P. Pankka, P. Poggi-Corradini and K. Rajala
Nagoya Math. J.
Vol. 192 (2008), 137–149

SHARP EXPONENTIAL INTEGRABILITY FOR TRACES OF MONOTONE SOBOLEV FUNCTIONS

PEKKA PANKKA, PIETRO POGGI-CORRADINI AND
KAI RAJALA

Abstract. We answer a question posed in [12] on exponential integrability of functions of restricted n-energy. We use geometric methods to obtain a sharp exponential integrability result for boundary traces of monotone Sobolev functions defined on the unit ball.

§1. Introduction

The following result answered a problem of A. Beurling, mentioned by J. Moser in a famous paper [10].

THEOREM A. (Chang-Marshall (1985), [1]) *There is a universal constant $C < \infty$ so that if f is analytic in the unit disc $\mathbb{D}$, $f(0) = 0$, and*

$$\int_{\mathbb{D}} |f'(z)|^2 \frac{\mathrm{d}A(z)}{\pi} \leq 1, \tag{1.1}$$

then

$$\int_0^{2\pi} \exp\big(|f^\star(e^{i\theta})|^2\big)\, \mathrm{d}\theta \leq C,$$

where $f^\star$ is the trace of f on $\partial\mathbb{D}$, i.e., $f^\star(\zeta) = \lim_{t\uparrow 1} f(t\zeta)$ for $\mathcal{H}^1$-a.e. $\zeta \in \partial\mathbb{D}$.

This result is moreover "sharp" in the following sense: the Beurling functions,

$$B_a(z) := \left(\log \frac{1}{1-az}\right)\left(\log \frac{1}{1-a^2}\right)^{-1/2} \qquad 0 < a < 1$$

Received March 6, 2007.
2000 Mathematics Subject Classification: Primary 46E35; Secondary 31C45.
The third author was supported by the Academy of Finland. The authors thank the Department of Mathematics at the University of Michigan where part of this research was conducted.

are analytic in $\mathbb{D}$, satisfy $B_a(0) = 0$ and (1.1), and have the property that for any given $\alpha > 1$, one can choose a so that the integral

$$\int_0^{2\pi} \exp\big(\alpha |B_a(e^{i\theta})|^2\big)\, \mathrm{d}\theta$$

is as large as desired.

The following is an easy corollary of the Chang-Marshall Theorem.

Corollary A. *There is a universal constant $C < \infty$ so that if $u : \mathbb{D} \to \mathbb{R}$ is harmonic with $u(0) = 0$ and*

$$\int_{\mathbb{D}} |\nabla u(z)|^2 \, \frac{\mathrm{d}A(z)}{\pi} \le 1,$$

then

$$\int_0^{2\pi} \exp\big(u^\star(e^{i\theta})^2\big)\, \mathrm{d}\theta \le C,$$

where $u^\star$ is the trace of u on $\partial\mathbb{D}$, i.e., $u^\star(\zeta) = \lim_{t\uparrow 1} u(t\zeta)$ for $\mathcal{H}^1$-a.e. $\zeta \in \partial\mathbb{D}$.

This can also be shown to be sharp by considering the real parts of the Beurling functions.

In [12] the last two authors generalized the Chang-Marshall theorem to quasiregular mappings in all dimensions. They asked in [12] whether Corollary A also generalizes, perhaps substituting "harmonic" with "n-harmonic". In this note we show that this is indeed possible. The key concept is that of a monotone Sobolev function, whose definition we recall below, and which is quite general, and includes for instance n-harmonic functions.

§2. Main results

Let Ω be an open and connected set. For a continuous function $u\colon \Omega \to \mathbb{R}$, we define the oscillation of u on a compact set $K \subset \Omega$ by

$$\operatorname*{osc}_K u = \max_{x,y\in K} |u(x) - u(y)|.$$

We say that $u\colon \Omega \to \mathbb{R}$ is *monotone* if $\operatorname{osc}_{\partial B} u = \operatorname{osc}_{\bar{B}} u$ for all n-balls B compactly contained in Ω.

