

*Proof.* Let us denote by  $p: X \rightarrow \mathbb{P}^2$  the projection from  $\mathbb{P}(\mathcal{E})$  down onto  $\mathbb{P}^2$ . It is clear that  $p$  cannot contract any curve from  $F$ , so  $\dim F = 2$ . Consider the restriction of  $\phi$  to  $H = p^{-1}(L) \cong \mathbb{P}(\mathcal{E}|L)$ . We see that  $\phi|_H$  contracts sections of the scroll  $p|_H: H \rightarrow L$  hence  $F \cap H$  consists of a sum of mutually disjoint sections of  $p|_H$ . Since it is true for any line  $L$ , we see that  $p|_F: F \rightarrow \mathbb{P}^2$  has to be unramified. Therefore  $F \cong \mathbb{P}^2$  and we have a section of  $p$  defined by  $F$ . Moreover,  $\xi_{\mathcal{E}}|_F = \mathcal{O}$  and we have an exact sequence

$$(2.14) \quad 0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{O} \longrightarrow 0.$$

To prove that this sequence splits it is enough to show that the induced map  $\kappa: H^0(\mathcal{E}) \rightarrow H^0(\mathcal{O})$  is onto. However,  $\kappa$  can be not onto only if  $F \subset Bs|\xi_{\mathcal{E}}|$ . In view of (2.11), (now proved unless  $c_1(\mathcal{E}) = 2$ ,  $c_2(\mathcal{E}) = 4$ ) we are done with exception of this case. But in the case of  $c_1(\mathcal{E}) = 2$ ,  $c_2(\mathcal{E}) = 4$ , by applying similar arguments concerning diagram (2.12), we conclude that  $\text{Im } \phi$  is of codimension at most 1 in  $E_x$ , hence  $Bs|\xi_{\mathcal{E}}|$  meets every fibre of  $p$  at one point at most, so  $F = Bs|\xi_{\mathcal{E}}|$ . In such a case however,  $\mathcal{E}'$  is spanned by the sections of  $\mathcal{E}$ . Therefore  $\mathcal{E}'$  is a Fano bundle, hence  $H^1(\mathcal{E}') = 0$ , see (2.10), and we see immediately that (2.14) splits which concludes the proof of (2.13).

Returning to the proof of (2.11) in the case  $c_1(\mathcal{E}) = 2$ ,  $c_2(\mathcal{E}) = 4$  let us note that, if the contraction of  $\mathbb{P}(\mathcal{E})$  defined by a multiple of  $\xi_{\mathcal{E}}$  has a fibre of dimension  $\geq 2$ , then in view of the above lemma  $\mathcal{E}$  is a direct sum of a trivial line bundle and a Fano bundle of smaller rank, so by



inductive arguments  $\mathcal{E}$  has to be spanned (for  $r = 2$  we have (1.12)). Therefore from (2.3) it follows that we have now to discuss i) and ii) of (2.3). The case ii) is impossible, since  $\xi_r^{r+1} = 0$ . In case i),  $\xi_r$  is the pullback of an ample Cartier divisor  $A$  on  $Y$ ,  $Y$  being the image of the contraction as in (2.3). Since  $H^0(\xi_r) = 2$ , it follows that  $A^r \leq 2$ . We also have  $\dim H^0(Y, \mathcal{O}(A)) = \dim H^0(\mathcal{O}(\xi)) = \dim H^0(\mathcal{E}) = r + 1$  and then we calculate the  $\Delta$ -genus of Fujita [4]:

$$\Delta(Y, A) = 0 \quad \text{if } A^r = 1, \quad \Delta(Y, A) = 1 \quad \text{if } A^r = 2.$$

By Fujita’s classification list of manifolds with sectional genera  $\leq 1$ , see [4], the only polarized manifolds  $(Y, A)$  with these  $\Delta$  and  $A^r$  are  $(\mathbf{P}^r, \mathcal{O}(1))$  or a double covering of  $\mathbf{P}^r$  with the pullback of  $\mathcal{O}(1)$ . In both cases  $A$  is spanned, hence  $\xi_r$  is spanned, too.

