14
Confinement versus screening

One of the most challenging problems of gauge dynamics in four dimensions is how to show that QCD admits confinement. One of the measures of confinement is the fact that the potential between an external quark and an external anti-quark placed at a separation distance L, as in Fig. 14.1, is dominated by a linear dependence, namely,

$$V = \sigma L. \tag{14.1}$$

The coefficient in this linear dependence is the *string tension*. Thus, a non-confining behavior, which will be referred to as a screening behavior, implies a vanishing string tension. Whereas in four dimensions the computation of the string tension is a formidable task, in two dimensions, as will be shown in this chapter, it is a fairly easy one. In this chapter we describe the extraction of the string tension in various two-dimensional gauge systems.

We start by calculating the string tension for the massive Schwinger model in both the fermionic and the bosonic languages. This is done in the small mass limit and then we discuss the corrections due to going beyond this limit. We then discuss the short range corrections to the confining potential. We focus on the abelian case, believing that the non-abelian case is very similar. Next we comment on the behavior of the string tension when finite temperature is introduced. Then we move to non-abelian generalization. We compute the string tension for the cases of matter in the fundamental and adjoint representations, followed by the symmetric and anti-symmetric representations.

Much of this chapter is based on [15] and [16].

The string tension of the massive Schwinger model was calculated using bosonizaton in [68]. The massless cases in gauge theories were analyzed in [116]. The next-to-leading order in small mass was computed by [4].

14.1 The string tension of the massive Schwinger model

We start with the derivation of the string tension in the massive Schwinger model, in the fermionic language. Consider the partition function of two dimensional massive QED_2,

$$Z = \int DA_\mu D\bar{\Psi} D\Psi \exp\left(i\int \mathrm{d}^2x \ \left(-\frac{1}{4e^2}F_{\mu\nu}^2 + \bar{\Psi}i\partial\!\!\!/\Psi - m\bar{\Psi}\Psi - q_{\mathrm{dyn}}A_\mu\bar{\Psi}\gamma^\mu\Psi\right)\right), \tag{14.2}$$

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Fig. 14.1. Quark anti-quark separated at a distance L.

where $q_{\rm dyn}$ is the charge of the dynamical fermions. Gauge fixing terms were not written explicitly. Let us add an external pair with charges $\pm q_{\rm ext}$ at $\pm L$, namely,

$$j_0^{\rm ext} = q_{\rm ext}(\delta(x+L) - \delta(x-L)), \tag{14.3}$$

so that the change of $\mathcal{L}$ is $-j_\mu^{\rm ext} A^\mu(x)$. Note that by choosing $j_\mu^{\rm ext}$ which is conserved, $\partial^\mu j_\mu^{\rm ext} = 0$, the action including the coupling to the external current is also gauge invariant.

Now, one can eliminate this charge by performing a local, space-dependent left-handed rotation,

$$\Psi \to \mathrm{e}^{i\alpha(x)\frac{1}{2}(1-\gamma_5)}\Psi \tag{14.4}$$

$$\bar{\Psi} \to \bar{\Psi}\mathrm{e}^{-i\alpha(x)\frac{1}{2}(1+\gamma_5)}, \tag{14.5}$$

where $\gamma^5 = \gamma^0\gamma^1$. We choose a left-handed rotation (or equally well a right-handed one) rather than an axial one, since in the non-abelian case the former will be easier to implement.

The new action is,

$$S = \int \mathrm{d}^2x \left[-\frac{1}{4e^2}F_{\mu\nu}^2 + \bar{\Psi}i\partial\!\!\!/\Psi - \bar{\Psi}\partial_\mu\alpha(x)\gamma^\mu\frac{1}{2}(1-\gamma_5)\Psi - m\bar{\Psi}\mathrm{e}^{-i\alpha(x)\gamma_5}\Psi \right.$$
$$\left. -\ q_{\rm dyn}A_\mu\bar{\Psi}\gamma^\mu\Psi - q_{\rm ext}(\delta(x+L) - \delta(x-L))A_0 + \frac{\alpha(x)q_{\rm dyn}}{2\pi}F \right], \tag{14.6}$$

where the last term is induced by the chiral anomaly,

$$\delta S = \int \mathrm{d}^2x \frac{\alpha(x)q_{\rm dyn}}{2\pi}F, \tag{14.7}$$

with F the dual of the electric field, $F = \frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu}$.

The external source and the anomaly term are similar, both being linear in the gauge potential. This is the reason that the θ-vacuum, to be discussed in Chapter 22, and electron-positron pair at the boundaries are the same in two dimensions.