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press

By integrating the gradient over radial segments and changing variables, we see that, for a continuous $u\colon B^n \to \mathbb{R}$ in the Sobolev space $W^{1,n}(B^n)$, the radial limit

$$\tilde{u}(y) = \lim_{r\to 1} u(ry)$$

exists at $\mathcal{H}^{n-1}$-a.e. point $y \in S^{n-1}$. We denote by $\tilde{u}$ the almost everywhere defined trace of u. Moreover, we denote the L^p-norm of a p-integrable $g\colon \Omega \to \mathbb{R}^n$ by $\|g\|_p = \|g\|_{\Omega,p}$. The surface measure $\mathcal{H}^{n-1}(S^{n-1})$ of the unit sphere S^{n-1} is ω_{n-1}. The notations $B^n(r) = B^n(0,r)$, $B^n = B^n(1)$ for n-dimensional balls will be used.

THEOREM 1. *There exists a constant $C = C(n) > 0$ so that if $u \in W^{1,n}(B^n)$ is a non-constant continuous monotone function such that $u(0) = 0$, then*

$$\int_{S^{n-1}} \exp\bigl(\alpha(|\tilde{u}(y)|/\|\nabla u\|_n)^{n/(n-1)}\bigr)\, \mathrm{d}\mathcal{H}^{n-1}(y) \le C, \tag{2.2}$$

where

$$\alpha = (n-1)\left(\frac{\omega_{n-1}}{2}\right)^{1/(n-1)}. \tag{2.3}$$

The continuity assumption in Theorem 1 is of technical nature. By a theorem of Manfredi [8], so-called weakly monotone functions in $W^{1,n}$ are always continuous and monotone in the above sense. In general, $W^{1,n}$-functions need not be continuous.

The monotonicity assumption in Theorem 1 cannot be dropped altogether, since the n-capacity of a point is zero. Indeed, if we define $u_i\colon B^n \to \mathbb{R}$,

$$u_i(x) = \begin{cases} \frac{\log(1/|x|)}{\log i}, & 1/i \le |x| < 1, \\ 1, & 0 \le |x| < 1/i, \end{cases}$$

and $v_i = 1 - u_i$, we see that $v_i(0) = 0$, $\tilde{v_i} = 1$ on the unit sphere, and $\|\nabla v_i\|_n \to 0$ as $i \to \infty$.

Our method of proof for Theorem 1 has a similar geometric flavor as in [9] and in [12], and the end-game is again to appeal to Moser's original one-dimensional proof. However, the so-called "egg-yolk" property, which was the hardest part to establish in the two papers cited above, can be quickly established in our present case. It might come as a surprise then that Theorem 1 is sharp, as we will see in Theorem 2 below, as opposed to the situation in [12].

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called *p-harmonic*, $1 < p < \infty$, if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, \mathrm{d}x = 0$$

for every C^∞-smooth test function ϕ with compact support in Ω, see [6]. Since p-harmonic functions are continuous and satisfy the maximum principle ([6, 6.5]), they are, in particular, monotone.

The next result shows that the constant α in Theorem 1 is sharp.

THEOREM 2. *Let α be as in Theorem 1. There exists a sequence of n-harmonic functions $u_i \in W^{1,n}(B^n)$ satisfying $\|\nabla u_i\|_n \le 1$ and $u_i(0) = 0$, so that*

$$\int_{S^{n-1}} \exp\big(\beta |\tilde{u}_i(y)|^{n/(n-1)}\big) \, \mathrm{d}\mathcal{H}^{n-1}(y) \to \infty \quad \text{as } i \to \infty$$

whenever $\beta > \alpha$.

§3. Proof of Theorem 1

In this section we assume that u satisfies the assumptions of Theorem 1. Moreover, by considering balls $B^n(0, 1 - 1/j)$, for j large, and using Fatou's lemma, we may assume that the function u in Theorem 1 is defined in a neighborhood of the unit ball.