**§ 3. The structure of  $\mathcal{E}$  and  $\mathbb{P}(\mathcal{E})$**

The following lemma provides almost complete information needed to describe the bundle  $\mathcal{E}$  from the main theorem. We formulate the lemma in a more general set-up.

(3.1) LEMMA. *Let  $M$  be a complex manifold of dimension  $n$  with  $H^1(M, \mathcal{O}_M) = 0$ , and  $\mathcal{E}_r$  be a vector bundle on  $M$  of rank  $r$ ,  $r \geq n$ , spanned by its global sections. Then there exists a resolution of  $\mathcal{E}_r$ :*

$$(*) \quad \mathcal{E}_r \twoheadrightarrow \mathcal{E}_k \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}_{n+1} \twoheadrightarrow \mathcal{E}_n$$

with each bundle  $\mathcal{E}_i$  spanned of rank  $i$  and the following is satisfied

- i)  $\mathcal{E}_r \cong \mathcal{E}_k \oplus \mathcal{O}_M^{r-k}$ ,
- ii) for each  $i = n + 1, \dots, k$ , the epimorphism  $\mathcal{E}_i \twoheadrightarrow \mathcal{E}_{i-1}$  is obtained from a non-trivial extension of  $\mathcal{E}_{i-1}$  by  $\mathcal{O}_M$ , i.e. we have a non-splitting sequence

$$0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{E}_i \longrightarrow \mathcal{E}_{i-1} \longrightarrow 0$$

- iii)  $k = h^1(\mathcal{E}_n^*) - h^1(\mathcal{E}_r^*)$ .

*Proof.* First we use non-vanishing sections of  $\mathcal{E}_r$  and its quotients (which are also spanned) to construct any resolution

$$(**) \quad \mathcal{E}_r = \mathcal{F}_r \twoheadrightarrow \mathcal{F}_{r-1} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}_{n+1} \twoheadrightarrow \mathcal{F}_n$$

with each  $\mathcal{F}_i$  fitting in a sequence

$$(***) \quad 0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_{i-1} \longrightarrow 0$$

Now we note that

$$h^1(\mathcal{F}_i^*) = \begin{cases} h^1(\mathcal{F}_{i-1}^*) & \text{if } (***) \text{ splits} \\ h^1(\mathcal{F}_{i-1}^*) - 1 & \text{otherwise} \end{cases} .$$

This is clear as from (\*\*\*) we have the exact sequence in cohomologies

$$\longrightarrow H^0(\mathcal{O}_M) \xrightarrow{\alpha} H^1(\mathcal{F}_{i-1}^*) \longrightarrow H^1(\mathcal{F}_i^*) \longrightarrow 0$$

and (\*\*\*) splits if and only if the map  $\alpha$  is zero.

Now we alternate the resolution (\*\*) inductively to obtain the one postulated in the lemma. First we take  $\mathcal{E}_n := \mathcal{F}_n$ . Then assume that  $\mathcal{F}_{i-1} \cong \mathcal{E}_j \oplus \mathcal{O}^{i-1-j}$  so we have (\*\*\*) replaced by

$$0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{E}_j \oplus \mathcal{O}^{i-1-j} \longrightarrow 0 .$$