In the following we assume $\theta = 0$, as otherwise we absorb it in to α. Choosing the $A_1 = 0$ gauge and integrating by parts, the anomaly term looks like an external source,

$$\frac{q_{\rm dyn}}{2\pi}A_0\partial_1\alpha(x). \tag{14.8}$$

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This term can cancel the external source by the choice,

$$\alpha(x) = 2\pi \frac{q_{\rm ext}}{q_{\rm dyn}} (\theta(x+L) - \theta(x-L)). \tag{14.9}$$

Let us take the limit $L \to \infty$. The form of the action, in the region B of $-L < x < L$ is,

$$S_B = \int_B \mathrm{d}^2x \left(-\frac{1}{4e^2} F_{\mu\nu}^2 + \bar{\Psi} i \partial\!\!\!/ \Psi - m\bar{\Psi} \mathrm{e}^{-i2\pi \frac{q_{\rm ext}}{q_{\rm dyn}} \gamma_5} \Psi - q_{\rm dyn} A_\mu \bar{\Psi} \gamma^\mu \Psi \right). \tag{14.10}$$

Thus the total impact of the external electron-positron pair is a chiral rotation of the mass term. This term can be written as,

$$\bar{\Psi} e^{-i2\pi \frac{q_{\rm ext}}{q_{\rm dyn}} \gamma_5} \Psi = \cos\left(2\pi \frac{q_{\rm ext}}{q_{\rm dyn}}\right) \bar{\Psi}\Psi - i \sin\left(2\pi \frac{q_{\rm ext}}{q_{\rm dyn}}\right) \bar{\Psi}\gamma_5\Psi. \tag{14.11}$$

The string tension is the vacuum expectation value (v.e.v.) of the Hamiltonian density in the presence of the external source relative to the v.e.v. of the Hamiltonian density without the external source, in the $L \to \infty$ limit,

$$\sigma = <\mathcal{H}> - <\mathcal{H}_0>_0, \tag{14.12}$$

where $|0>_0$ is the vacuum state with no external sources. The change in the vacuum energy is due to the mass term. The change in the kinetic term which appears in (14.6) does not contribute to the vacuum energy.

Thus,

$$\sigma = m \cos\left(2\pi \frac{q_{\rm ext}}{q_{\rm dyn}}\right) <\bar{\Psi}\Psi> - m \sin\left(2\pi \frac{q_{\rm ext}}{q_{\rm dyn}}\right) <\bar{\Psi} i\gamma_5 \Psi> - m <\bar{\Psi}\Psi>_0 . \tag{14.13}$$

The values of the condensates $<\bar{\Psi}\Psi>$ and $<\bar{\Psi}\gamma_5\Psi>$ are needed. The easiest way to compute these condensates is bosonization, but it can also be computed directly in the fermionic language. We state here the final result for the $m = 0$ case (the derivation can be found in the references of this chapter),

$$<\bar{\Psi}\Psi>_{m=0} = -e\frac{\exp(\gamma)}{2\pi^{3/2}} \tag{14.14}$$

$$<\bar{\Psi}\gamma_5\Psi>_{m=0} = 0. \tag{14.15}$$

Equation (14.15) is due to parity invariance (with our choice $\theta = 0$). The resulting string tension, to first order in m,

$$\sigma = me\frac{\exp(\gamma)}{2\pi^{3/2}} \left(1 - \cos\left(2\pi \frac{q_{\rm ext}}{q_{\rm dyn}}\right)\right). \tag{14.16}$$

Though this expression is only the leading term in a m/e expansion and might be corrected, when $q_{\rm ext}$ is an integer multiple of $q_{\rm dyn}$ the string tension is *exactly* zero, since in this case the rotated action (14.10) is not changed from the original one (14.2).

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14.2 The Schwinger model in bosonic form

Next we derive the same result in the bosonized formulation.

The bosonized Lagrangian, in the gauge $A_1 = 0$, is given by,

$$\mathcal{L} = \frac{1}{2e^2}(\partial_1 A_0)^2 + \frac{1}{2}(\partial_\mu \phi)^2 + M^2 \cos(2\sqrt{\pi}\phi) + \frac{q_{\rm dyn}}{\sqrt{\pi}} A_0 \partial_1 \phi - A_0 j_{\rm ext}, \quad (14.17)$$

where $M^2 = m\mu$, $\mu = \frac{\exp(\gamma)}{2\pi}\mu_{(\phi)}$ with $\mu_{(\phi)} = \frac{e}{\sqrt{\pi}} q_{\rm dyn}$ the mass of the photon, for $e \gg m$.

