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SMALL VARIETIES OF FINITE SEMIGROUPS AND EXTENSIONS

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Abstract

We find the atoms of certain subclasses of varieties of finite semigroups and the corresponding varieties of languages. For example we give a new description of languages whose syntactic monoids are $\mathcal{R}$-trivial and idempotent. We also describe the least variety containing all commutative semigroups and at least one non-commutative semigroup. Finally we extend to varieties of finite semigroups some classical results about semilattice decomposition of semigroups.

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Following Eilenberg [4] we define a variety of finite semigroups to be a class of finite semigroups closed under taking quotients, subsemigroups and finite direct products. Eilenberg has shown the existence of a one-to-one correspondence between varieties of finite semigroups and certain classes of recognizable languages, called varieties of languages.

The main purpose of this paper is to detail this correspondence for some small varieties of semigroups (or monoids) where the term "small" refers to the inclusion ordering. Indeed a classical question in the theory of varieties of semigroups in the sense of Birkhoff was to find the atoms of certain subclasses of the whole lattice of varieties [6]. Fortunately most of these results on atoms proved for varieties in Birkhoff's sense are easily adapted for varieties of finite semigroups [3, 6, 9, 15]. The next step is to find the corresponding varieties of languages. This is already known for the atoms of the class of all varieties [4]. Here we investigate three new examples. First of all we consider the class of all non-commutative varieties, that is varieties containing at least one non-commutative semigroup (or monoid). The atoms of this class are described in [8]. We

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complete these results by giving equations of these atoms (except for the minimal varieties of groups) and new descriptions of the corresponding varieties of languages. We show in particular that the variety generated by the monoid U_2^r (three elements: one unit and two left zeros) is in fact the variety $\mathbf{R}_1$ of $\mathcal{R}$-trivial and idempotent monoids. Furthermore we show that the syntactic monoid of a language L is in $\mathbf{R}_1$ if and only if L is a disjoint union of languages of the form $a_1a_1^*a_2\{a_1, a_2\}^* \cdots a_n\{a_1,\ldots,a_n\}^*$ where the a_i's are distinct letters. Of course the dual variety $\mathbf{R}_1^r$ is also an atom in the class of non-commutative varieties. We go a step further and give equations for the variety $\mathbf{R}_1 \vee \mathbf{R}_1^r$ and describe the corresponding languages. As a byproduct we obtain a result of Tamura [14] showing that there exists no variety between $\mathbf{R}_1$ and $\mathbf{R}_1 \vee \mathbf{R}_1^r$. Finally we extend to varieties of finite semigroups a result stated in [6] for varieties in the sense of Birkhoff: the variety defined by the equations $x_1x_2x_3 = x_{1\sigma}x_{2\sigma}x_{3\sigma}$ for all permutations σ of $\{1,2,3\}$ is the least non-commutative variety containing all commutative semigroups. A description of the corresponding variety of languages is also given.

The last section is independent of the rest of the paper: we show that there exists an order-preserving mapping from the subvarieties of the variety of finite nil-simple semigroups to the subvarieties of the variety of finite monoids whose regular $\mathcal{D}$-classes are subsemigroups. This result extends to varieties of finite semigroups some classical results [13] about semilattice decomposition of semigroups. We also give some well-known examples of this correspondence.

1. Preliminaries and notation

Following Eilenberg [4] we define *a variety of finite semigroups* (*monoids*) to be a class of finite semigroups (monoids) closed under taking subsemigroups (submonoids), quotients, that is morphic images, and finite direct products. If $\mathbf{V}$ is a variety, $\mathbf{V}^r$ denotes the reverse of $\mathbf{V}$ which is the variety consisting of the reverse semigroups S^r, $S \in \mathbf{V}$. Varieties are ordered by inclusion and form a lattice under the operations of intersection and join: the join $\mathbf{V}_1 \vee \mathbf{V}_2$ of $\mathbf{V}_1$ and $\mathbf{V}_2$ is the smallest variety containing $\mathbf{V}_1$ and $\mathbf{V}_2$.

In the sequel the term "finite" is often omitted and thus "variety of semigroups" means "variety of finite semigroups".

If $\mathcal{C}$ is a class of semigroups we denote by $(\mathcal{C})$ the variety of semigroups generated by $\mathcal{C}$. If $\mathcal{C}$ is a class of monoids we denote by $(\mathcal{C})$ $((\mathcal{C})_S)$ the variety of monoids (semigroups) generated by $\mathcal{C}$. If $\mathcal{C}$ consists of only one semigroup S we shall use the notation (S) instead of $(\{S\})$.

Let Σ^+ be the free semigroup over a denumerable alphabet Σ and let $u = v \in \Sigma^+$. We say that a semigroup S *satisfies the equation* $u = v$ if $u\phi = v\phi$ for all morphisms $\phi: \Sigma^+ \to S$. It is easy to see that the class of all (finite) semigroups satisfying the equation $u = v$ is a variety of semigroups, denoted $\mathbf{V}(u, v)$.

Let $(u_n, v_n)_{n>0}$ be a sequence of pairs of words in Σ^* and let us consider the following varieties: $\mathbf{V}_n = \mathbf{V}(u_n, v_n)$, $\mathbf{V}' = \bigcap_{n>0} \mathbf{V}_n$ and $\mathbf{V}'' = \liminf \mathbf{V}_n = \bigcup_{m>0} \bigcap_{n \geq m} \mathbf{V}_n$.