If  $h^1(\mathcal{F}_i^*) = h^1(\mathcal{E}_j^*)$ , then by the above arguments the sequence splits and  $\mathcal{F}_i \cong \mathcal{E}_j \oplus \mathcal{O}^{i-j}$ . Otherwise we consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & H^0(\mathcal{F}_i) \otimes \mathcal{O} & \longrightarrow & H^0(\mathcal{E}_j) \otimes \mathcal{O} \oplus \mathcal{O}^{i-j-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_M & \longrightarrow & \mathcal{F}_i & \longrightarrow & \mathcal{E}_j \oplus \mathcal{O}^{i-1-j} \longrightarrow 0 \end{array}$$

with exact rows and vertical arrows being evaluation maps. The upper row splits as it comes from a sequence of vector spaces. But this embeds the trivial bundle  $\mathcal{O}^{i-1-j}$  as a component in  $\mathcal{F}_i$ , so we have  $\mathcal{F}_i \cong \mathcal{E}_{j+1} \oplus \mathcal{O}^{i-j-1}$  for some bundle  $\mathcal{E}_{j+1}$  of rank  $j + 1$ . One checks easily that the bundles  $\mathcal{E}_i$  constructed above form a resolution satisfying conditions i)–iii).

Now, using (2.1), (3.1) and also (1.12), by inductive arguments we obtain the description of  $\mathcal{E}$  as in the third column of the table from our theorem. The only cases that need some extra explanation are  $c_1 = 2, c_2 = 2$  and  $c_1 = 2, c_2 = 3$ . In the former case we claim that the only non-trivial extension of rank-2 Fano bundle  $\mathcal{E}_2$  with  $c_1(\mathcal{E}_2) = 2, c_2(\mathcal{E}_2) = 2$  by the trivial line bundle is isomorphic to  $T(-1) \oplus \mathcal{O}(1)$ . Indeed,  $T(-1) \oplus \mathcal{O}(1)$  has a non-vanishing section so it fits into a non-splitting sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow T(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

with  $\mathcal{E}_2$  as above. On the other hand we know that  $\dim H^1(\mathcal{E}_2^*) = 1$ , see (1.12), and because all semistable rank 2 bundles with  $c_1 = 2, c_2 = 2$  are

projectively equivalent (which follows from their presentation as extensions of  $J_x(1)$  by  $\mathcal{O}(1)$ ) we conclude that a bundle which is a non-trivial extension of such a  $\mathcal{E}_2$  by  $\mathcal{O}$  is defined uniquely up to projective equivalence, thus it must be  $T(-1) \oplus \mathcal{O}(1)$ . In the latter case where  $c_1(\mathcal{E}) = 2, c_2 = 3$  we claim that the kernel of the evaluation map  $ev: \mathcal{O}^{r+2} \rightarrow \mathcal{E}$  is isomorphic to  $\mathcal{O}(-1)^2$ . But the bundle dual to  $\ker(ev)$  is then generated by global sections and has Chern classes  $c_1 = 2, c_2 = 1$ , so it is Fano hence is isomorphic to  $\mathcal{O}(1)^2$  as follows from the other part of our classification.

Now we want to examine the geometric structure of  $X = \mathbb{P}(\mathcal{E})$  in terms of the contraction defined by the linear system  $|\xi_s|$ , see the list in the Theorem. The description is quite clear if  $\mathcal{E}$  splits or if  $\dim H^0(\mathcal{E}) = r + 1$ . By standard arguments, in the former case  $X = \mathbb{P}(\mathcal{E})$  is a blow-up of an appropriate cone along its vertex; whereas in the latter it is a divisor in  $P^2 \times P^r$ . The only two delicate cases are again  $c_1(\mathcal{E}) = 2, c_2(\mathcal{E}) = 2$  and  $c_1(\mathcal{E}) = 2, c_2(\mathcal{E}) = 3$ . From (2.10) and (1.3) we obtain

(3.2) If  $c_1(\mathcal{E}) = 2$ , then

$$\begin{cases} \xi^{r+1} = 2, \dim H^0(\xi) = r + 3 & \text{if } c_2(\mathcal{E}) = 2, \\ \xi^{r+1} = 1, \dim H^0(\xi) = r + 2 & \text{if } c_2(\mathcal{E}) = 3. \end{cases}$$