Chiral rotation corresponds to a shift in the field ϕ. Upon the transformation,

$$\phi = \tilde{\phi} + \sqrt{\pi}\frac{q_{\rm ext}}{q_{\rm dyn}} \left(\theta(x+L) - \theta(x-L)\right). \quad (14.18)$$

The Lagrangian (14.17) takes, in the region B, the form,

$$\mathcal{L}_B = \frac{1}{2e^2}(\partial_1 A_0)^2 + \frac{1}{2}(\partial_\mu \tilde{\phi})^2 + M^2 \cos\left(2\sqrt{\pi}\tilde{\phi} + 2\pi\frac{q_{\rm ext}}{q_{\rm dyn}}\right) + \frac{q_{\rm dyn}}{\sqrt{\pi}} A_0 \partial_1 \tilde{\phi}. \quad (14.19)$$

Hence, similarly to the previous derivation, a local chiral rotation was used to eliminate the external source. The calculation of the string tension is exactly the same as in the previous section.

The relevant part of the Hamiltonian density is,

$$\mathcal{H} = -M^2 \cos\left(2\sqrt{\pi}\tilde{\phi} + 2\pi\frac{q_{\rm ext}}{q_{\rm dyn}}\right). \quad (14.20)$$

To zeroth order in $(\frac{M}{e})^2$, the vacuum is $\tilde{\phi} = 0$. Setting this choice in (14.20) and subtracting the v.e.v. of the free Hamiltonian, we arrive at,

$$\sigma_{QED} = m\mu\left(1 - \cos\left(2\pi\frac{q_{\rm ext}}{q_{\rm dyn}}\right)\right), \quad (14.21)$$

where m is the electron mass, $\mu = e\frac{\exp(\gamma)}{2\pi^{3/2}}$, e the gauge coupling, γ the Euler number and $q_{\rm ext}$, $q_{\rm dyn}$ are the external and dynamical charges, respectively (we measure charge in units of e, thus $q_{\rm ext}$ and $q_{\rm dyn}$ are dimensionless).

14.3 Beyond the small mass abelian string tension

The expression (14.21) contains only the leading $\frac{m}{e}$ contribution to the abelian string tension. This expression was computed in the previous section, using a classical average. However, as we used the normal ordering scale μ_ϕ which is the photon mass for $e \gg m$, taking $\tilde{\phi} = 0$ actually gives the full quantum answer, as is evident by comparing with the fermionic calculation in the section before that.

The full perturbative (in m) string tension can be written as,

$$\sigma_{QED} = m\mu \sum_{l=1}^{\infty} C_l \left(\frac{m}{e q_{\rm dyn}}\right)^{l-1} \left(1 - \cos\left(2\pi l \frac{q_{\rm ext}}{q_{\rm dyn}}\right)\right). \quad (14.22)$$

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The value of the first coefficient is $C_1 = 1$ and the next was found to be $C_2 = -8.9\frac{\exp(\gamma)}{8\pi^{1/2}}$. Higher coefficients are not calculated yet.

Note that for finite $\frac{m}{e}$ we have to minimize the potential,

$$V = M^2\left(1 - \cos\left(2\sqrt{\pi}\phi + 2\pi\frac{q_{\text{ext}}}{q_{\text{dyn}}}\right)\right) + \frac{1}{2}\mu_\phi^2\phi^2. \tag{14.23}$$

The minimum $\phi = \phi_m$ obeys,

$$2\sqrt{\pi}M^2 \sin\left(2\sqrt{\pi}\phi_m + 2\pi\frac{q_{\text{ext}}}{q_{\text{dyn}}}\right) + \mu_\phi^2\phi_m = 0. \tag{14.24}$$

Thus, for the first-order $(\frac{m}{eq_{\text{dyn}}})$ correction, we get a C_2 which is $-(\frac{1}{2})\sqrt{\pi}(\exp\gamma)$. This has the same sign, but a factor 1.41 larger, than the instanton contribution.

Note that all above results for the string tension are symmetric under change of sign of the external charge, as expected on general grounds. However, when a θF term is introduced, we get odd terms as well, like $\sin(l\theta)\sin(2\pi l\frac{q_{\text{ext}}}{q_{\text{dyn}}})$. The even terms are multiplied by $\cos(l\theta)$.

Finally, let us remark that for very large $\frac{m}{e}$, the abelian case has a string tension which is $\frac{1}{2}e^2q_{\text{ext}}^2$.

14.4 Correction to the leading long distance abelian potential

The potential (14.1) is the dominant long-range term. However, there are, of course, corrections. In this section we present these corrections.

