ON THE STABILITY OF RESONANT ASTEROID ORBITS

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ABSTRACT. The stability of the asteroid orbits has been studied by the method of surface of section. Families of simple symmetric periodic orbits of the asteroid and their stability have been computed and this served as a guide for the selection of the energy levels for the surface of section. In this way all possible cases for the structure of phase space have been obtained. It was found that the region in phase space around the resonant orbits at the resonances 1/3, 3/5, 5/7,.... is unstable, but small stability regions of doubly symmetric periodic orbits near the above resonances are also present. At the resonances 1/2, 2/3, 3/4, ... it was found that there exist two separate regions in phase space at about the same resonance 1/2, 2/3, 3/4,...., respectively, one being stable and the other unstable. At certain energy levels only the stable region appears. The above results are consistent with the observed distribution of the asteroids.

1. INTRODUCTION

The purpose of this paper is to study the stability of the asteroid orbits and in particular those orbits whose mean motion is in resonance with that of Jupiter. It is well known that the distribution of the asteroids is not smooth but gaps exist at some resonances, the most conspicuous being at 1/3, 1/2, 3/5 and also, to a lesser extend, at 2/5, 3/7. These are the well known Kirkwood gaps whose explanation is not yet clear despite the fact that much work has been done. Several mechanisms have been proposed for the explanation of the Kirkwood gaps, but recent work based on statistical analysis (Dermott and Murray, 1981) supports the gravitational hypothesis, i.e. that the gaps are due to instabilities in the asteroid orbits produced by the gravitational perturbation of Jupiter.

Several papers have been published, in which the gravitational hypothesis is studied, both from the analytical and the numerical point of view. A review of this work is made by Hagihara (1972). We note in this respect that the appearance of small divisors in the resonant cases

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cannot explain the gaps since there do exist groups of resonant asteroids, for example the Hecuba group and the Hilda group at the 1/2 and 2/3 resonances, respectively. On the other hand, numerical integrations at or near the resonant cases do not always produce the gaps that we would expect (Froeschlé and Scholl, 1974, 1975, 1979, Lecar and Franklin, 1973, Sinclair, 1969, 1970). So, the explanation of the Kirkwood gaps is still an open problem.

In this paper we present a global view for the totality of asteroid orbits. The orbit of Jupiter will be considered as circular and the orbits of the asteroids will be considered as coplanar with Jupiter, moving in the same direction. This will be done by giving the structure of the phase space by the method of surface of section. In this way the stable and unstable regions will be presented clearly and the relation between the various resonant cases with stability or instability and the corresponding generation of gaps will become evident.

A different approach to the study of the stability of the asteroid orbits can be made by using Hill's stability criterium. This has been done by Szebehely, Vicente and Lundberg, 1983.

We present here the qualitative aspects and the main results of this work. The complete work with all the numerical results will be presented elsewhere.

2. FAMILIES OF PERIODIC ORBITS

The periodic orbits and their stability characteristics determine criticaly the structure of the phase space. For this reason we present here families of periodic orbits of asteroids, moving under the gravitational attraction of the Sun and Jupiter. The mass of the asteroid will be considered negligible and Jupiter will be assumed to describe a circular orbit around the Sun. The orbit of the asteroid will be considered with respect to a rotating frame whose origin is at the center of mass of Sun-Jupiter and the x-axis is the line from Sun to Jupiter. This is the well known circular restriced 3-body problem. We shall study planar motion only and the orbit of the asteroid will be considered inside the orbit of Jupiter and moving in the same direction.

Families of periodic orbits for the asteroid, in the planar circular restricted 3-body problem have been computed by Colombo, Franklin and Munford (1968) and Broucke (1968). We have recomputed these families to a high accuracy in order to have reliable results for the stability, especially at some critical cases, which have been predicted analytically (Hadjidemetriou, 1982b). The value of the small parameter $\mu=m_j/(m_s+m_j)$ is taken equal to $\mu=0.001$, where m_s, m_j are the masses of the Sun and Jupiter, respectively.

