BULL. AUSTRAL. MATH. SOC. 33E05, 11F11, 11F20, 11F27
VOL. 71 (2005) [353–358]

A SIMPLE PROOF OF AN EXPANSION OF AN ETA-QUOTIENT AS A LAMBERT SERIES

SHAUN COOPER

We give a simple proof of the identity

$$\prod_{n=1}^{\infty} \frac{(1-q^{3n})^{10}}{(1-q^n)^3(1-q^{9n})^3} = 1 + 3\sum_{\substack{n=1 \\ 9\nmid n}}^{\infty} \frac{nq^n}{1-q^n}.$$

The proof uses only a few well-known properties of the cubic theta functions $a(q)$, $b(q)$ and $c(q)$. We show this identity implies the interesting definite integral

$$\int_0^{e^{-2\pi/3}} \prod_{n=1}^{\infty} \frac{(1-q^{3n})^{10}}{(1-q^n)^6} dq = \frac{1}{3\sqrt{3}}.$$

1. INTRODUCTION

The purpose of this article is to give a direct proof of the following identity.

THEOREM 1.1. *Let q be a complex number satisfying $|q| < 1$. Then*

$$\prod_{n=1}^{\infty} \frac{(1-q^{3n})^{10}}{(1-q^n)^3(1-q^{9n})^3} = 1 + 3\sum_{\substack{n=1 \\ 9\nmid n}}^{\infty} \frac{nq^n}{1-q^n}.$$

The summation is over all positive integers n excluding multiples of 9.

This result was discovered using symbolic computation by Borwein and Garvan [**4**], and it was used to produce a ninth order iteration that converges to $1/\pi$. The proof of Theorem 1.1 in [**4**] appeals to two entries in Ramanujan's Notebook [**16**, Chapter 20 Entry 1(iv) and Chapter 21 Entry 7(i)]. The proofs of these entries in Berndt's excellent book [**1**] take several pages, and appeal to several earlier results in Ramanujan's Notebook.

Two proofs of Theorem 1.1 were given by Berndt, Chan, Liu and Yesilyurt [**3**]. The first is essentially the same as the one in [**4**]. The second proof in [**3**] uses less sophisticated machinery, but is more than three pages long, and depends on another entry in Ramanujan's Notebook [**16**, Chapter 20 Entry 1(v)].

Received May 9, 2005

Another proof of Theorem 1.1 of a completely different nature was obtained by Farkas and Kra [**9**, p. 307]. Their proof uses meromorphic functions defined on Riemann surfaces.

In view of the importance of Theorem 1.1, it is desirable to have as direct a proof as possible. We give such a proof, which depends only on the well-known properties satisfied by the cubic theta functions $a(q)$, $b(q)$ and $c(q)$ given in Lemma 2.1 below.

We conclude by showing that Theorem 1.1 leads to an evaluation of a definite integral. Three similar integrals were given by Fine [**10**, pp. 86–91].

2. Proof

The three cubic theta functions are defined by

$$a(q) = \sum_m \sum_n q^{m^2+mn+n^2},$$
$$b(q) = \sum_m \sum_n q^{m^2+mn+n^2} \omega^{m-n},$$
$$c(q) = \sum_m \sum_n q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2},$$

where $\omega = \exp(2\pi i/3)$ and $q = e^{-2\pi t}, \mathrm{Re}(t) > 0$. The summation indices m and n range over all integer values. The following are some well known properties of the cubic theta functions.

Lemma 2.1.

(2.2) $$a(q)^3 = b(q)^3 + c(q)^3,$$

(2.3) $$b(q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^{3n})},$$

(2.4) $$c(q) = 3q^{1/3} \prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{(1-q^n)},$$

(2.5) $$a(q) = a(q^3) + 2c(q^3),$$

(2.6) $$b(q) = a(q^3) - c(q^3),$$

(2.7) $$a(q) = 1 + 6 \sum_{n=1}^{\infty} \left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}} \right),$$

(2.8) $$a(q)^2 = 1 + 12 \sum_{\substack{n=1 \\ 3 \nmid n}}^{\infty} \frac{nq^n}{1-q^n}.$$

Equation (2.2) was discovered and proved by Borwein and Borwein [**5**]. Additional proofs have since been given by Borwein, Borwein and Garvan [**6**], Chapman [**7**], Garvan [**11**], Hirschhorn, Garvan and Borwein [**12**], Liu [**14**] and Solé [**18**]. Proofs of (2.3)–(2.6) can be found in [**6, 11, 12**]. Equation (2.7) was known to Lorenz and Ramanujan; see [**13**]. A beautiful and elementary proof of (2.8) using (2.7) was given by Ramanujan [**15**, equation 19].