LEMMA 3. *There exists a constant $r_0 = r_0(n) > 0$ so that if $M_0 := \max_{\bar{B}^n(r_0)} |u|$, then*

$$\int_{\{|u| \le M_0\}} |\nabla u|^n \, \mathrm{d}x \ge M_0^n.$$

Proof. For $0 < r < 1$ let $m := \max_{\bar{B}^n(r)} |u|$ and set $v := \min\{|u|, m\}$. By monotonicity, and since $u(0) = 0$, $\operatorname{osc}_{S^{n-1}(t)} v = m$ for every $t \ge r$. By the Sobolev embedding theorem on spheres, see e.g. [5, Lemma 1] or [11], there exists a constant a depending only on n such that

$$\begin{aligned} \int_{B^n \setminus \bar{B}^n(r)} |\nabla v|^n \, \mathrm{d}x &= \int_r^1 \left(\int_{S^{n-1}(t)} |\nabla v|^n \, \mathrm{d}\mathcal{H}^{n-1} \right) \mathrm{d}t \\ &\ge \int_r^1 \frac{\big(\operatorname{osc}_{S^{n-1}(t)} v\big)^n}{at} \, \mathrm{d}t = \frac{m^n}{a} \log \frac{1}{r}. \end{aligned}$$

The claim follows by choosing $r_0 := \exp(-a)$. ∎

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press

Let Γ be a family of paths in an open and connected set Ω. The *n-modulus* $\mathsf{M}_n(\Gamma)$ of Γ is defined as follows:

$$\mathsf{M}_n(\Gamma) = \inf_{\rho} \int_{\Omega} \rho^n \, \mathrm{d}x,$$

where $\rho\colon \Omega \to [0,\infty]$ is *an admissible function* for Γ, i.e. a Borel function satisfying

$$\int_{\gamma} \rho \, \mathrm{d}s \geq 1 \tag{3.4}$$

for every locally rectifiable $\gamma \in \Gamma$. The family of all paths joining two sets $A, B \subset \bar{\Omega}$ in Ω is denoted by $\Delta(A, B; \Omega)$. We say that a given property holds for n-almost every path in a path family Γ if the property holds for all paths in $\Gamma \setminus \Gamma_0$, where Γ_0 is a subfamily of Γ having n-modulus zero.

LEMMA 4. *For every $r \in (0,1)$, there exists a constant $c = c(n,r)$, so that*

$$\mathcal{H}^{n-1}\big(\{y \in S^{n-1} : |u(y)| \geq s\}\big) \leq c \exp\big(-\alpha I_M^s(u)\big) \tag{3.5}$$

for $s \geq M$. Here α is as in (2.3), *$M = M(r,u) = \max_{S^{n-1}(r)} |u|$, and*

$$I_M^s(u) = \int_M^s \frac{\mathrm{d}t}{\left(\int_{\{|u|=t\}} |\nabla u|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1}\right)^{1/(n-1)}}.$$

Proof. Fix $r \in (0,1)$ and $s > M = M(r,u)$. Write

$$E = E_s := \{y \in S^{n-1} : |u(y)| \geq s\}$$

and

$$U_M := \{x \in B^n : M \leq |u(x)| \leq s\}.$$

Also, in what follows we write

$$A_t := \int_{\{|u|=t\}} |\nabla u|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1}. \tag{3.6}$$

The fact that A_t is a Borel function of t is standard, see for instance [2] p. 117. By the coarea formula, cf. [7], and the n-integrability of $|\nabla u|$, A_t is an integrable function of t.

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press

We construct an admissible function ρ for $\Delta(B^n(r), E; B^n)$ as follows: Let $I = I_M^s(u)$, and set

$$\rho(x) := \frac{|\nabla u(x)|}{I A_{|u(x)|}^{1/(n-1)}} \chi_{U_M}(x).$$

Recall that, by Fuglede's theorem [3, Theorem 3], u is absolutely continuous on n-almost every path. So, for n-almost every $\gamma \in \Delta(B^n(r), E; B^n)$ parameterized by arc length $\ell(\gamma)$, we have, by change of variables

$$\begin{aligned}\int_\gamma \rho \, \mathrm{d}s &= \int_0^{\ell(\gamma)} \frac{|\nabla u(\gamma(t))|}{I A_{|u(\gamma(t))|}^{1/(n-1)}} \chi_{U_M}(\gamma(t)) \, \mathrm{d}t \\ &\geq I^{-1} \int_0^{\ell(\gamma)} \frac{|(u \circ \gamma)'(t)|}{A_{|(u \circ \gamma)(t)|}^{1/(n-1)}} \chi_{U_M}(\gamma(t)) \, \mathrm{d}t \geq I^{-1} \int_M^s \frac{\mathrm{d}t}{A_t^{1/(n-1)}} = 1.\end{aligned}$$