We say that $\mathbf{V}'$ ($\mathbf{V}''$) is *defined* (*ultimately defined*) by the equations $u_n = v_n$ $(n > 0)$: this corresponds to the fact that a semigroup is in $\mathbf{V}'$ ($\mathbf{V}''$) if it satisfies the equations $u_n = v_n$ for all n (for all sufficiently large n).

Replacing Σ^+ by Σ^*, the free monoid over Σ, we can define in the same way equations for varieties of monoids. The main result on equations, due to Eilenberg and Schützenberger [5] is

PROPOSITION 1.1. *Every variety of semigroups (monoids) is ultimately defined by a sequence of equations. Every variety of semigroups (monoids) generated by a finite number of semigroups (monoids) is defined by a sequence of equations.*

Here is a list of varieties which play a role in the sequel. In this list we denote by $E(S)$ the set of idempotents of a semigroup S.

Varieties of finite semigroups

Nil nilpotent semigroups.

K (**K**′) left (right) nil-simple aperiodic semigroups (or reverse definite (definite) semigroups [4]). A semigroup S is in **K** if and only if $eS = e$ for all $e \in E(S)$.

$\mathbf{K}_1$ ($\mathbf{K}'_1$) left (right) simple aperiodic semigroups.

LI nil-simple aperiodic (or "locally trivial", "generalized definite" semigroups [4]). A semigroup S is locally trivial if and only if $eSe = e$ for all $e \in E(S)$.

$\mathbf{LI}_1$ simple aperiodic semigroups.

LG nil-simple semigroups. A semigroup is nil-simple if and only if it is locally a group, that is if eSe is a group for all $e \in E(S)$.

$\mathbf{LG}_1$ simple semigroups.

Varieties of finite monoids

J $\mathcal{J}$-trivial monoids. A monoid M is $\mathcal{J}$-trivial if and only if $MaM = MbM$ implies $a = b$, for all a, b.

R (**R**′) $\mathcal{R}$-trivial ($\mathcal{L}$-trivial) monoids. A monoid is $\mathcal{R}$-trivial ($\mathcal{L}$-trivial) if and only if $aM = bM$ ($Ma = Mb$) implies $a = b$, for all a, b.

$\mathbf{J}_1$ idempotent and commutative monoids, or semilattices.

$\mathbf{R}_1$ ($\mathbf{R}'_1$) $\mathcal{R}$-trivial ($\mathcal{L}$-trivial) and idempotent monoids.

DA monoids whose regular $\mathcal{D}$-classes are aperiodic semigroups.

$\mathbf{A}_1$ idempotent monoids, or bands.

DS monoids whose regular $\mathcal{D}$-classes are semigroups.

$\mathbf{DS}_1$ monoids which are union of groups.

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We conclude this section by giving some useful notations. If u is a word of A^* and a is a letter of A, $|u|_a$ denotes the number of occurrences of a in u. The integer $|u| = \sum_{a \in A} |u|_a$ is the length of u. If $u = a_1 \cdots a_n$ where the a_i's are letters, the *reverse* of u is the word $\tilde{u} = a_n \cdots a_1$. Two words u and v are *commutatively equivalent* if and only if $|u|_a = |v|_a$ for all $a \in A$. Commutative equivalence is a congruence on A^*. A language L is *commutative* if it is saturated with respect to this congruence; that is, if L is a union of congruence classes.

If S is a semigroup we denote by S^1 the monoid constructed as follows: $S^1 = S$ if S is a monoid and $S^1 = S \cup \{1\}$ where 1 is a unit added to S if S is not a monoid.

2. Small varieties

Varieties of finite semigroups or monoids are ordered by inclusion. In this section we study varieties which are "small" relative to this ordering i.e. varieties which are minimal elements of certain subclasses of the class of all varieties. We first summarize some well-known results. (Some of them, see Evans [6], were formulated for varieties in the sense of Birkhoff but can be easily adapted for our purpose.)

Let $\mathcal{C}$ be a class of varieties. A variety $\mathbf{V}$ is minimal in $\mathcal{C}$ if $\mathbf{W} \subset \mathbf{V}$ and $\mathbf{V} \in \mathcal{C}$ implies $\mathbf{W} = \mathbf{V}$.

PROPOSITION 2.1 [5, 6]. *The minimal non-trivial varieties of monoids are*
(1) *the variety* $\mathbf{J}_1$ *of idempotent and commutative monoids,*
(2) *the variety* $(\mathbf{Z}_p)$ *for every prime* p.

PROPOSITION 2.2 [6]. *The minimal non-trivial varieties of semigroups are*
(1) *the variety* $\mathbf{J}_1$ *of idempotent and commutative semigroups,*
(2) *the variety* $(\mathbf{Z}_p)_S$ *for every prime* p,
(3) *the variety* $\mathbf{K}_1$ *of left simple aperiodic semigroups,*
(4) *the variety* $\mathbf{K}_1^r$ *of right simple aperiodic semigroups,*
(5) *the variety* (N_2) *where* N_2 *is the two-element nilpotent semigroup.*

Equations for these varieties are also well known.