The above families are the continuation, for $\mu\neq0$, from families of periodic orbits of the asteroid, in the rotating frame defined above, for the case $\mu=0$ (no perturbation from Jupiter). We describe now briefly the families for $\mu=0$:

a). There exists a family of circular orbits, in the rotating frame, symmetric with respect to the x-axis. At t=0 we have $y=\dot{x}=0$, so the initial conditions of such a periodic orbit are the values of x and $\dot{y}$ at t=0. Instead of $\dot{y}$ we shall use the value C of the Jacobi constant (e.g. Szebehely 1967). This family is shown in Fig. 1. The normalization of the variables is such that the mean motion of Jupiter is n=1. Along this family the value n/n' varies, and several resonant (unperturbed) orbits exist, where n' is the mean motion of the asteroid.

b). There exist families of elliptic orbits for the asteroid, which bifurcate from the circular resonant orbits 1/2, 2/3, 3/4, ... These are simple periodic orbits, i.e. they close after the first intersection with the x-axis. From Keplerian theory it can be proved that these elliptic families (Fig.1) are symmetric with respect to a line normal to the x-axis, passing through the corresponding circular resonant orbit. All along such a family, the resonance is constant, equal to the resonance of the generating circular orbit.

c). There exist also families of elliptic multiple periodic orbits (for $\mu=0$) which bifurcate from resonant circular orbits at the resonance p/q ($q-p\neq1$), in the same way as the families described in (b) above. The multiplicity of such an orbit is equal to q-p. It is clear that a dense set of such resonant orbits exists, though of measure zero.

When the perturbation from Jupiter comes into effect, i.e. for $\mu\neq0$, the above families are continued to a set of families of periodic orbits of the restricted circular 3-body problem, as shown in Fig. 1. The existence proof for the continuation of the circular orbits is given by Birkhoff (1927) and for the elliptic orbits by Arenstorf (1963) and Schmidt (1972a). Also, the form of the continued orbits at the vicinity of the resonant circular orbits 1/2, 2/3, 3/4, ... has been studied by Guillaume (1969) and Schmidt (1972b).

We note that there exist several distinct families of periodic orbits for $\mu\neq0$, three of them shown in Fig. 1. Each family has a part which is the continuation of the circular orbits for $\mu=0$ (periodic orbits of the first kind) and two branches (one for the first family, to the lower left of the Figure) of resonant elliptic orbits at the resonances 1/2, 2/3, 3/4, ... Along these latter branches the eccentricity increases as we move away from the resonant circular orbit 1/2, 2/3,... .

In particular, we have two resonant elliptic branches at the 1/2 resonance, denoted by A_1 and A_2 respectively. The branch A_1 corresponds to a mean motion n' of the asteroid such that $n/n'\approx1/2$ but $n/n'<1/2$ and the branch A_2 corresponds to $n/n'\approx1/2$ but $n/n'>1/2$.

Apart from the above mentioned families of periodic orbits, which are all simple periodic orbits, there also exist families of multiple