Thus ρ is an admissible function for $\Delta(B^n(r), E; B^n)$, by the definition of n-modulus. By the coarea formula, we have

$$\begin{aligned}\mathsf{M}_n(\Delta(B^n(r), E; B^n)) &\leq \int_{U_M} \rho^n \, \mathrm{d}x = I^{-n} \int_{U_M} \frac{|\nabla u(x)|^n}{A_{|u(x)|}^{n/(n-1)}} \, \mathrm{d}x \\ &= I^{-n} \int_M^s \int_{\{|u|=t\}} \frac{|\nabla u(y)|^{n-1}}{A_t^{n/(n-1)}} \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}t \\ &= I^{-n} \int_M^s \frac{A_t}{A_t^{n/(n-1)}} \, \mathrm{d}t = I^{1-n}.\end{aligned}$$

By the conformal invariance of n-modulus, taking inversion with respect to the unit sphere yields

$$2\mathsf{M}_n(\Delta(B^n(r), E; B^n)) \geq \mathsf{M}_n(\Delta(S^{n-1}(r) \cup S^{n-1}(1/r), E; \mathbb{R}^n)).$$

By spherical symmetrization and [4, Theorem 4],

$$\begin{aligned}2I^{1-n} &\geq \mathsf{M}_n(\Delta(S^{n-1}(r) \cup S^{n-1}(1/r), E; \mathbb{R}^n)) \\ &\geq \omega_{n-1} \left(\log \frac{c_2}{\mathcal{H}^{n-1}(E)^{1/(n-1)}} \right)^{1-n},\end{aligned}$$

where c_2 depends only on n and r. See [12] for further details. This implies (3.5). ∎

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press

Proof of Theorem 1. We will use the following result of Moser [10, Equation (6)]: If $\omega\colon [0,\infty) \to [0,\infty)$ is absolutely continuous and satisfies $\omega(0) = 0$, $\omega' \geq 0$ almost everywhere, and

$$\int_0^\infty (\omega'(t))^n \, \mathrm{d}t \leq 1,$$

then

$$\int_0^\infty \exp\bigl(\omega(t)^{n/(n-1)} - t\bigr) \, \mathrm{d}t \leq C, \tag{3.7}$$

where $C > 0$ depends only on n. By scaling invariance of (2.2), we may assume that

$$\int_{B^n} |\nabla u|^n \, \mathrm{d}x = 1. \tag{3.8}$$

Moreover, we fix $r = r_0$ and $M = M_0$ as in Lemma 3. Then, in particular, $M \leq 1$.

By the Cavalieri principle,

$$\begin{aligned}\int_{S^{n-1}} &\exp\bigl(\alpha |u(x)|^{n/(n-1)}\bigr) \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \omega_{n-1} + \frac{\alpha n}{n-1} \int_0^\infty s^{1/(n-1)} \mathcal{H}^{n-1}(E_s) \exp\bigl(\alpha s^{n/(n-1)}\bigr) \, \mathrm{d}s,\end{aligned}$$

where

$$E_s = \{y \in S^{n-1} : |u(y)| \geq s\}.$$

Then, by Lemma 4, it suffices to bound

$$\int_0^{\|u\|_\infty} s^{1/(n-1)} \exp\bigl(\alpha(s^{n/(n-1)} - I_M^s(u))\bigr) \, \mathrm{d}s, \tag{3.9}$$

where $\|u\|_\infty = \max_{y \in S^{n-1}} |u(y)|$, and $I_M^s(u) = 0$ for $0 < s < M$. We define a function $\psi : [0,\infty) \to [0,\infty)$,

$$\psi(s) = \begin{cases} \mu s, & 0 < s < M \\ \alpha I_M^s(u) + \mu M, & M \leq s \leq \|u\|_\infty \\ \alpha I_M^{\|u\|_\infty}(u) + \mu M, & s > \|u\|_\infty, \end{cases}$$