The equations of motions which follow from the bosonized Lagrangian (14.17) are, in the static case,

$$-\frac{1}{e^2}\partial_1^2 A_0 + \frac{q_{\text{dyn}}}{\sqrt{\pi}}\partial_1\phi - j_{\text{ext}} = 0 \tag{14.25}$$

$$-\partial_1^2\phi + 2\sqrt{\pi}M^2 \sin 2\sqrt{\pi}\phi + \frac{q_{\text{dyn}}}{\sqrt{\pi}}\partial_1 A_0 = 0. \tag{14.26}$$

In order to solve these equation, it is useful to eliminate the bosonized matter field ϕ. Using the approximation $\sin 2\sqrt{\pi}\phi \sim 2\sqrt{\pi}\phi$, we arrive at (in momentum space),

$$A_0(k) = \frac{e^2(k^2 + 4\pi M^2)}{k^2\left(k^2 + \left(4\pi M^2 + \frac{e^2}{\pi}q_{\text{dyn}}^2\right)\right)} j_{\text{ext}}(k), \tag{14.27}$$

where k is the Fourier transform of the space coordinate. We will discuss the validity of our approximation for ϕ later in this section. The last equation can be rewritten as,

$$A_0(k) = \left(\frac{m_1^2}{m_2^2}\frac{1}{k^2} + \left(1 - \frac{m_1^2}{m_2^2}\right)\frac{1}{k^2 + m_2^2}\right)e^2 j_{\text{ext}}(k), \tag{14.28}$$

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where,

$$m_1^2 = 4\pi M^2, \tag{14.29}$$

$$m_2^2 = 4\pi M^2 + \frac{e^2}{\pi} q_{\text{dyn}}^2. \tag{14.30}$$

Note that the photon propagator has two poles, a massless pole that reproduces the string tension and a massive pole which adds a screening term to the potential. Note that there is no $\frac{\text{const.}}{L}$ correction, which appears in higher dimensions, since in the present case the string cannot fluctuate in transverse directions.

Note also that in the massless case, when $M^2 = 0$, only the second term survives and the photon has only one pole with mass square $\frac{e^2}{\pi} q_{\text{dyn}}^2$. This result is of course exact, independent of our approximation.

The resulting gauge field is,

$$\begin{aligned} A_0(x) = {} & \frac{2\pi^2 M^2 q_{\text{ext}}}{q_{\text{dyn}}^2} (|\, x+L \,| - |\, x-L \,|) \\ & - \frac{e\sqrt{\pi}}{2} \frac{q_{\text{ext}}}{q_{\text{dyn}}} \left(\mathrm{e}^{-\frac{e}{\sqrt{\pi}} q_{\text{dyn}} |x+L|} - \mathrm{e}^{-\frac{e}{\sqrt{\pi}} q_{\text{dyn}} |x-L|} \right), \end{aligned} \tag{14.31}$$

where we took $M^2 \ll e^2$ for simplicity.

In order to calculate the potential we will use,

$$V = \frac{1}{2} \int A_0(x) j_{\text{ext}}(x) \mathrm{d}x. \tag{14.32}$$

Hence the potential is,

$$V = 2\pi^2 M^2 \frac{q_{\text{ext}}^2}{q_{\text{dyn}}^2} \times 2L + \frac{e\sqrt{\pi}}{2} \frac{q_{\text{ext}}^2}{q_{\text{dyn}}} \left(1 - e^{-\frac{e}{\sqrt{\pi}} q_{\text{dyn}} 2L} \right). \tag{14.33}$$

The first term is the confining potential which exists whenever the quark mass is non-zero. On top of this, there is always a screening potential.

The string tension which results from the above potential is,

$$\sigma = m\mu \times 2\pi^2 \frac{q_{\text{ext}}^2}{q_{\text{dyn}}^2}, \tag{14.34}$$

which is exactly (14.21) in the approximation $2\pi \frac{q_{\text{ext}}}{q_{\text{dyn}}} \ll 1$. This turns out to be also the condition for $\sin 2\sqrt{\pi}\phi \sim 2\sqrt{\pi}\phi$ that we assumed at the start of this section. To see that, we solve for ϕ from eqn. (14.25) as,

$$\phi(k) = -ik \frac{q_{\text{dyn}}}{\sqrt{\pi}} \frac{e^2}{m_2^2} \left(\frac{1}{k^2} - \frac{1}{k^2 + m_2^2} \right) j_{\text{ext}}(k). \tag{14.35}$$

Define $\phi = \phi_1 + \phi_2$, where ϕ_1 is the part with $\frac{1}{k^2}$, and ϕ_2 with $\frac{1}{k^2+m_2^2}$. The ϕ_2 part goes to zero at long distances, i.e. $k \to 0$. As for the ϕ_1 part, its x-space

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form is,

$$\phi_1(x) = \frac{e^2}{\sqrt{\pi} m_2^2} q_{\rm dyn} q_{\rm ext} (\theta(x+L) - \theta(x-L)), \tag{14.36}$$

which for small $\frac{m}{e}$ reduces to,

$$\phi_1(x) \sim \sqrt{\pi} \frac{q_{\rm ext}}{q_{\rm dyn}} (\theta(x+L) - \theta(x-L)). \tag{14.37}$$

Thus $2\sqrt{\pi}\phi$ small means,

$$(2\pi) \frac{q_{\rm ext}}{q_{\rm dyn}} \ll 1, \tag{14.38}$$

the condition mentioned before.