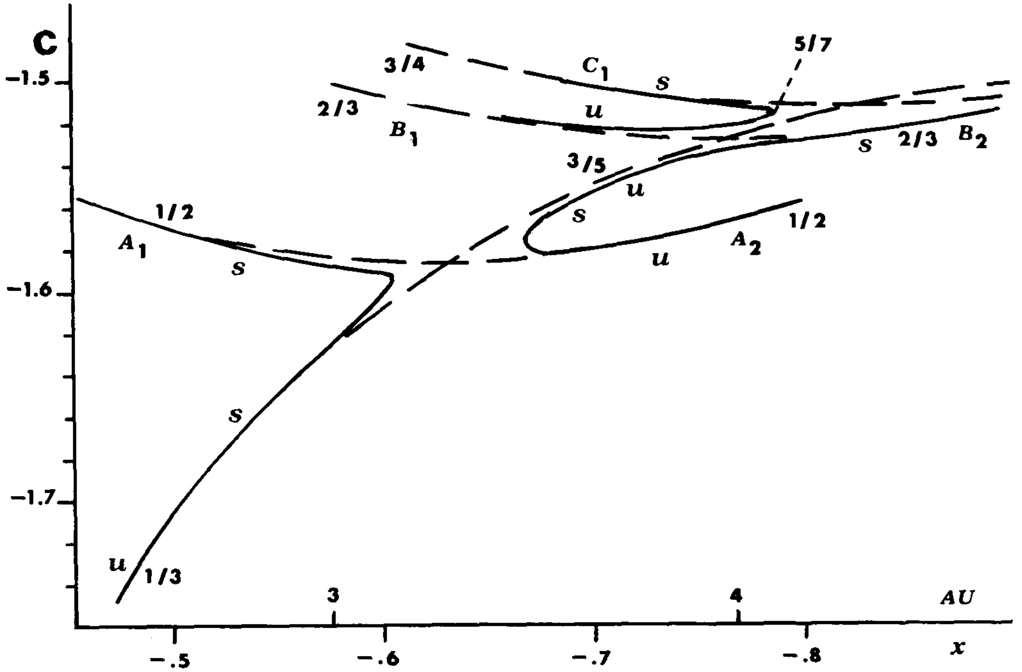

Figure 1. Families of simple periodic orbits for the asteroid (μ=0.001). The dotted lines represent the unperturbed families (μ=0). The resonance on the elliptic branches A_1, A_2, B_1, B_2, C_1, and the stability (s) or instability (u) is indicated.

resonant periodic orbits of the form p/q (q-p≠1) which fill the whole space densely. They are not shown in Fig. 1 but some of them will appear on the surface of section, as we shall see in the next section.

The resonant branches 1/2 and 2/3 can be identified with the Hecuba and the Hilda group of the asteroids, respectively. This will become clear in the next section when the stability will be studied.

3. STABILITY

It can be proved analytically (Hadjidemetriou 1982b) that all the resonant circular orbits at the resonances 1/3, 3/5, 5/7, ... are continued, for $\mu\neq0$, to periodic orbits of the first kind which are unstable. Note from Fig. 1 that to each family for $\mu\neq0$ there belongs only one such unstable resonant orbit. This implies that on the extended family for $\mu\neq0$ there exists a small region, around the corresponding resonant orbit, which is unstable. It was also found that this unstable region extends as the value of μ (i.e. the perturbation) increases, and also the magnitude of the unstable eigenvalue increases.

All the other simple periodic orbits of the first kind are stable and no Hamiltonian perturbation exists that could make them unstable.

The resonant branches A_1, A_2, B_1, B_2, C_1,.... can be proved to be unstable, in the sense that there always exists a Hamiltonian perturbation on the unperturbed elliptic orbit which generates instability. This however does not mean that all the above branches are unstable when $\mu \neq 0$. In fact, it is found by numerical integrations that the branches A_1, B_2 and C_1, at the resonances 1/2, 2/3, and 3/4, respectively, are stable.

A similar situation holds for the resonant branches p/q ($q-p \neq 1$) of multiple elliptic periodic orbits.

A measure of the instability can be provided by the magnitude of the unstable eigenvalue, $|\lambda|>1$. We obtained that for the same value of μ the magnitude of $|\lambda|$ increases, for the resonant orbits $n/n' = (2\nu-1)/(2\nu+1)$ as ν increases. This is shown in the Table I below (the max. value of $|\lambda|$ in the corresponding unstable area is given):

Table I

n/n'	λ
1/3	-1.005
3/5	-1.076
5/7	-1.257

As far as the unstable resonant branches A_2 and B_1 are concerned, we found that $|\lambda|$ increases as we proceed outwards to higher eccentricities, for the same branch, and also $|\lambda|$ is larger as ν in $n/n'=\nu/(\nu+1)$ increases. The value of $|\lambda|$ is much larger on these brances than at the resonant orbits 1/3, 3/5, 5/7, in Table I. This is shown in Table II (each orbit on a branch is identified by the initial value x, as can be seen from Fig. 1):

Table II

Branch A_2: $n/n' \simeq 1/2$		Branch B_1: $n/n' \simeq 2/3$	
x	λ	x	λ
-.681	1.101	-.727	1.876
-.691	1.202	-.716	2.745
-.737	1.517	-.697	4.841
-.767	1.823	-.666	29.6

4. THE STRUCTURE OF PHASE SPACE

The best way to obtain a global view of the totality of orbits and the stable or unstable regions in phase space is to consider a mapping on the surface of section. We have in our case two degrees of freedom and consequently a 4-dimensional phase space. The surface of section is now defined by C=constant, y=0, which is the 2-dimensional space, on which we use the cartesian coordinates x, $\dot{x}$. The periodic orbits are the fixed points of this mapping.