$\mathbf{J}_1$	$xy = yx$ and $x = x^2$.
$(\mathbf{Z}_p)$	$x^p = 1$ and $xy = yx$.
$\mathbf{K}_1$	$xy = x$.
$\mathbf{K}_1^r$	$yx = x$.
(N_2)	$x_1x_2 = y_1y_2$.

The following result is more involved. Recall that a variety **V** is noncommutative if it contains at least one non-commutative semigroup.

PROPOSITION 2.3 [8]. *The minimal non-commutative varieties of semigroups are*

(1) *the variety* $\mathbf{K}_1$,

(2) *the variety* $\mathbf{K}_1^r$,

(3) *the variety* (N) *where* N *is the syntactic semigroup of the language* $\{ab\}$ *over the alphabet* $\{a, b\}$,

(4) *certain varieties of the form* $(G)_S$ *where* G *is a group whose derived group is commutative.*

PROPOSITION 2.4 [8]. *The minimal non-commutative varieties of monoids consist of*

(1) *the variety* (U_2),

(2) *the variety* (U_2^r),

(3) *the variety* (N^1) *where* N^1 *is the syntactic monoid of the language* $\{ab\}$ *over the alphabet* $\{a, b\}$,

(4) *certain varieties of the form* (G) *where* G *is a group whose derived group is commutative.*

The classification of minimal non-commutative varieties of groups has not been completed. In particular equations of these varieties are not yet known (see [2] for recent progress on varieties of finite groups). For the other minimal varieties however we have some more precise results. Edmunds [3] has shown that the variety (N^1) is defined by the equations $x^2 = x^3$ and $x^2y = yx^2 = xyx$. Again this was originally formulated for varieties in the sense of Birkhoff.

The equations of the varieties (U_2) and (U_2^r) are probably also well-known. However these equations are not explicitly given in the classical papers [3, 6, 9, 15]. Thus for the convenience of the reader we give here a complete proof.

PROPOSITION 2.5. (a) $(U_2) = \mathbf{R}_1^r$, *the variety of idempotent and* $\mathfrak{L}$*-trivial monoids, defined by the equation* $xyx = yx$.

(b) $(U_2^r) = \mathbf{R}_1$, *the variety of idempotent and* $\mathfrak{R}$*-trivial monoids, defined by the equation* $xyx = xy$.

By duality it is sufficient to prove (b). Let $\mathbf{V} = (U_2^r)$,. By Proposition 1.1, **V** is defined by a sequence of equations. Since $U_2^r \in \mathbf{R}_1$, **V** is contained in $\mathbf{R}_1$ and thus satisfies the equation

(1) $xyx = xy$ of $\mathbf{R}_1$.

If $\mathbf{V} \neq \mathbf{R}_1$, one can find an equation

(2) $u = v$

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satisfied by $\mathbf{V}$ which cannot be deduced from (1). Choose such an equation with $|u|+|v|$ minimal. Then neither u nor v contains a factor of the form xyx with $x \neq 1$. Otherwise one could use (1) to get an equation $u' = v'$ with $|u'|+|v'|<|u|+|v|$. Therefore both u and v contain at most one occurrence of each letter. Let x be a letter of u. Setting $y = 1$ for all $y \neq x$ in (2) yields $x = x^{|v|_x}$. Since $\mathbf{V}$ is not trivial $|v|_x \neq 0$ and thus x occurs in v. The same argument shows that every letter of v occurs in u. Thus u and v contains the same letters and therefore $u = x_1 \cdots x_n$ and $v = x_{1\sigma} \cdots x_{n\sigma}$ where the x_i's are all different and σ is a permutation of $\{1,\ldots,n\}$. If σ is not the identity, one can find two indices $i<j$ such that $j\sigma^{-1} < i\sigma^{-1}$. Taking $x_k = 1$ for $k \neq i, j$ in (2) yields $x_i x_j = x_j x_i$. Thus $\mathbf{V}$ satisfies the equation $xy = yx$, a contradiction since U_2^r is not commutative. Therefore $\mathbf{V} = \mathbf{R}_1$.

The next theorem summarizes some properties of the variety $\mathbf{R}_1$. Equivalences (2)–(3) and (1)–(4) were known to Eilenberg [4]. The new statement (5) provides a useful unambiguous description of languages in $\mathcal{R}_1$, the variety of languages corresponding to $\mathbf{R}_1$. Note also that the proof below furnishes a new demonstration of Proposition 2.5 and of Eilenberg's results.

THEOREM 2.6. *Let L be a language of A^* and let $M = M(L)$ be the syntactic monoid of L. The following conditions are equivalent*

(1) *M belongs to (U_2^r).*

(2) *M belongs to $\mathbf{R}_1$.*

(3) *For all $x, y \in M$, $xy = xyx$.*

(4) *L belongs to the boolean algebra generated by languages of the form B^*aA^* where $a \in A$ and $B \subset A$.*

(5) *L is a disjoint union of languages of the form*

$$a_1a_1^*a_2\{a_1, a_2\}^*a_3\{a_1, a_2, a_3\}^* \cdots a_n\{a_1, a_2,\ldots,a_n\}^*$$

where the a_i's are distinct letters of A.

PROOF. (1) implies (2) since $U_2^r \in \mathbf{R}_1$.