Note that we could generalize the argument to values of $2\pi \frac{q_{\rm ext}}{q_{\rm dyn}}$ that are close to $2\pi n$, with integer n.

14.5 Finite temperature

In this section we would like to comment on the behavior of the string tension in the presence of finite temperature. It is interesting to check whether the string is torn due to high temperature and whether the system undergoes a phase transition from confinement to deconfinement.

The prescription for calculating quantities at finite temperature T is to formulate the theory on a circle in Euclidean time with circumference $\beta = T^{-1}$.

For the purpose of calculating the string tension, we can follow the same steps which we employed previously, leading to a modification of eqn. (14.16) as,

$$\sigma = -m <\bar{\Psi}\Psi>_T \left(1 - \cos 2\pi \frac{q_{\rm ext}}{q_{\rm dyn}}\right). \tag{14.39}$$

It is enough to calculate $<\bar{\Psi}\Psi>_T$, the condensate at finite temperature, in the massless Schwinger model.

The chiral condensate behaves as,

$$<\bar{\Psi}\Psi>_{(T\to 0)} \to -\frac{e}{2\pi^{3/2}} e^{\gamma}, \tag{14.40}$$

and

$$<\bar{\Psi}\Psi>_{(T\to\infty)} \to -2T \mathrm{e}^{-\frac{\pi^{3/2} T}{e}}. \tag{14.41}$$

This result indicates that the string is not torn even at very high temperatures. The explicit expression shows that $<\bar{\Psi}\Psi>_T$ is non-zero for all T. Thus, the system does not undergo a phase transition. It is just energetically favorable to have the electron-positron pair confined.

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14.6 Two-dimensional QCD

The action of bosonized QCD_2 with massive quarks in the fundamental representation of $SU(N)$ (see Chapter 8) is

$$\begin{aligned}S_{\text{fundamental}} = & \frac{1}{8\pi}\int_\Sigma \mathrm{d}^2x \ \mathrm{tr}\left(\partial_\mu g \partial^\mu g^\dagger\right) \\ & +\frac{1}{12\pi}\int_B \mathrm{d}^3y\epsilon^{ijk} \ \mathrm{tr}(g^\dagger\partial_i g)(g^\dagger\partial_j g)(g^\dagger\partial_k g) \\ & +\frac{1}{2}m\mu_{fund}\int \mathrm{d}^2x \ \mathrm{tr}(g+g^\dagger) - \int \mathrm{d}^2x\frac{1}{4e^2}F^a_{\mu\nu}F^{a\mu\nu} \\ & -\frac{1}{2\pi}\int \mathrm{d}^2x \ \mathrm{tr}(ig^\dagger\partial_+ gA_- + ig\partial_- g^\dagger A_+ + A_+gA_-g^\dagger - A_+A_-),\end{aligned} \tag{14.42}$$

where e is the gauge coupling, m is the quark mass, $\mu = e\frac{\exp(\gamma)}{(2\pi)^{\frac{3}{2}}}$, g is an $N\times N$ unitary matrix, A_μ is the gauge field and the trace is over $U(N)$ indices. Note, however, that only the $SU(N)$ part of the matter field g is gauged.

When the quarks transform in the adjoint representation, the expression for the action is,

$$\begin{aligned}S_{\text{adjoint}} = & \frac{1}{16\pi}\int_\Sigma \mathrm{d}^2x \ \mathrm{tr}\left(\partial_\mu g \partial^\mu g^\dagger\right) \\ & + \frac{1}{24\pi}\int_B \mathrm{d}^3y\epsilon^{ijk} \ \mathrm{tr}\left(g^\dagger\partial_i g\right)\left(g^\dagger\partial_j g\right)\left(g^\dagger\partial_k g\right) \\ & + \frac{1}{2}m\mu_{\text{adj}}\int \mathrm{d}^2x \ \mathrm{tr}\left(g+g^\dagger\right) - \int \mathrm{d}^2x\frac{1}{4e^2}F^a_{\mu\nu}F^{a\mu\nu} \\ & - \frac{1}{4\pi}\int \mathrm{d}^2x \ \mathrm{tr}\left(ig^\dagger\partial_+ gA_- + ig\partial_- g^\dagger A_+ + A_+gA_-g^\dagger - A_+A_-\right).\end{aligned} \tag{14.43}$$

The action (14.43) differs from (14.42) by a factor of one half in front of the WZW and interaction terms, because g is real and represents Majorana fermions. Another difference is that g now is an $(N^2-1)\times(N^2-1)$ orthogonal matrix. The two actions (14.42) and (14.43) can be schematically represented by one action,

$$\begin{aligned}S = S_0 + \frac{1}{2}m\mu_R\int \mathrm{d}^2x \ \mathrm{tr}\left(g+g^\dagger\right) \\ -\frac{ik_{\text{dyn}}}{4\pi}\int \mathrm{d}^2x \ \left(g\partial_- g^\dagger\right)^a A^a_+,\end{aligned} \tag{14.44}$$

where $A_- = 0$ gauge was used, S_0 stands for the WZW action and the kinetic action of the gauge field, k_{dyn} is the level (the chiral anomaly) of the dynamical charges ($k=1$ for the fundamental representation of $SU(N)$ and $k=N$ for the adjoint representation).