In order to understand the structure of the phase space on the surface of section, we start with the unperturbed case $\mu=0$ (Keplerian, circular or elliptic orbit of the asteroid, referred to the rotating frame). From Fig. 1 we can see that for each energy level C we have only one circular orbit, which corresponds to a central fixed point on the surface of section. It can be proved that this is a central fixed point which is surrounded by smooth invariant curves, topologically equivalent to circles (Hadjidemetriou 1982a). A dense subset of them corresponds to resonant orbits p/q, though in general p and q are large integers. These resonant invariant curves correspond to the intersections of the line C=constant in Fig. 1 with the resonant elliptic branches p/q mentioned in section 2. As we go outward, starting from the central fixed point, the ratio n/n' of the mean motions decreases. The mapping around the central fixed point is a twist mapping.

When the perturbation from Jupiter is applied, $\mu\neq 0$, the nonresonant invariant curves, p/q=irrational, survive as smooth invariant curves, as expected by the Kolmogorof-Arnold-Moser theorem. The resonant invariant curves evolve to a set of stable and unstable fixed points, and thus dissolution occurs, (e.g. Arnold and Avez, 1968). This dissolution however is in most cases negligible, except at the low order resonances. Chaotic behavior appears near the unstable fixed points with large value of the eigenvalue λ, as is indeed the case with the orbits of the branch B_1 (see Table II).

We present now some representative cases: In all the figures, together with the invariant curves we have plotted the curve $\dot{y}^2=0$ (dotted line), which is the boundary of the motion on the surface of section.

(a). Energy level C=-1.7367295694

The value of C is so selected that the central fixed point is an unstable fixed point at the resonance 1/3. In order to save computer time, we used the value $\mu=0.01$ instead of $\mu=0.001$, so that the eigenvalue λ is larger at the central fixed point and its effects appear in a shorter time. The value of λ is in this case $\lambda=-1.056$. As expected, the mapping around the central fixed point is a hyperbolic twist mapping (Fig. 2). One doubly symmetric stable periodic orbit appears. Also, resonant orbits at the 2/7 and 3/11 resonances are clearly seen. The stable fixed points are surrounded by islands, shown in Fig. 2 and the unstable fixed points are indicated (schematically) at the place we should expect them.

From this diagram we can deduce that the asteroids cannot stay near the unstable resonant orbit 1/3, and thus a gap is expected at that resonance. A few multiple periodic orbits can still exist at about that resonance, trapped around the doubly symmetric periodic orbit at 1/3, as shown in Fig. 2.

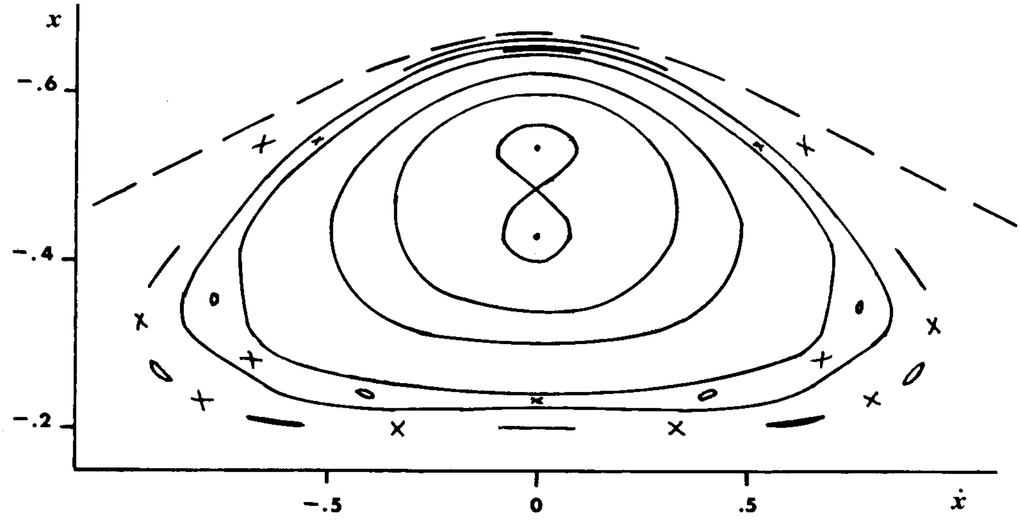

Figure 2. Invariant curves at C=−1.7367295694 for μ=0.01. The central unstable fixed point is at the resonance 1/3. Resonant orbits at 2/7 and 3/11 are also present.

At the 2/7 and 3/11 resonance stable regions exist, trapped around the stable periodic orbits at that resonance, but unstable regions also exist. Thus a smaller density of asteroids is expected at the above resonance cases, which results in minor gaps in the distribution of the asteroids.