LEMMA 2.9. *Let* $x = c(q)^3/a(q)^3$, $z = a(q)$, $X = c(q^3)^3/a(q^3)^3$, $Z = a(q^3)$. *Then*

$$\begin{aligned} x &= 1 - \left(\frac{1 - X^{1/3}}{1 + 2X^{1/3}}\right)^3, \\ z &= Z(1 + 2X^{1/3}), \\ X &= \left(\frac{1 - (1-x)^{1/3}}{1 + 2(1-x)^{1/3}}\right)^3, \\ Z &= \frac{z}{3}(1 + 2(1-x)^{1/3}). \end{aligned}$$

PROOF: From Lemma 2.1 we have

$$\begin{aligned} 1 - x &= 1 - \frac{c(q)^3}{a(q)^3} \\ &= \frac{b(q)^3}{a(q)^3} \\ &= \left(\frac{a(q^3) - c(q^3)}{a(q^3) + 2c(q^3)}\right)^3 \\ &= \left(\frac{1 - X^{1/3}}{1 + 2X^{1/3}}\right)^3. \end{aligned}$$

This proves the first part. Similarly,

$$\begin{aligned} z &= a(q) \\ &= a(q^3) + 2c(q^3) \\ &= a(q^3)\left(1 + 2\frac{c(q^3)}{a(q^3)}\right) \\ &= Z(1 + 2X^{1/3}). \end{aligned}$$

This proves the second part. The third and fourth parts are obtained by rearranging the first two parts and solving for X and Z. ∎

REMARK 2.10. The first two formulas in Lemma 2.9 are called the trimidiation formulas, and the last two are called the triplication formulas. See [**2**, pp. 101–102] for another proof and further explanation.

PROOF OF THEOREM 1.1: Using Lemmas 2.1 and 2.9, we have

$$\begin{aligned}
1+3\sum_{\substack{n=1\\ 9\nmid n}}^{\infty}\frac{nq^n}{1-q^n} &= \frac{1}{4}\left(a(q)^2+3a(q^3)^2\right)\\
&= \frac{1}{4}\left(z^2+\frac{z^2}{3}(1+2(1-x)^{1/3})^2\right)\\
&= \frac{z^2}{3}\left(1+(1-x)^{1/3}+(1-x)^{2/3}\right)\\
&= \frac{z^2x}{3(1-(1-x)^{1/3})}\\
&= \frac{c(q)^3}{3a(q)(1-b(q)/a(q))}\\
&= \frac{c(q)^3}{3(a(q)-b(q))}\\
&= \frac{c(q)^3}{9c(q^3)}\\
&= \prod_{n=1}^{\infty}\frac{(1-q^{3n})^{10}}{(1-q^n)^3(1-q^{9n})^3}.
\end{aligned}$$

□

3. A DEFINITE INTEGRAL

In this section we state and prove the value of an interesting definite integral. We use the same method of proof as Fine [10, pp. 86–91], who gave three similar integrals.

THEOREM 3.1.

$$\int_0^{e^{-2\pi/3}}\prod_{n=1}^{\infty}\frac{(1-q^{3n})^{10}}{(1-q^n)^6}\,dq=\frac{1}{3\sqrt{3}}.$$

PROOF: From Theorem 1.1 we have

$$\prod_{n=1}^{\infty}\frac{(1-q^{3n})^{10}}{(1-q^n)^3(1-q^{9n})^3}=1+3\sum_{\substack{n=1\\ 9\nmid n}}^{\infty}\frac{nq^n}{1-q^n}.$$

If we multiply by $\prod_{n=1}^{\infty}\frac{(1-q^{9n})^3}{(1-q^n)^3}$, we get

$$\begin{aligned}
\prod_{n=1}^{\infty}\frac{(1-q^{3n})^{10}}{(1-q^n)^6} &= \prod_{n=1}^{\infty}\frac{(1-q^{9n})^3}{(1-q^n)^3}\ \left\{1+3\sum_{\substack{n=1\\ 9\nmid n}}^{\infty}\frac{nq^n}{1-q^n}\right\}\\
&= \frac{d}{dq}\left\{q\prod_{n=1}^{\infty}\frac{(1-q^{9n})^3}{(1-q^n)^3}\right\},
\end{aligned}$$

or equivalently

$$\int_0^q \prod_{n=1}^{\infty} \frac{(1-s^{3n})^{10}}{(1-s^n)^6}\, ds = q \prod_{n=1}^{\infty} \frac{(1-q^{9n})^3}{(1-q^n)^3}. \tag{3.2}$$