This shows that the system  $|\xi_s|$  defines a birational map  $\phi$  from  $X$  onto  $P^{r+1}$  if  $c_2(\mathcal{E}) = 3$ , whereas the map  $\phi$  is onto a (possibly singular) quadric  $Q_{r+1} \subseteq P^{r+2}$  for  $c_2(\mathcal{E}) = 2$ . We claim that  $\phi$  is a contraction of  $X$  of divisorial type. This is clearly seen if we consider the restriction of  $\phi$  to an intersection of  $r - 2$  divisors from  $|\xi_s|$ -what we get is a divisorial contraction of a Fano 3-fold  $\mathbb{P}(\mathcal{E}_2)$  with  $\mathcal{E}_2$  as in (3.1). The divisor  $E$  that is blown down by  $\phi$  has the intersection number with  $\xi_s^r$  equal to 0 and together with  $\xi$  it generates  $\text{Pic}(X)$ . Therefore, by simple calculation  $E = \xi_s - H$  if  $c_2(\mathcal{E}) = 2$  and  $E = 2\xi_s - H$  if  $c_2(\mathcal{E}) = 3$ . The degree of the image  $Z = \phi(E)$  in  $P^{r+1}$  (resp. in  $Q_{r+1} \subset P^{r+2}$ ) is equal to  $-E^2 \xi^{r-1}$  which is 3 or, respectively, 1. Hence in case  $c_1(\mathcal{E}) = 2, c_2(\mathcal{E}) = 2, Z$  is a linear subspace. The singularity of  $Q_{r+1}$  depends on the rank of the trivial component of  $\mathcal{E}$ , see (2.3) and (2.13). In the case  $c_1(\mathcal{E}) = 2, c_2(\mathcal{E}) = 3$  let us first note that  $Z$  is not contained in any hyperplane in  $P^{r+1}$ . Indeed, if it were the case, then  $E$  would be contained in a divisor from  $|\xi_s|$  which is impossible since  $\xi_s - E = -\xi_s + H$ . Therefore  $Z$  is a minimal variety in the sense of [3] and by the classification list from [3] we infer that it is a rational normal scroll of degree 3, as stated in our Theorem.

The type of the scroll that we obtain by contracting  $E$  depends again on the rank of the trivial component of  $\mathcal{E}$  (see again (2.3) and (2.13))

*Remark.* For  $\mathcal{E} = \mathcal{T}(-1) \oplus \mathcal{O}(1)$  the diagram

$$\begin{array}{ccc} X = \mathbb{P}(\mathcal{E}) & \xrightarrow{\phi} & \mathbf{Q}_4 \\ p \downarrow & & \\ \mathbf{P}^2 & & \end{array}$$

has a nice geometric interpretation. Namely, we can see  $\mathbf{Q}_4$  as  $\text{Grass}(1, \mathbb{P}(V))$ , i.e., as the Grassmann variety of lines on a three-dimensional projective space  $\mathbb{P}(V)$  and our plane  $\mathbf{P}^2 \cong Z \subset \mathbf{Q}_4$  can be understood as the Schubert variety of lines which are contained in a fixed projective plane  $\mathbb{P}(W) \subset \mathbb{P}(V)$ . Now, the above diagram is a resolution of a rational map

$$\mathbf{Q}_4 = \text{Grass}(1, \mathbb{P}(V)): \dashrightarrow \mathbb{P}(W) = \mathbf{P}^2,$$

which to a general point on  $\text{Grass}(1, \mathbb{P}(V))$  representing a line  $L \subset \mathbb{P}(V)$  associates the point where  $L$  meets  $\mathbb{P}(W)$ . It may be worthwhile to note that similar construction applies to other Grassmann varieties whose blow-ups along some sub-Grassmannians (some special Schubert cycles) can be presented as projective bundles over other Grassmann varieties.

*Remark.* Using the presentation of Fano bundles given by the Theorem, one may obtain a description of the moduli space for Fano bundles. The technique developed in this paper will also work for studying vector bundles over any Del Pezzo surface. As in the case of rank-2 bundles, see [8], the results seem to be less interesting.

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