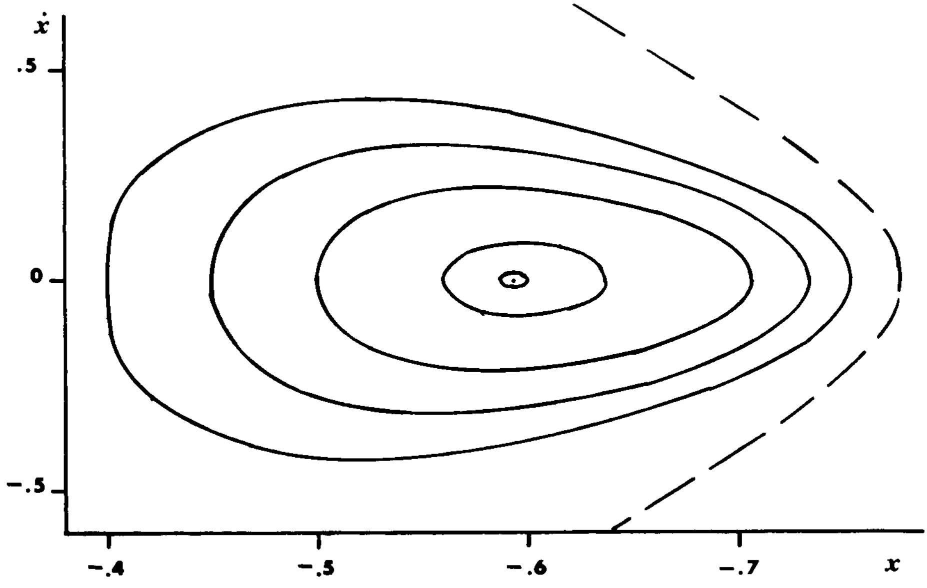

Figure 3. Invariant curves at C=−1.59. Only the stable resonant orbit 1/2 appears.

(b). Energy level C=-1.59

At this value of the energy the line C=-1.59 intersects only the stable branch A_1 at the resonance 1/2 and not the unstable branch A_2 at about the same resonance. This happens because the perturbed branch A_1 evolves to a position "lower" than that of the unperturbed one (Fig.1) while the unstable branch A_2 evolves to a position "above" the corresponding unperturbed branch. As a consequence, only the stable fixed point at the resonance 1/2 appears on the surface of section. Clearly, trapping at this resonance is possible, and this corresponds to the Hecuba group of the asteroids. A similar situation also appears at the 2/3 resonance where the line C=-1.525 intersects only the stable branch B_2 (Hilda group of the asteroids).

(c). Energy level C=-1.574982425

The central fixed point corresponds to a nonresonand periodic orbit of the first kind, which belongs to the second family of periodic orbits in Fig. 1. The value of n/n' at this central fixed point is larger than 1/2 and consequently, as we go outwards, there exists an invariant curve at the resonance 1/2 (for $\mu=0$) which dissolves when the perturbation $\mu\neq0$ is applied into a stable and an unstable fixed point (Fig.4). This can be clearly seen from Fig.1 where the line C=-1.574982425 intersects both resonant elliptic branches A_1 and A_2. The intersection with the stable resonant branch A_1 corresponds to the stable fixed point and the