Recall the modular transformation for the Dedekind eta function, for example, see [8, Theorem 4.11], which may be written in the form

$$q^{1/24} \prod_{n=1}^{\infty} (1-q^n) = \frac{1}{\sqrt{t}} p^{1/24} \prod_{n=1}^{\infty} (1-p^n),$$

where $q = e^{-2\pi t}$, $p = e^{-2\pi/t}, \mathrm{Re}(t) > 0$. If we take $t = 1/3$, then $p = q^9$, and so in this case the modular transformation implies

$$\begin{aligned} q \prod_{n=1}^{\infty} \frac{(1-q^{9n})^3}{(1-q^n)^3} &= q \prod_{n=1}^{\infty} \frac{(1-p^n)^3}{(1-q^n)^3} \\ &= q \Big(\frac{q}{p}\Big)^{1/8} \big(\sqrt{t}\big)^3 \\ &= t^{3/2} \\ &= \frac{1}{3\sqrt{3}}. \end{aligned}$$

Using this in (3.2) we complete the proof. ∎

References

[1] B.C. Berndt, *Ramanujan's Notebooks, Part III* (Springer-Verlag, New York, 1991).

[2] B.C. Berndt, *Ramanujan's Notebooks, Part V* (Springer-Verlag, New York, 1998).

[3] B.C. Berndt, S.H. Chan, Z.-G. Liu and H. Yesilyurt, 'A new identity for $(q;q)_\infty^{10}$ with an application to Ramanujan's partition congruence modulo 11', *Q.J. Math.* **55** (2004), 13–30.

[4] J.M. Borwein and F.G. Garvan, 'Approximations to π via the Dedekind eta function', *CMS Conf. Proc.* **20** (1997), 89–115.

[5] J.M. Borwein and P.B. Borwein, 'A cubic counterpart of Jacobi's identity and the AGM', *Trans. Amer. Math. Soc.* **323** (1991), 691–701.

[6] J.M. Borwein, P.B. Borwein and F.G. Garvan, 'Some cubic modular identities of Ramanujan', *Trans. Amer. Math. Soc.* **343** (1994), 35–47.

[7] R. Chapman, 'Cubic identities for theta series in three variables', *Ramanujan J.* **8** (2004), 459–465.

[8] S. Cooper, 'Cubic theta functions', *J. Comput. Appl. Math.* **160** (2003), 77–94.

[9] H.M. Farkas and I. Kra, *Theta constants, Riemann surfaces and the modular group*, Graduate Studies in Mathematics **37** (Amer. Math. Soc., Providence, RI, 2001).

[10] N.J. Fine, *Basic hypergeometric series and applications*, Mathematical Surveys and Monographs **27** (American Mathematical Society, Providence, RI, 1988).

https://doi.org/10.1017/S000497270003834X Published online by Cambridge University Press

[11] F. Garvan, 'Cubic modular identities of Ramanujan, hypergeometric functions and analogues of the arithmetic-geometric mean iteration', *Contemp. Math.* **166** (1994), 245–264.

[12] M. Hirschhorn, F. Garvan and J. Borwein, 'Cubic analogues of the Jacobian theta functions $\theta(z,q)$', *Canad. J. Math.* **45** (1993), 673–694.

[13] M. Hirschhorn, 'Three classical results on representations of a number', *Sém. Lothar. Combin.* **42** (1999). Art. B42f, 8 pp. (electronic).

[14] Z-G. Liu, 'Some Eisenstein series identities associated with the Borwein functions', *Dev. Math.* **4** (2001), 147–169.

[15] S. Ramanujan, 'On certain arithmetical functions', *Trans. Camb. Phil. Soc.* **22** (1916), 159–184. Reprinted in [**17**, 136–162].

[16] S. Ramanujan, *Notebooks*, (2 volumes) (Tata Institute of Fundamental Research, Bombay, 1957).

[17] S. Ramanujan, *Collected papers* (AMS Chelsea Publishing, Providence, RI, 2000).

[18] P. Solé, 'D_4, E_6, E_8 and the AGM', in *Applied algebra, algebraic algorithms and error-correcting codes (Paris, 1995)*, Lecture Notes in Computer Science **948** (Springer, Berlin, 1995), pp. 448–455.

Institute of Information and Mathematical Sciences
Massey University - Albany
Private Bag 102904
North Shore Mail Centre
Auckland
New Zealand
e-mail: s.cooper@massey.ac.nz