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press

where

$$\mu = \alpha\left(\frac{M}{\int_{\{|u|\le M\}} |\nabla u|^n \, \mathrm{d}x}\right)^{1/(n-1)}. \tag{3.10}$$

Then, by Lemma 3, $\mu M \le \alpha$, and thus we may consider

$$\int_0^{\|u\|_\infty} s^{1/(n-1)} \exp\big(\alpha s^{n/(n-1)} - \psi(s)\big) \, \mathrm{d}s \tag{3.11}$$

instead of (3.9). We define ϕ by $\phi(y) = \psi^{-1}(y)$ for $0 < y < \|\psi\|_\infty$, and $\phi(y) = \|u\|_\infty$ for $y \ge \|\psi\|_\infty$. Then, changing variables $y = \psi(s)$ in (3.11) yields

$$\int_0^\infty \exp\big(\alpha\phi(y)^{n/(n-1)} - y\big)\phi'(y)\phi(y)^{1/(n-1)} \, \mathrm{d}y. \tag{3.12}$$

Integrating by parts, we then have that (3.12) equals $C_1(n) + C_2(n)T$,

$$T = \int_0^\infty \exp\big((\alpha^{(n-1)/n}\phi(y))^{n/(n-1)} - y\big) \, \mathrm{d}y.$$

Now, since ϕ is absolutely continuous and increasing, and $\phi(0) = 0$, Theorem 1 follows from Moser's result (3.7) if we can show that

$$\int_0^\infty \big(\alpha^{(n-1)/n}\phi'(y)\big)^n \, \mathrm{d}y \le 1. \tag{3.13}$$

We have

$$\alpha^{(n-1)/n}\phi'(y) = \begin{cases} \alpha^{(n-1)/n}\mu^{-1}, & 0 < y < \mu M \\ \alpha^{-1/n} A_{\phi(y)}^{1/(n-1)}, & \mu M < y < \|\psi\|_\infty \\ 0, & y > \|\psi\|_\infty, \end{cases}$$

where $A_{\phi(y)}$ as in (3.6). Hence,

$$\alpha^{n-1}\int_0^\infty \phi'(y)^n \, \mathrm{d}y = \alpha^{n-1}\mu^{1-n}M + \alpha^{-1}\int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}^{n/(n-1)} \, \mathrm{d}y. \tag{3.14}$$

By our choice of μ, the first term equals $\int_{\{|u|\le M\}} |\nabla u|^n \, \mathrm{d}x$. Also, by changing variables $\phi(y) = s$ in the right hand integral, and using the coarea formula,

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press

we have

$$\alpha^{-1}\int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}^{n/(n-1)}\,\mathrm{d}y = \int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}\phi'(y)\,\mathrm{d}y = \int_{M}^{\|u\|_\infty} A_s\,\mathrm{d}s = \int_{\{|u|\geq M\}} |\nabla u|^n\,\mathrm{d}x. \tag{3.15}$$

Combining (3.14), (3.15), (3.10) and (3.8) then yields (3.13). The proof is complete. ∎

§4. Proof of Theorem 2

Fix $\beta > \alpha$. For notational convenience, we consider first functions in $B^n(e_n, 1)$ instead of B^n. Fix $2 \leq i \in \mathbb{N}$, and denote $\varepsilon = \varepsilon_i = i^{-1}$. Define $v = v_i\colon B^n(-\varepsilon e_n, 2+\varepsilon) \to \overline{\mathbb{R}}$,

$$v(x) = -\log|x + \varepsilon e_n|.$$

Then v is n-harmonic in $B^n(-\varepsilon e_n, 2+\varepsilon) \setminus \{-\varepsilon e_n\}$. We first show that

$$\int_{B^n(e_n,1)} |\nabla v|^n\,\mathrm{d}x \leq \frac{\omega_{n-1}}{2}\log\frac{2+\varepsilon}{\varepsilon}. \tag{4.16}$$