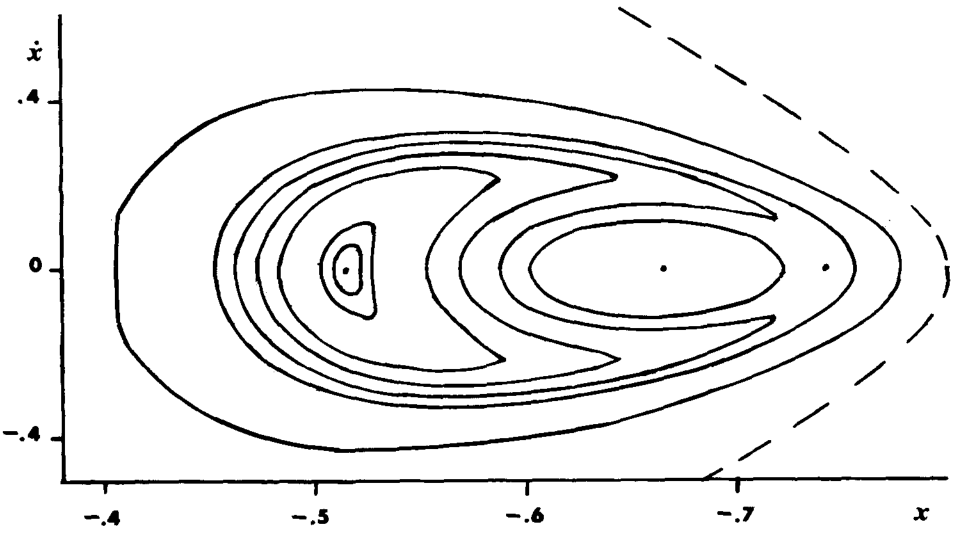

Figure 4. Invariant curves at C=-1.574982425. The central fixed point is a nonresonant periodic orbit of the first kind. Two resonant orbits near 1/2 appear, one stable and the other unstable.

intersection with the unstable resonant branch A_2 corresponds to the unstable fixed point. Both these latter points are at a resonance $n/n' \approx 1/2$. The stable point is however at a resonance slightly smaller than 1/2, as already mentioned in section 2.

As a consequence of the existence of the above mentioned stable fixed point at the resonance $\approx 1/2$, asteroids are expected there. This is the Hecuba group. On the other hand, at a resonance n/n slightly larger than 1/2 a quite extended unstable region exists, which corresponds to the 1/2 Kirkwood gap.

(d). Energy level C=-1.54696142.

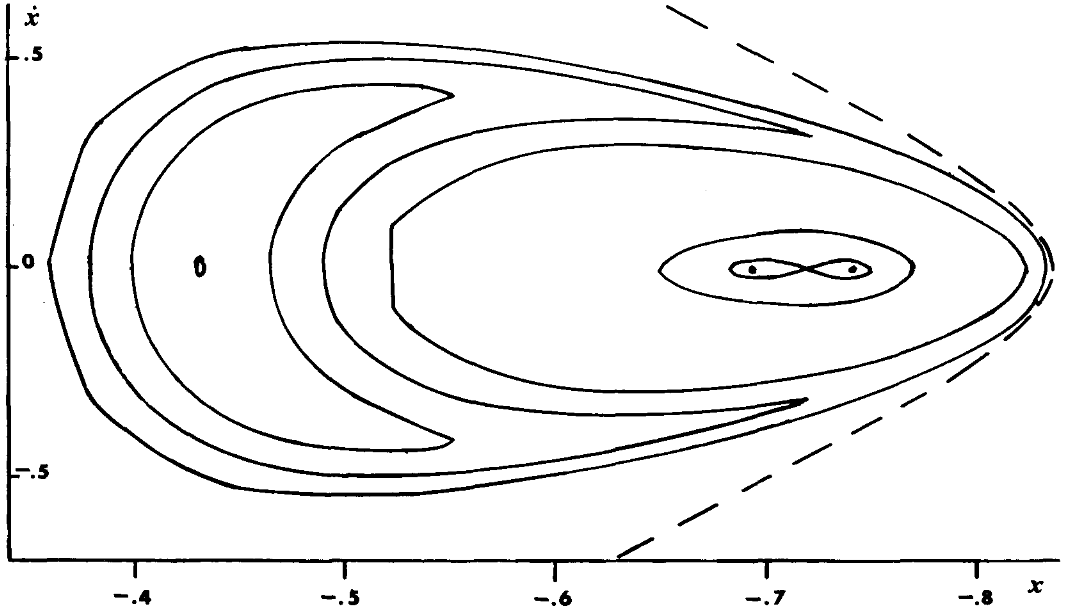

Figure 5. Similar to Fig.4, at C=-1.54696142. The central fixed point is the resonant periodic orbit at 3/5. The stable and unstable resonances at 1/2 are present.

This case, shown in Fig.5, is completely similar to the case (c) above. The only difference is that the central fixed point is now in the unstable area at the resonance 3/5 (see Fig.1). As a consequence, a hyperbolic twist mapping appears, which prevents the concentration of asteroid orbits at this resonance. This is indeed observed in the distribution of the asteroids. This situation is similar to the resonant gap at $n/n' \approx 1/3$, shown in Fig.2, but in this case the gap is expected to be wider, as the unstable eigenvalue λ is, absolutely, larger (see Table I). Also, the instability area at the resonance $n/n' \approx 1/2$ is more prominent than that in Fig. 4. This is so because the unstable eigenvalue λ increases along the unstable resonant branch A_2 (i.e. as the eccentricity increases).