(2) $\Rightarrow$ (3). Let $x, y \in M$. Since M is idempotent $xy = (xy)^2$ and thus $xy \mathcal{R} xyx$. But M is $\mathcal{R}$-trivial and therefore $xy = xyx$.

(3) $\Rightarrow$ (5). Let ρ: $A^* \to A^*$ be the function which associates to any word u the sequence of all distinct letters of U in the order in which they first appear when u is read from left to right. For example if $u = caabacb$ then $u\rho = cab$. In fact ρ is a sequential function: a sequential transducer [1] realizing ρ is $\mathcal{C} = (2^A, A, \delta, \lambda, \varnothing)$ where the transition function δ and the output function λ are defined by

$$(B, a)\delta = B \cup \{a\},$$

$$(B, a)\lambda = \begin{cases} 1 & \text{if } a \in B, \\ a & \text{if } a \notin B. \end{cases}$$

For example for $A = \{a, b\}$, $\mathcal{C}$ can be pictured as in Figure 1.

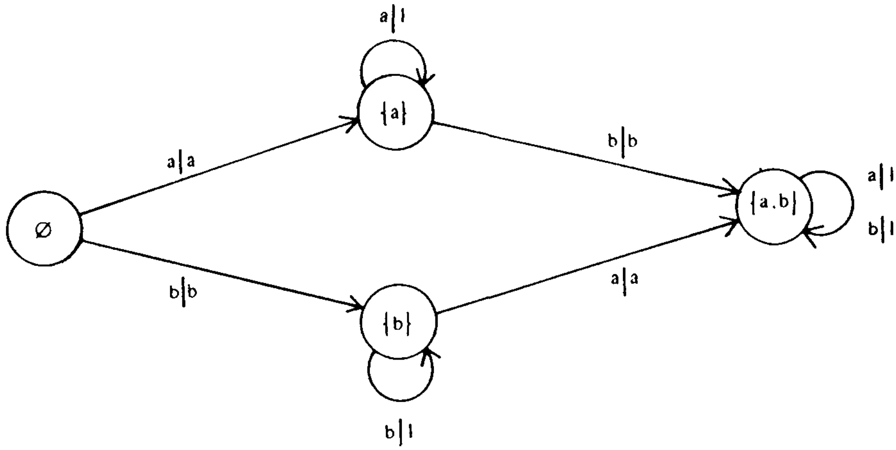

FIGURE 1

Define an equivalence $\sim$ on A^* by $u \sim v$ if $u \sim v$ if $u\rho = v\rho$. It is not difficult to see that the equivalence classes of A^* are the (disjoint) sets

$$L_{a_1,\ldots,a_n} = a_1 a_1^* a_2 \{a_1, a_2\}^* \cdots a_n \{a_1, a_2, \ldots, a_n\}^*$$

where $\{a_1, \ldots, a_n\} \subset A$. Thus if $u \sim v$, then u and v belong to some language $L_{a_1,\ldots,a_n}$. Let $a \in A$. If $a = a_i$ for some i, then ua, $va \in L_{a_1,\ldots,a_n}$ and au, $av \in L_{a, a_1,\ldots,a_{i-1}, a_{i+1},\ldots,a_n}$. Thus $ua \sim va$ and $au \sim av$. If $a \notin \{a_1, \ldots, a_n\}$, then ua, $va \in L_{a_1,\ldots,a_n, a}$ and au, $av \in L_{a, a_1,\ldots,a_n}$ and thus again $ua \sim va$ and $au \sim av$. Therefore $\sim$ is a congruence on A^*.

Let η: $A^* \to M$ be the syntactic morphism of L. If $u \in L_{a_1,\ldots,a_n}$ then $u = a_1 u_1 a_2 \cdots a_n u_n$ where $u_i \in \{a_1, \ldots, a_i\}^*$ for $i = 1, \ldots, n$ and thus by (3) $u\eta = (a_1 \cdots a_n)\eta$. It follows that $u \sim v$ implies $u\eta = v\eta$ and therefore L is a (disjoint) union of equivalence classes of $\sim$, that is of languages of the form $L_{a_1 \cdots a_n}$.

(5) $\Rightarrow$ (4). It is sufficent to observe that

$$L_{a_1,\ldots,a_n} = A_n^* \cap \bigcap_{i=1}^{n} A_{i-1}^* a_i A_i^* \quad \text{where } A_i = \{a_1, \ldots, a_i\}^* \text{ and } A_0 = \varnothing.$$

(4) $\Rightarrow$ (1). By [4, Proposition 2.2 page 189], it is sufficient to show that U_2^r recognizes B^*aA^*, where $B \subset A$ and $a \in A$. Let $U_2^r = \{1, x, y\}$ and let ϕ: $A^* \to U_2^r$ be a morphism defined by

$$\begin{aligned} b\phi &= 1 \quad \text{for } b \in B \backslash \{a\}, \\ a\phi &= x, \\ b\phi &= y \quad \text{for } b \in A \backslash (B \cup \{a\}). \end{aligned}$$

Then $x\phi^{-1} = (B \backslash \{a\})^* a A^* = B^* a A^*$ which proves the claim.

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Of course a dual statement holds for the variety $\mathbf{R}_1^r$.