Let us add an external charge to the action. We choose a static charge (with respect to the light-cone coordinate x^+) and therefore we can omit its kinetic

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term from the action. Thus an external charge coupled to the gauge field would be represented by,

$$-\frac{ik_{\text{ext}}}{4\pi}\int \mathrm{d}^2x \;\left(u\partial_- u^\dagger\right)^a A_+^a .$$

Suppose that we want to put a quark and an anti-quark at a very large separation. A convenient choice of the charges would be a direction in the algebra in which the generator has a diagonal form. The simplest choice is a generator of an $SU(2)$ subalgebra. Since a rotation in the algebra is always possible, the results are insensitive to this specific choice. As an example we write down the generator in the case of fundamental and adjoint representations,

$$T^3_{\text{fund}} = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, \underbrace{0, 0, \ldots, 0}_{N-2}\right)$$

$$T^3_{\text{adj}} = \text{diag}\left(1, 0, -1, \underbrace{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}}_{2(N-2)\ doublets}, \underbrace{0, 0, \ldots, 0}_{(N-2)^2}\right).$$

Generally T^3 can be written as

$$T^3 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, 0, 0, \ldots),$$

where $\{\lambda_i\}$ are the 'isospin' components of the representation under the $SU(2)$ subgroup.

We take the $SU(N)$ part of u as,

$$u = \left[\exp -i4\pi \left(\theta(x^- + L) - \theta\left(x^- - L\right)\right)\right] T^3_{\text{ext}}, \tag{14.45}$$

for $N > 2$, and a similar expression but with a 2π factor for $N = 2$. T^3_{ext} represents the '3' generator of the external charge and u is static with respect to the light-cone time coordinate x^+. The theta function is used as a limit of a smooth function which interpolates between 0 and 1 over a very short distance. In that limit $u = 1$ everywhere except at isolated points, where it is not well defined.

The form of the action (14.44) in the presence of the external source is,

$$\begin{aligned} S = S_0 + \frac{1}{2} m\mu_R \int \mathrm{d}^2x \Big\{ &\operatorname{tr}\left(g + g^\dagger\right) \\ &+ \left[-\frac{ik_{\text{dyn}}}{4\pi}\left(g\partial_- g^\dagger\right)^a + k_{\text{ext}}\delta^{a3}\left(\delta\left(x^- + L\right) - \delta\left(x^- - L\right)\right)\right] A_+^a \Big\}. \end{aligned}$$

The external charge can be eliminated from the action by a transformation of the matter field. A new field $\tilde{g}$ can be defined as follows,

$$-\frac{ik_{\text{dyn}}}{4\pi}\left(\tilde{g}\partial_- \tilde{g}^\dagger\right)^a = -\frac{ik_{\text{dyn}}}{4\pi}\left(g\partial_- g^\dagger\right)^a + k_{\text{ext}}\delta^{a3}\left(\delta(x^- + L) - \delta(x^- - L)\right).$$

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This definition leads to the following equation for $\tilde{g}^\dagger$,

$$\partial_-\tilde{g}^\dagger = \tilde{g}^\dagger \left(g\partial_- g^\dagger + i4\pi \frac{k_{\rm ext}}{k_{\rm dyn}} (\delta(x^- + L) - \delta(x^- - L)) T^3_{\rm dyn} \right) . \qquad (14.46)$$

The solution of 14.46 is,

$$\begin{aligned} \tilde{g}^\dagger &= P \exp \left\{ \int {\rm d}x^- \left(g\partial_- g^\dagger + i4\pi \frac{k_{\rm ext}}{k_{\rm dyn}} (\delta(x^- + L) - \delta(x^- - L)) T^3_{\rm dyn} \right) \right\} \\ &= {\rm e}^{i4\pi \frac{k_{\rm ext}}{k_{\rm dyn}} \theta(x^- + L) T^3_{\rm dyn}} g^\dagger {\rm e}^{-i4\pi \frac{k_{\rm ext}}{k_{\rm dyn}} \theta(x^- - L) T^3_{\rm dyn}} , \end{aligned} \qquad (14.47)$$

where P denotes path ordering and we assume that $T^3_{\rm dyn}$ commutes with $g\partial_- g^\dagger$ for $x^- \geq L$ and with $g^\dagger$ for $x^- = -L$ (as we shall see, this assumption is self consistent with the vacuum configuration).