(e). Energy level C=-1.5201681

This is shown in Fig. 6. The central fixed point is in the unstable area at the resonance 5/7 and consequently we have a hyperbolic twist mapping, as in the cases 1/3 and 3/5. We have also two fixed points at the resonance 2/3, which correspond to the intersection of the line C=-1.5201681 with the two resonant elliptic branches B_1 and B_2. The intersection with the stable branch B_2 corresponds to the stable fixed

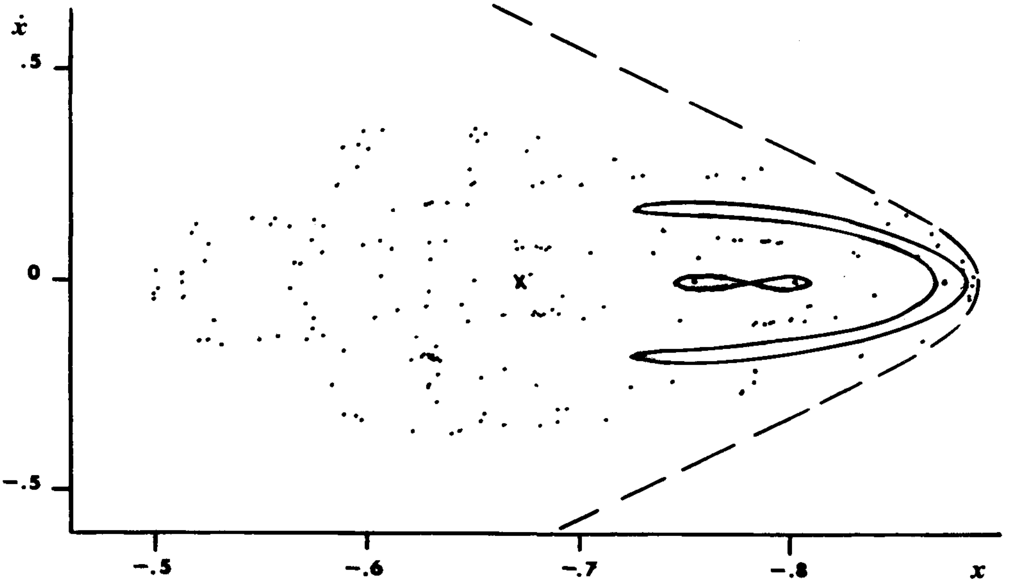

Figure 6. Invariant curves at C=-1.5201681. The central fixed point is at the resonance 5/7. Two resonant fixed points appear near the resonance 2/3. The stable one is surrounded by smooth invariant curves while dissolution appears at the unstable fixed point (indicated by x).

point and the intersection with the unstable branch B_1 corresponds to the unstable fixed point. Contrary however to the previous cases where smooth invariant curves appeared, even in the vicinity of the unstable fixed point, in this case we have chaotic behavior generated by the unstable fixed point. This is so because the unstable eigenvalue λ is large, as shown in Table II. Note that all scattered points belong to the same orbit. But around the stable fixed point a stable area clearly exists, which can explain a trapping at the resonance 2/3. This is the Hilda group of the asteroids, in accordance with the observations.

4. DISCUSSION

From the above study we see that all the observed Kirkwood gaps can be explained, at least qualitatively, on the assumption that the gaps are due to instabilities generated by the gravitational attraction of Jupiter.

Note that no other gaps appear in this study than those observed in the actual motion of the asteroids.

It is also clear that not all resonances are of equal importance, and this is not directly related to the order of the resonance. The 1/3 and 3/5 resonances are more important than the 1/2 or 2/3 resonances. The latter type of resonance has been studied extensively, but little work has been done for the 1/3, 3/5 resonances, especially the 3/5 resonance. Recent work on the 1/3 and 3/5 resonances is by Colombo and Franklin (1982). From the present study we find that both these resonances are of the same nature qualitatively, but the gap expected at the 3/5 resonance is wider, because the corresponding unstable eigenvalue is larger. This is indeed the case in the actual situation. Note however that stable areas at about the same resonances, 1/3 and 3/5, do exist, though small. So one would expect a few such resonant orbits of asteroids, librating around the above mentioned stable doubly symmetric periodic orbits at the corresponding resonances. These stable areas are clearly seen in Figs 2,5 and 6.