Clearly,

$$\int_{B^n(e_n,1)} |\nabla v|^n\,\mathrm{d}x \leq \frac{1}{2}\int_A |\nabla v|^n\,\mathrm{d}x,$$

where

$$A = B^n(-\varepsilon e_n, 2+\varepsilon) \setminus \bar{B}^n(-\varepsilon e_n, \varepsilon).$$

Since

$$|\nabla v(x)|^n = |x + \varepsilon e_n|^{-n},$$

we have

$$\frac{1}{2}\int_A |\nabla v|^n\,\mathrm{d}x = \frac{1}{2}\int_{B^n(0,2+\varepsilon)\setminus\bar{B}^n(0,\varepsilon)} |x|^{-n}\,\mathrm{d}x = \frac{\omega_{n-1}}{2}\log\frac{2+\varepsilon}{\varepsilon}.$$

Hence (4.16) holds.

To study exponential integrability of v, set

$$\gamma = \beta\left(\frac{\omega_{n-1}}{2}\log\frac{2+\varepsilon}{\varepsilon}\right)^{1/(1-n)}$$

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press

and $\tau = \gamma/(n-1)$.

By the choice of γ, (4.16), and the Cavalieri principle,

$$\begin{aligned}\int_{S^{n-1}(e_n,1)} &\exp\bigl(\beta\,(|v|/\|\nabla v\|_n)^{n/(n-1)}\bigr)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\geq \omega_{n-1} + \frac{n\gamma}{n-1}\int_0^\infty \mathcal{H}^{n-1}(E_s)s^{1/(n-1)}\exp\bigl(\gamma s^{n/(n-1)}\bigr)\,\mathrm{d}s,\end{aligned} \tag{4.17}$$

where

$$E_s = \{x \in S^{n-1}(e_n,1) : |v(x)| \geq s\}.$$

Since

$$\begin{aligned}E_s = S^{n-1}(e_n,1) &\cap \bar{B}^n(-\varepsilon e_n, \exp(-s)) \\ &\cup\ S^{n-1}(e_n,1)\setminus B^n(-\varepsilon e_n,\exp(s)),\end{aligned}$$

we have

$$\mathcal{H}^{n-1}(E_s) \geq C(n)(\exp(-s))^{n-1} = C(n)\exp((1-n)s) \tag{4.18}$$

for $0 \leq s \leq \log(1/(2\varepsilon))$.

Combining (4.17) and (4.18) yields

$$\begin{aligned}\frac{1}{C(n)}&\int_{S^{n-1}(e_n,1)} \exp\bigl(\beta(|v|/\|\nabla v\|_n)^{n/(n-1)}\bigr)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\geq \frac{n\gamma}{n-1}\int_0^{\log(1/(2\varepsilon))} s^{1/(n-1)}\exp\bigl(\gamma s^{n/(n-1)} + (1-n)s\bigr)\,\mathrm{d}s \\ &= n\tau\int_0^{\log(1/(2\varepsilon))} s^{1/(n-1)}\exp\bigl((n-1)(\tau s^{n/(n-1)} - s)\bigr)\,\mathrm{d}s \\ &= \int_0^{\log(1/(2\varepsilon))} (n\tau s^{1/(n-1)} - (n-1))\exp\bigl((n-1)(\tau s^{n/(n-1)} - s)\bigr)\,\mathrm{d}s \\ &\qquad + (n-1)\int_0^{\log(1/(2\varepsilon))} \exp\bigl((n-1)(\tau s^{n/(n-1)} - s)\bigr)\,\mathrm{d}s \\ &\geq \exp\left((n-1)\left(\tau\left(\log\frac{1}{2\varepsilon}\right)^{n/(n-1)} - \log\frac{1}{2\varepsilon}\right)\right) - 1.\end{aligned}$$

Since

$$\left(\log\frac{2+\varepsilon}{\varepsilon}\right)^{1/(1-n)}\left(\log\frac{1}{2\varepsilon}\right)^{1/(n-1)} \geq 1 - \delta(\varepsilon),$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, we have

$$\int_{S^{n-1}(e_n,1)} \exp\big(\beta(|v|/\|\nabla v\|_n)^{n/(n-1)}\big)\,\mathrm{d}\mathcal{H}^{n-1} \geq C(n)\varepsilon^{-T} - C(n), \tag{4.19}$$

where

$$T = (\beta - \alpha)(2/\omega_{n-1})^{1/(n-1)} - \delta'(\varepsilon),$$

and $\delta'(\varepsilon) \to 0$ when $\varepsilon \to 0$.