Let us take the limit $L \to \infty$. For $-L < x^- < L$, the above relation simply means that,

$$g = \tilde{g} {\rm e}^{i4\pi \frac{k_{\rm ext}}{k_{\rm dyn}} T^3_{\rm dyn}} .$$

Since the Haar measure is invariant (and finite, unlike the fermionic case) with respect to unitary transformations, the form of the action in terms of the new variable $\tilde{g}$ reads,

$$\begin{aligned} S = S_{WZW}(\tilde{g}) + S_{\rm kinetic}(A_\mu) - \frac{ik_{\rm dyn}}{4\pi} \int {\rm d}^2x \ \left(\tilde{g}\partial_- \tilde{g}^\dagger \right)^a A^a_+ \\ + \frac{1}{2} m\mu_R \int {\rm d}^2x \ {\rm tr} \left(\tilde{g} e^{i4\pi \frac{k_{\rm ext}}{k_{\rm dyn}} T^3_{\rm dyn}} + e^{-i4\pi \frac{k_{\rm ext}}{k_{\rm dyn}} T^3_{\rm dyn}} \tilde{g}^\dagger \right) , \end{aligned} \qquad (14.48)$$

which is QCD_2 with a chiraly rotated mass term.

The string tension can be calculated easily from (14.48). It is simply the vacuum expectation value (v.e.v.) of the Hamiltonian density, relative to the v.e.v. of the Hamiltonian density of the theory without an external source,

$$\sigma = <H> - <H_0> .$$

The vacuum of the theory is given by $\tilde{g} = 1$. In terms of the variable g, this configuration points in the '3' direction and hence satisfies our assumptions while solving eqn. (14.46). The v.e.v. is,

$$\begin{aligned} <H> &= -\frac{1}{2} m\mu_R \ {\rm tr} \left({\rm e}^{i4\pi \frac{k_{\rm ext}}{k_{\rm dyn}} T^3_{\rm dyn}} + {\rm e}^{-i4\pi \frac{k_{\rm ext}}{k_{\rm dyn}} T^3_{\rm dyn}} \right) \\ &= -m\mu_R \ \sum_i \cos \left(4\pi\lambda_i \frac{k_{\rm ext}}{k_{\rm dyn}} \right) . \end{aligned}$$

Therefore the string tension is,

$$\sigma = m\mu_R \ \sum_i \left(1 - \cos \left(4\pi\lambda_i \frac{k_{\rm ext}}{k_{\rm dyn}} \right) \right) , \qquad (14.49)$$

which is the desired result.

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We expect that similar corrections as those in eqn. (14.22) will occur also in non-abelian systems. For the fundamental/adjoint case, the following expression may correct the leading term,

$$\sigma_{QCD} = m\mu_R \sum_{l=1}^{\infty} \tilde{C}_l \left(\frac{m}{ek_{\mathrm{dyn}}}\right)^{l-1} \sum_j \left(1 - \cos\left(4\pi\lambda_j l \frac{k_{\mathrm{ext}}}{k_{\mathrm{dyn}}}\right)\right). \tag{14.50}$$

A few remarks should be made:

(i) The string tension (14.49) reduces to the abelian string tension (14.21) when abelian charges are considered. It follows that the non-abelian generalization is realized by replacing the charge q with the level k.

(ii) The string tension was calculated in the tree level of the bosonized action. Perturbation theory (with m as the coupling) may cause changes, eqn. (14.49), since the loop effects may add $O(m^2)$ contributions. However, we believe that it would not change its general character. In fact, one feature is that the string tension vanishes for any m when $\frac{k_{\mathrm{ext}}}{k_{\mathrm{dyn}}}$ is an integer, as follows from eqn. (14.48), since the action does not depend then on k_{ext} at all.

(iii) When no dynamical mass is present, the theory exhibits screening. This is simply because non-abelian charges at the end of the world interval can be eliminated from the action by a chiral transformation of the matter field.

(iv) When the test charges are in the adjoint representation $k_{\mathrm{ext}} = N$, eqn. (14.49) predicts screening by the fundamental charges (with $k_{\mathrm{dyn}} = 1$).

(v) String tension appears when the test charges are in the fundamental representation and the dynamical charges are in the adjoint. The value of the string tension is

$$\sigma = m\mu_{\mathrm{adj}} \left(2\left(1 - \cos\frac{4\pi}{N}\right) + 4(N-2)\left(1 - \cos\frac{2\pi}{N}\right)\right) \tag{14.51}$$

as follows from eqn. (14.49) for this case.

The case of $SU(2)$ is special. The 4π which appears in eqn. (14.49) is replaced by 2π, since the bosonized form of the external $SU(2)$ fundamental matter differs by a factor of a half with respect to the other $SU(N)$ cases. Hence, the string tension in this case is $4m\mu_{\mathrm{adj}}$.