The 1/2 and 2/3 resonances are of different nature than the 1/3 and 3/5 resonances in that the stable and unstable areas in the 1/2, 2/3 case are comparable, contrary to the 1/3, 3/5 resonances where the instability character on the surface of section dominates. Compare for example Fig.5 where both the 1/2 and 3/5 resonance show clearly. As a consequence, stable areas at the resonances 1/2, 2/3 do exist. In particular, the stable region at the 1/2 resonance corresponds to mean motions such that n/n' is slightly smaller than 1/2 (corresponding to the Hecuba group) and the unstable region at 1/2 corresponds to n/n' slightly larger than 1/2, (corresponding to the observed gap at 1/2). The same situation, more pronounced, occurs at the resonance 2/3 (Hilda group.

Finally, we note that all resonances of the form $(2\nu-1)/(2\nu+1)$ are unstable (Hadjidemetriou, 1982a), i.e. the resonances 1/3, 3/5, 5/7,... These resonances have an accumulation point at the orbit of Jupiter, and for this reason the whole space near Jupiter is dominated by the instabilities generated by the above orbits. Thus, no asteroids are expected near Jupiter, as is indeed the case. Note that the continuation from the above periodic orbits, for $\mu=0$, to $\mu\neq0$ is always possible, but the max. value of μ for which the continuation can be extended decreases as ν increases. This means that no periodic orbits for $\mu=0.001$ exist beyond a certain value of ν. This however seems to enhance the chaotic situation at the area near Jupiter, but we do not have detailed numerical examples.

Note: The energy constant has been computed from the formula

$$C=\frac{1}{2}(\dot{x}^2+\dot{y}^2)-\frac{1}{2}[\mu r_1^2+(1-\mu)r_2^2]-\frac{\mu}{r_1}-\frac{1-\mu}{r_2}$$

where

$$r_1 = [(x-1+\mu)^2+y^2]^{\frac{1}{2}} , \qquad r_2 = [(x+\mu)^2+y^2]^{\frac{1}{2}} .$$

REFERENCES

Arenstorf , R. F., 1963, Amer. J. Math. 83, 27.
Arnold V. I. and Avez, A., 1968, "Ergodic Problems in Classical Mechanics", Benjamin
Birkhoff, G. D., 1927, "Dynamical Systems", Amer, Math. Soc. New York.
Broucke, R. A., 1968, NASA-JPL Technical Report, 32-1168.
Colombo G., Franklin, F. A., 1982, in V. Szebehely (ed.) "Applications of Modern Dynamics to Celestial Mechanics and Astrodynamics", p.339, D. Reidel Publ. Co.
Colombo G., Franklin, F. A. and Munford, C. M., 1968, Astron. J. 73,111.
Dermott, S. F. and Murray, C.D., 1981, Nature 290, 664.
Froeschlé, C. and Scholl H., 1979, in V. Szebehely (ed.), "Instabilities in Dynamical Systems", p.115, Reidel Publ. Co.
Guillaume, P.: 1969, Astron. Astrophys. 3, 155.
Hadjidemetriou, J. D., 1982a in V. Szebehely (ed.), "Applications of Modern Dynamics to Celestial Mechanics and Astrodynamics", p.25, Reidel Publ. Co.
Hadjidemetriou, J. D., 1982b, Celes. Mech. 27, 305.
Hagihara, Y., 1972,"Celestial Mechanics", Vol.2, part 1, MIT Press.
Lecar, M. and Franklin, F. A., 1973, Icarus 20, 422.
Schmidt, D., 1972a, in L. Weiss (ed.) "Ordinary Differential equations", p.553, A.P.
Schmidt, D., 1972b, SIAM J. Appl. Math. 22, No 1.
Scholl, H. and Froeschlé, C.: 1974, Astron. Astrophys. 33, 455.
Scholl, H. and Froeschlé, C.: 1975, Astron. Astrophys. 42, 457.
Sinclair, A. T., 1969, Monthly Not. Roy. Astron. Soc. 142, 289.
Sinclair, A. T., 1970, Monthly Not. Roy. Astron. Soc. 148, 325.
Szebehely, V., 1967, "Theory of Orbits", Academic Press.
Szebehely, V., Vicente, R. and Lundberg, J.B.: 1983, this volume, p.123.