THEOREM (2.6)^r. *Let L be a language of A^* and let $M = M(L)$ be the syntactic monoid of L. The following conditions are equivalent*

(1) *M belongs to (U_2).*

(2) *M belongs to $\mathbf{R}_1^r$.*

(3) *For all $x, y \in M$, $xyx = yx$.*

(4) *L belongs to the boolean algebra generated by languages of the form A^*aB^* where $a \in A$ and $B \subset A$.*

(5) *L is a disjoint union of languages of the form*

$$\{a_1,\ldots,a_n\}^* a_n \{a_1,\ldots,a_{n-1}\}^* \cdots a_1^* a_1$$

where the a_k's are pairwise different letters of A.

The join variety $\mathbf{R}_1 \vee \mathbf{R}_1^r$ also admits many characterizations.

THEOREM 2.7. *Let M be a finite monoid. The following conditions are equivalent*

(1) *$M \in \mathbf{R}_1 \vee \mathbf{R}_1^r$.*

(2) *M satisfies the equations $x = x^2$ and $xyxzx = xyzx$.*

(3) *M divides a direct product $S_1^1 \times \cdots \times S_n^1$ where the S_i's are simple aperiodic semigroups.*

(4) *M is idempotent and for all $s, t \in M$ and $x \in M^{-1}sM^{-1} \cap M^{-1}tM^{-1}$ $sxt = st$.*

(5) *M is idempotent and for all $s, t \in M$ and $x \in SM^{-1} \cap M^{-1}t$ $sxt = s(t)$.*

PROOF. (1) ⇒ (3). Let $\mathbf{V}$ be the variety generated by all monoids S^1 where S is a simple aperiodic semigroup. Clearly U_2 and U_2^r belong to $\mathbf{V}$. By Proposition 2.4, $(U_2) = \mathbf{R}_1^r$ and $(U_2^r) = \mathbf{R}_1$ and thus $\mathbf{R}_1$, $\mathbf{R}_1^r$ and $\mathbf{R}_1 \vee \mathbf{R}_1^r$ are contained in $\mathbf{V}$. Therefore every $M \in \mathbf{R}_1 \vee \mathbf{R}_1^r$ satisfies (3).

(3) ⇒ (2). Let $M = S^1$ where S is a simple aperiodic semigroup. Then clearly M is idempotent. Moreover let $x, y, z \in M$. If x, y or $z = 1$ then clearly $xyxzx = xyzx$. If $x, y, z \in S$ then $xyxzx = x = xyzx$ by the Green-Rees theorem.

(2) ⇒ (4). Since $x = x^2$ holds in M, M is idempotent. Moreover if $x \in M^{-1}sM^{-1} \cap M^{-1}tM^{-1}$ then $s = axb$ and $t = cxd$ for some $a, b, c, d \in M$. Therefore by (2) $sxt = axbxcxd = axbcxd = st$.

(4) ⇒ (5). Trivial.

(5) ⇒ (2). Since M is idempotent, M satisfies the equation $x = x^2$. On the other hand since $x \in (xy)M^{-1} \cap M^{-1}(zx)$, $xyxzx = xyzx$.

(2) ⇒ (1). We shall use the notations introduced in the proof of Theorem 2.6. Moreover we shall denote by $u\alpha$ the set of letters appearing in a word u: $u\alpha = \{a \in A \,\|\, |u|_a > 0\}$. Let ϕ: $A^* \to M$ be a surjective morphism. We claim that for all $u \in A^*$

(1) $u\phi = (u\rho[(u^r)\rho]^r)\phi$ where u^r denotes the reverse of the word u.

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First we introduce the following lemma.

LEMMA 2.8. *Let $u, v \in (A \backslash a)^*$ with $v\alpha \subset u\alpha$. Then*
$$(uava)\phi = (uva)\phi.$$

PROOF (induction on $|v|$). Since $a\phi = a^2\phi$ the case $|v| = 0$ is trivial. Assume $v = bv'$ for some $b \in A \backslash a$. Since $v\alpha \subset u\alpha$, there exxist $u_0, u_1 \in (A \backslash a)^*$ such that $u = u_0 b u_1$. Now we apply the equation $xyxzx = xyzx$ twice to obtain the equalities

$$(uava)\phi = (u_0bu_1abv'a)\phi = (u_0bu_1babv'a)\phi = (u_0bu_1babav'a)\phi.$$

But $(ba)\phi = (ba)^2\phi$ and thus $(uava)\phi = (ubav'a)\phi = (ubv'a)\phi$ by induction. Thus $(uava)\phi = (uva)\phi$ as required.

We now prove the claim (1) by induction on $|u|$. The result is clear for $|u| = 0$. Consider the word ua.