To prove Theorem 2, we consider the sequence $u_i\colon \bar{B}^n \to \mathbb{R}$,

$$u_i(x) = v_i(x + e_n) - v_i(e_n),$$

where $v_i(e_n) = -\log(1+\varepsilon_i) \geq -\log 2$ for all i. We fix M such that

$$\beta' = \beta\left(\frac{M - \log 2}{M}\right)^{n/(n-1)} > \alpha.$$

Set also $E_i = \{y \in S^{n-1}(e_n,1) : |v_i(y)| \geq M\}$. Then

$$\beta|v_i(y) - v_i(e_n)|^{n/(n-1)} \geq \beta'|v_i(y)|^{n/(n-1)}$$

on E_i for every i. Thus

$$\begin{aligned}
&\int_{S^{n-1}} \exp\big(\beta(|u_i|/\|\nabla u_i\|_n)^{n/(n-1)}\big)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad = \int_{S^{n-1}(e_n,1)} \exp\big(\beta(|v_i(y) - v_i(e_n)|/\|\nabla v_i\|_n)^{n/(n-1)}\big)\,\mathrm{d}\mathcal{H}^{n-1}(y) \\
&\quad \geq \int_{E_i} \exp\big(\beta'(|v_i(y)|/\|\nabla v_i\|_n)^{n/(n-1)}\big)\,\mathrm{d}\mathcal{H}^{n-1}(y) \\
&\quad \geq \int_{S^{n-1}(e_n,1)} \exp\big(\beta'(|v_i(y)|/\|\nabla v_i\|_n)^{n/(n-1)}\big)\,\mathrm{d}\mathcal{H}^{n-1}(y) \\
&\qquad\qquad - \omega_{n-1}\exp\big(\beta'(M/\|\nabla v_i\|_n)^{n/(n-1)}\big).
\end{aligned}$$

Since $\beta' > \alpha$ and $\varepsilon_i = i^{-1}$ in (4.19), the claim now follows from (4.19).

Acknowledgements. We thank the referee for valuable comments.

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press

References

[1] S.-Y. A. Chang and D. E. Marshall, *On a sharp inequality concerning the Dirichlet integral*, Amer. J. Math., **107(5)** (1985), 1015–1033.

[2] L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[3] B. Fuglede, *Extremal length and functional completion*, Acta Math., **98** (1957), 171–219.

[4] F. W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc., **101** (1961), 499–519.

[5] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc., **103** (1962), 353–393.

[6] J. Heinonen, T. Kilpeläinen, and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1993.

[7] J. Malý, D. Swanson, and W. P. Ziemer, *The co-area formula for Sobolev mappings*, Trans. Amer. Math. Soc., **355(2)** (2003), 477–492 (electronic).

[8] J. J. Manfredi, *Weakly monotone functions*, J. Geom. Anal., **4(3)** (1994), 393–402.

[9] D. E. Marshall, *A new proof of a sharp inequality concerning the Dirichlet integral*, Ark. Mat., **27(1)** (1989), 131–137.

[10] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J., **20**, (1970/71), 1077–1092.

[11] G. D. Mostow, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math., **(34)** (1968), 53–104.

[12] P. Poggi-Corradini and K. Rajala, *An egg-yolk principle and exponential integrability for quasiregular mappings*, J. London Math. Soc. (2), **76(2)** (2007), 531–544.

Pekka Pankka
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
USA

pankka@umich.edu

Pietro Poggi-Corradini
Department of Mathematics
Cardwell Hall
Kansas State University
Manhattan, KS 66506
USA

pietro@math.ksu.edu

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press

Kai Rajala
Department of Mathematics and Statistics
P. O. Box 35 (MaD), FI-40014 Univ. of Jyväskylä
Finland
kirajala@maths.jyu.fi

https://doi.org/10.1017/S0027763000026015 Published online by Cambridge University Press