(vi) We would like to add, that when computing the string tension in the pure YM case with external sources in representation R, the Wilson loop gives $\frac{1}{2}e^2 C_2(R)$, while our way of defining external source gives $\frac{1}{2}e^2 k_{\mathrm{ext}}^2$. Thus we need a factor $\frac{C_2(R)}{k_{\mathrm{ext}}^2}$ to bring our result to the Wilson loop case. Analogous factors should be computed for the other cases, when dynamical matter is also present.

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14.7 Symmetric and antisymmetric representations

The generalization of (14.49) to arbitrary representations is not straightforward. However, we can comment about its nature (without rigorous proof).

Let us focus on the interesting case of the antisymmetric representation. One can show that the WZW action with g taken to be $\frac{1}{2}N(N-1) \times \frac{1}{2}N(N-1)$ unitary matrices, is a bosonized version of QCD_2 with fermions in the antisymmetric representation.

The antisymmetric representation is described in the Young-tableaux notation by two vertical boxes. Its dimension is $\frac{1}{2}N(N-1)$ and its diagonal $SU(2)$ generator is,

$$T^3_{\text{as}} = \text{diag}\Big(\underbrace{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2}}_{(N-2)\ \text{doublets}}, 0, 0, ..., 0\Big), \tag{14.52}$$

and consequently $k = N-2$. When the dynamical charges are in the fundamental and the external in the antisymmetric the string tension should vanish because the tensor product of two fundamentals include the antisymmetric representation. Indeed, (14.49) predicts this result.

The more interesting case is when the dynamical charges are antisymmetric and the external are fundamentals. In this case the value of the string tension depends on whether N is odd or even.

When N is odd the string tension should vanish because the anti-fundamental representation can be built by tensoring the antisymmetric representation with itself $\frac{1}{2}(N-1)$ times. When N is even string tension must exist.

Note that (14.49) predicts,

$$\sigma = 2m\mu_{\text{as}}(N-2)\left(1 - \cos\frac{2\pi}{N-2}\right), \tag{14.53}$$

which is not zero when N is odd, contrary to expectation.

The resolution of the puzzle seems to be the following. Non-abelian charge can be static with respect to its spatial location. However, its representation may change in time due to emission or absorption of soft gluons (without cost of energy). Our semi-classical description of the external charge as a c-number is insensitive to this scenario. We need an extension of (14.45) which takes into account the possibilities of all various representations. One possible extension is,

$$j^a_{\text{ext}} = \delta^{a3}k_{\text{ext}}\left(1 + lN\right)\left(\delta\left(x^- + L\right) - \delta\left(x^- - L\right)\right), \tag{14.54}$$

where l is an arbitrary positive integer. This extension takes into account the cases which correspond to $1 + lN$ charges multiplied in a symmetric way. The resulting string tension is,

$$\sigma = m\mu_R \sum_i \left(1 - \cos\left(4\pi\lambda_i \frac{k_{\text{ext}}}{k_{\text{dyn}}}\left(1 + lN\right)\right)\right), \tag{14.55}$$

which includes the arbitrary integer l. What is the value of l that we should pick?

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The dynamical charges are attracted to the external charges in such a way that the total energy of the configuration is minimal. Therefore the value of l which is needed, is the one that guarantees minimal string tension.

Thus the extended expression for string tension is the following,

$$\sigma = \min_{l} \left\{ m\mu_R \sum_i \left(1 - \cos\left(4\pi\lambda_i \frac{k_{\text{ext}}}{k_{\text{dyn}}}(1 + lN)\right)\right)\right\}. \tag{14.56}$$

In the case of dynamical antisymmetric charges and external fundamentals and odd N, $l = \frac{1}{2}(N-3)$ gives zero string tension. When N is even the string tension is given by (14.53).

The expression (14.56) yields the right answer in some other cases also, like the case of dynamical charges in the symmetric representation. The bosonization for this case can be derived in a similar way to that of the antisymmetric representation, and T^3 is given by

$$T^3_{\text{symm}} = \text{diag}\Big(1, 0, -1, \underbrace{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}}_{(N-2)\ \text{doublets}}, 0, 0, \dots, 0\Big). \tag{14.57}$$

Hence $k = N + 2$. When the external charges transform in the fundamental representation and N is odd, eqn (14.56) predicts zero string tension (as it should). When N is even the string tension is given by

$$\sigma = 2m\mu_{\text{symm}} \left(\left(1 - \cos\frac{4\pi}{N+2}\right) + (N-2)\left(1 - \cos\frac{2\pi}{N+2}\right)\right). \tag{14.58}$$

We have discussed only the cases of the fundamental, adjoint, anti-symmetric and symmetric representations, since we used bosonization techniques which are applicable to a limited class of representations.

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