First case: $a \in u\alpha$. Then $u\rho = a_1 \cdots a_n$, $u^r\rho = b_1 \cdots b_n$ and $a = a_i = b_j$ for some i and j. Therefore $(ua)\rho = a_1 \cdots a_n$, $(ua)^r\rho = ab_1 \cdots b_{j-1}b_{j+1} \cdots b_n$ and finally $(ua)\rho[(ua)^r\rho]^r = a_1 \cdots a_{i-1}aa_{i+1} \cdots a_nb_n \cdots b_{j+1}b_{j-1} \cdots b_1a$. Using the equation $xyzx = xyxzx$ we obtain

$$\begin{aligned}\big((ua)\rho\big[(ua)^r\rho\big]^r\big)\phi &= \big(a_1 \cdots a_{i-1}aa_{i+1} \cdots a_nb_n \cdots b_{j+1}ab_{j-1} \cdots b_1a\big)\phi \\ &= \big[(u\rho)(u^r\rho)^r a\big]\phi = (u\phi)(a\phi) = (ua)\phi.\end{aligned}$$

Second case: $a \notin ua$. Then $(ua)\rho = (u\rho)a$, $(ua)^r\rho = a(u^r\rho)$ and $(ua)\rho((ua)^r\rho)^r = (u\rho)a(u^r\rho)a$. Since $u\rho\alpha = u^r\rho\alpha$ we can apply Lemma 2.8. Thus $[(ua)\rho((ua)^r\rho)^r]\phi = (u\rho u^r\rho a)\phi = u\phi a\phi = (ua)\phi$ and this proves the claim.

It follows from (1) that if $u\rho = v\rho$ and $u^r\rho = v^r\rho$ then $u\phi = v\phi$. Therefore M divides $A^*/\sim \times A^*/\sim^r$ where $\sim^r$ is the congruence on A^* defined by $u \sim^r v$ if and only if $u^r\rho = v^r\rho$. But $A^*/\sim$ satisfies the equations $x = x^2$ and $xyx = xy$ and thus $A^*/\sim \in \mathbf{R}_1$. Dually $A^*/\sim^r \in \mathbf{R}_1^r$ and therefore $M \in \mathbf{R}_1 \vee \mathbf{R}_1^r$.

COROLLARY 2.9. *$\mathbf{R}_1 \vee \mathbf{R}_1^r$ is the smallest variety containing all monoids S^1 where S is a simple aperiodic semigroup.*

The next proposition shows that there exist no varieties "between" $\mathbf{R}_1$ (resp. $\mathbf{R}_1^r$) and $\mathbf{R}_1 \vee \mathbf{R}_1^r$.

PROPOSITION 2.10 [14]. *If $\mathbf{R}_1 \subset \mathbf{V} \subset \mathbf{R}_1 \vee \mathbf{R}_1^r$ ($\mathbf{R}_1^r \subset \mathbf{V} \subset \mathbf{R}_1 \vee \mathbf{R}_1^r$) then $\mathbf{V} = \mathbf{R}_1$ ($\mathbf{R}_1^r$) or $\mathbf{V} = \mathbf{R}_1 \vee \mathbf{R}_1^r$.*

PROOF. Assume $\mathbf{R}_1 \subset \mathbf{V} \subset \mathbf{R}_1 \vee \mathbf{R}_1^r$ and $\mathbf{R}_1 \neq \mathbf{V}$. Then there exists $M \in \mathbf{V} \backslash \mathbf{R}_1$. M is idempotent but not $\mathcal{R}$-trivial and thus there exist in M two elements a and b which are $\mathcal{R}$-related. It follows that $\{1, a, b\}$ is a copy of U_2 which divides M. Therefore $U_2 \in \mathbf{V}$ and $\mathbf{R}_1^r \subset \mathbf{V}$ by Proposition 2.5. Since $\mathbf{R}_1 \subset \mathbf{V}$, $\mathbf{R}_1 \vee \mathbf{R}_1^r \subset \mathbf{V}$ and finally $\mathbf{V} = \mathbf{R}_1 \vee \mathbf{R}_1^r$. The statement in brackets is dual.

We turn now to non-commutative varieties of (finite) semigroups which contain all commutative semigroups. The next theorem extends to varieties of finite semigroups a result stated in [4] for varieties in the sense of Birkhoff.

THEOREM 2.11. *There is one and only one minimal variety* $\mathbf{V}$ *in the class of non-commutative varities of finite semigroups containing all commutative semigroups.* $\mathbf{V}$ *is defined by the equations* $x_1x_2x_3 = x_{1\sigma}x_{2\sigma}x_{3\sigma}$ *for all permutations* σ *of* $\{1,2,3\}$. *The corresponding variety of languages* $\mathcal{V}$ *is described as follows: a recognizable language belongs to* $A^+\mathcal{V}$ *if and only if* $L \cap A^2A^+$ *is a commutative language.*

PROOF. Let **Com** be the variety of all commutative semigroups and let $\mathbf{W}$ be a non-commutative variety containing **Com**. Let $\mathcal{W}$ be the corresponding variety of languages. Since $\mathbf{W}$ is non-commutative, there exists an alphabet A such that $A^+\mathcal{W}$ contains a non-commutative language L. That is there exist two commutatively equivalent words u and v such that $u \in L$ and $v \notin L$. Now $A^{|u|}$ is a commutative language and thus $A^{|u|} \in A^+\mathcal{W}$ since $\mathbf{Com} \subset \mathbf{W}$. Therefore $L \cap A^{|u|}$ is a finite non-commutative language contained in $A^+\mathcal{W}$. It follows [10] that $\mathbf{W}$ contains N. Thus $\mathbf{V} = \mathbf{Com} \vee (N)$ is the smallest non-commutative variety containing **Com**. Let $\mathbf{U}$ be the variety defined the equations (1) $x_1x_2x_3 = x_{1\sigma}x_{2\sigma}x_{3\sigma}$ for all $\sigma \in S_3$ (the symmetric group on three letters). Since N and every commutative semigroup satisfy these equations, $\mathbf{V}$ is contained in $\mathbf{U}$. We also note that the equations (1) imply $x_i(x_{i+1} \cdots x_{j-1})x_j = x_j(x_{i+1} \cdots x_{j-1})x_i$ and thus $x_1 \cdots x_n = x_{1\tau} \cdots x_{n\tau}$ for any transposition τ. Since S_n is generated by the transpositions $x_1 \cdots x_n = x_{1\sigma} \cdots x_{n\sigma}$ holds in $\mathbf{U}$ for all $\sigma \in S_n$, $n \geqslant 3$. It follows that if a language L is recognized by a semigroup $S \in \mathbf{U}$ then $L \cap A^2A^+$ is commutative.

Conversely let L be a language such that $L_1 = L \cap A^2A^+$ is commutative. Then L is union of L_1 and of languages of the form $\{w\}$ with $|w| \leqslant 2$. If $w = a$ or a^2 for some $a \in A$, $\{w\}$ is a commutative language and if $w = ab$ for some $a \neq b$, $\{w\}$ is recognized by N. Since L_1 is also commutative, we have shown that L belongs to the variety of languages corresponding to $\mathbf{V}$. Thus $\mathbf{U}$ is contained in $\mathbf{V}$ and this concludes the proof of the theorem.

3. Extensions of idempotent and commutative semigroups

In this section we study an operation on varieties which is closely related to the semilattice decomposition of a semigroup. We first summarize some definitions (see the Chapters XI and XII of [4], written by Tilson, or [10] for more details).

A *relational morphism* between two semigroups S and T is a relation $\tau: S \to T$ such that

(1) $s\tau \neq \varnothing$ for all $s \in S$,

(2) $(s_1\tau)(s_2\tau) \subset (s_1s_2)\tau$ for all $s_1, s_2 \in S$.

If T' is a subsemigroup of T, then $T'\tau^{-1} = \{s \in S \mid s\tau \cap T' \neq \varnothing\}$ is a subsemigroup of S'. Let $\mathbf{V}$ be a variety of finite semigroups. A (relational) morephism $\tau: S \to T$ is a (relational) $\mathbf{V}$-morphism if and only if for all subsemigroups T' of T, $T' \in \mathbf{V}$ implies $T'\tau^{-1} \in \mathbf{V}$.

If $\mathbf{W}$ is a variety of finite monoids we denote by $\mathbf{V}^{-1}\mathbf{W}$ the variety of finite monoids M such that there exists a relational $\mathbf{V}$-morphism $\tau: M \to N$ for some $N \in \mathbf{W}$. It is not difficult to prove that $\mathbf{V}^{-1}\mathbf{W}$ is generated by the finite monoids M such that there exists a $\mathbf{V}$-morphism $\tau: M \to N$ for some $N \in \mathbf{W}$. Here we consider the case $\mathbf{W} = \mathbf{J}_1$.

PROPOSITION 3.1. *The mapping* $\mathbf{V} \to \mathbf{V}^{-1}\mathbf{J}_1$ *induces an order preserving mapping from the varieties of semigroups contained in* $\mathbf{LG}$ *to the varieties of monoids contained in* $\mathbf{DS}$.

PROOF. Let $\mathbf{V}$ be a variety of semigroups contained in $\mathbf{LG}$ and let $\phi: S \to T$ be a relational $\mathbf{V}$-morphism with $T \in \mathbf{J}_1$. Since the only subsemigroups of T which are in $\mathbf{V}$, hence in $\mathbf{LG}$, are trivial, ϕ is a relational $\mathbf{V}$-morphism if and only if for every idempotent $e \in T$, $e\phi^{-1} \in \mathbf{V}$. Consequently if $\mathbf{V}_1 \subset \mathbf{V}_2 \subset \mathbf{LG}$, $\mathbf{V}_1^{-1}\mathbf{J}_1 \subset \mathbf{V}_2^{-1}\mathbf{J}_1$.

Assume now that $\tau: S \to T$ is a $\mathbf{LG}$-morphism and let D be a regular $\mathcal{D}$-class of S. Then for all $s, t \in D$, $s\tau \mathcal{D} t\tau$ and thus $s\tau = t\tau = e$. Therefore $D\tau = e$ and $D \subset e\tau^{-1}$. Since τ is a $\mathbf{LG}$-morphism, $e\tau^{-1}$ is nil-simple and thus D is a semigroup. It follows that $S \in \mathbf{DS}$ and hence $\mathbf{LG}^{-1}\mathbf{J}_1 \subset \mathbf{DS}$.

The operation $\mathbf{V} \to \mathbf{V}^{-1}\mathbf{J}_1$ is the extension to varieties of the well-known "semilattice decomposition" of semigroups. Here are some classical examples (see [13] for an overview in terms of semilattice decomposition):

PROPOSITION 3.2. *The following equalities hold*:

(1) $\mathbf{R} = \mathbf{K}^{-1}\mathbf{J}_1$, $\mathbf{R}^r = (\mathbf{K}^r)^{-1}\mathbf{J}_1$, $\mathbf{R}_1 = \mathbf{K}_1^{-1}\mathbf{J}_1$, $\mathbf{R}_1^r = (\mathbf{K}_1^r)^{-1}\mathbf{J}_1$,

(2) $\mathbf{J} = \mathbf{Nil}^{-1}\mathbf{J}_1$,

(3) $\mathbf{DA} = \mathbf{LI}^{-1}\mathbf{J}_1$, $\mathbf{DS} = \mathbf{LG}^{-1}\mathbf{J}_1$,

(4) $\mathbf{A}_1 = \mathbf{LI}_1^{-1}\mathbf{J}_1$, $\mathbf{DS}_1 = \mathbf{LG}_1^{-1}\mathbf{J}_1$.

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PROOF. Let $T \in \mathbf{J}_1$ and let ϕ: $S \to T$ be a $\mathbf{K}$-morphism. Let R be a regular $\mathcal{R}$-class of S. Then for all $s, t \in R$, $s\phi \mathcal{R} t\phi$ and hence $s\phi = t\phi = e$. Thus $R \subset e\phi^{-1}$ and since ϕ is a $\mathbf{K}$-morphism, $e\tau^{-1} \in \mathbf{K}$. Therefore R is trivial and $S \in \mathbf{R}$. It follows that $\mathbf{K}^{-1}\mathbf{J}_1$ is contained in $\mathbf{R}$. The other inclusions of the type $\mathbf{V}^{-1}\mathbf{J}_1 \subset \mathbf{W}$ can be proved in the same way.

We now prove the opposite inclusions. Let $S \in \mathbf{R}$ and let $T = (2^{E(S)}, \cap)$ be the semigroup of all subsets of $E(S)$ with intersection as multiplication. Clearly $T \in \mathbf{J}_1$. Define π: $S \to T$ by

$$s\pi = \{e \in E(S) \mid es = e\}.$$

We claim that π is a $\mathbf{K}$-morphism. Indeed we have $e \in (s_1 s_2)\pi$ if and only if $e = es_1s_2$. Since S is $\mathcal{R}$-trivial this is equivalent to $e = es_1 = es_2$. It follows that $(s_1s_2)\pi = (s_1\pi)(s_2\pi)$.

Since S is $\mathcal{R}$-trivial, hence aperiodic, there exists $n > 0$ such that $s^n = s^{n+1}$ for all $s \in S$. Fix an element $A \in T$ and let $x, y \in A\pi^{-1}$. Then $x^n \in A\pi = x$, since $x^n x = x^n$. Therefore $x^n y = x^n$ because $A = y\pi$. It follows that the semigroup $A\pi^{-1}$ satisfies the equation $x^n y = x^n$ so $A\pi^{-1} \in \mathbf{K}$. This proves the claim and the inclusion $\mathbf{R} \subset \mathbf{K}^{-1}\mathbf{J}_1$ follows easily. Moreover if $S \in \mathbf{R}_1$, π is clearly a $\mathbf{K}_1$-morphism. Thus $\mathbf{R}_1 \subset \mathbf{K}_1^{-1}\mathbf{J}_1$. The inclusions $\mathbf{R}^r \subset (\mathbf{K}^r)^{-1}\mathbf{J}_1$ and $\mathbf{R}_1^r \subset (\mathbf{K}_1^r)^{-1}\mathbf{J}_1$ are dual. The inclusion $\mathbf{J} \subset \mathbf{Nil}^{-1}\mathbf{J}_1$ is obtained in the same way by considering for $S \in \mathbf{J}$, the morphism π: $S \to T = (2^{E(S)}, \cap)$ defined $s\pi = \{e \in E(S) \mid es = e = se\}$.

For the inclusions $\mathbf{DA} \subset \mathbf{LI}^{-1}\mathbf{J}_1$ and $\mathbf{A}_1 \subset \mathbf{LI}_1^{-1}\mathbf{J}_1$ we consider the morphism π: $S \to T$ defined by $s\pi = \{e \in E(S) \mid ese = e\}$. Finally for the inclusion $\mathbf{DS} \subset \mathbf{LG}^{-1}\mathbf{J}_1$ we consider the map π: $S \to T$ defined by $s\pi = \{e \in E(S) \mid (ese)^\omega = e\}$ where x^ω denotes the (unique) idempotent contained in the semigroup of S generated by x. We claim that π is an $\mathbf{LG}$-morphism: we have $e \in (s_1s_2)\pi$ if and only if $(es_1s_2e)^\omega = e$. But this condition implies $es_1e \mathcal{H} e \mathcal{H} es_2e$ and hence $(es_1e)^\omega = (es_2e)^\omega = e$. Conversely if $(es_1e)^\omega = (es_2e)^\omega = e$, we get $es_1 \mathcal{R} e \mathcal{L} s_2e$ and hence $es_1e_2 \mathcal{J} e$. It follows $es_1s_2e \mathcal{H} e$ and thus $(es_1s_2e)^\omega = e$. Therefore π is a morphism.

Fix an element $A \in T$ and let $x, y \in A\pi^{-1}$. Then $x^\omega \in A$ since $(xx^\omega x)^\omega = x^\omega$ and thus $(x^\omega y x^\omega)^\omega = x^\omega$ since $A = y\pi$. It follows that $A\pi^{-1} \in \mathbf{LG}$ and π is a $\mathbf{LG}$-morphism.

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