This is a ``preproof'' accepted article for *Canadian Mathematical Bulletin*
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DOI: 10.4153/S000843952400095X

A NOTE ON HAYMAN'S PROBLEM

JIAXING HUANG [1] AND YUEFEI WANG [1,2]

Abstract. In this note, it is shown that the differential polynomial of the form $Q(f)^{(k)}-p$ has infinitely many zeros and particularly $Q(f)^{(k)}$ has infinitely many fixed points for any positive integer k, where f is a transcendental meromorphic function, p is a nonzero polynomial and Q is a polynomial with coefficients in the field of small functions of f. The results are traced back to Problem 1.19 and Problem 1.20 in the book of research problems by Hayman and Lingham [*Research Problems in Function Theory, Springer, 2019*]. As a consequence, we give an affirmative answer to an extended problem on the zero distribution of $(f^n)'-p$, proposed by Chiang and considered by Bergweiler [*Bull. Hong Kong Math. Soc.* **1**(1997), p. 97–101].

1. Introduction and main results

In this paper, we focus on the zero distributions of differential polynomials in a meromorphic function f with small meromorphic coefficients. We assume that the reader is familiar with the standard notations and some basic results in Nevanlinna theory (see [16, 19]).

In 1959, Hayman [15, 17] conjectured that if f is a transcendental meromorphic function and $n \geq 2$ is an integer, then $(f^n)'$ assumes every nonzero complex number infinitely often. He proved this conjecture for $n \geq 4$. The case for $n=3$ was settled by Mues [23] in 1979, and the remaining for $n=2$ was obtained by Bergweiler and Eremenko [6], Chen and Fang [9] and Pang and Zalcman [26]. One principal extension was studied in [6], the authors showed the Hayman conjecture is valid if $(f^n)'$ is replaced by $(f^n)^{(k)}$ for $n>k\geq 1$.

Two related questions arising in connection with Hayman's problem are as follows.

The first one is to consider the zero distribution of $(f^n)'-p$ where p is a small function of f. In fact, this extension originates from the study of the zero distribution of $ff'-p$, proposed by Chiang in 1994 (see [4]). Bergweiler [4] gave a positive answer when p is a nonzero polynomial and f is of finite order. This result (and a more general

2000 *Mathematics Subject Classification.* 30D30, 30D20, 30D35, 30D35, 34K12.
Key words and phrases. Hayman's problem, differential polynomials, meromorphic functions, zero distributions.
The authors supported by NSF of China (No.12231013, 12201420) and NCAMS.
Corresponding author: Yuefei Wang.

form $f'f^n - R$, where R is a rational function) was completely solved by Bergweiler and Pang (see [7, Theorem 1.1]) from the perspectives of normal families and dynamic arguments [6].

The second question given by Eremenko and Langley [11] is whether one can consider a more general differential polynomial of f such as a linear differential polynomial

$$F := f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f$$

with suitable small function coefficients a_j instead of $f^{(k)}$ only.

All known proofs of generalizations of Hayman's conjecture rely on the method of either the dynamic argument in [6], or a much later result of Yamanoi [29]. Indeed, the deep and highly technical result of Yamanoi is crucial to the resolution of some difficult conjectures in value distribution theory, for example, the Gol'dberg conjecture [14, Page 456 (B.4)] and Mues' conjecture [24]. Very recently, applying Yamanoi's result ([29, Theorem 1.2]), An and Phuong [2] investigated this question for a differential polynomial $Q(f)$ with some restrictive conditions.

Theorem 1 ([2], Theorem 1)**.** *Let f be a transcendental meromorphic function, and $Q(z) = b(z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_l)^{m_l}$ be a polynomial of degree q, where $b \in \mathbb{C}^*$ and $a_1, \ldots, a_l \in \mathbb{C}$. If $q \geq l+1$, then $(Q(f))^{(k)}$ takes every finite nonzero value infinitely often, for any positive integer k.*

In 2013, Fang and Wang [13] relied on the Yamanoi result to derive some new consequences, one thing they obtained is that the pole order of a transcendental meromorphic function f can be controlled by the zero order of $f^{(k)} - p$ (see Lemma 2 below), where p is a polynomial.

In this paper, by Fang and Wang's result (an outgrowth of generalizations of Yamanoi's result), we generalize An and Phuong's result. In fact, we consider more general situations in which the coefficients of the polynomial Q are allowed to be functions meromorphic in $\mathbb{C}$, and the nonzero value is replaced by a polynomial. Our main result is the following.

Theorem 2. *Let f be a transcendental meromorphic function, p be a nonzero polynomial and $Q(z) = b(z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_l)^{m_l}$ be a polynomial of degree q, where $b \not\equiv 0, a_1, \ldots, a_l$ are small functions of f. If $q \geq l+1$, then $(Q(f))^{(k)} - p$ has infinitely many zeros, and particularly $(Q(f))^{(k)}$ has infinitely many fixed points, for any positive integer k.*

Remark 1. *Recently, based on Yamanoi's result* [29, Theorem 1.2]*, Fang et al.* [12] *announced that Theorem 1 is still true if the condition $q \geq l+1$ is replaced by $q \geq 2$. Hence, it is natural to ask if the above theorem also works under such a condition $q \geq 2$. In general, this question is not true, the condition on $q \geq l+1$ is necessary. For* *example, let* $f = \exp(z)$*,* $Q(w) = (w - z)(w + z)$ *and* $p = -2z$*, in this case, we have*

$$Q(f)' - p = (f^2 - z^2)' + 2z = 2ff' = 2\exp(2z)$$

which has no zeros.

As a consequence of Theorem 2, we also give an affirmative answer to the question of Chiang when p is a polynomial, without any growth restriction on f.

Corollary 1 (Hayman's problem for polynomials)**.** *If* f *is a transcendental meromorphic function and* p *is a nonzero polynomial, then* $f^n f' - p$ *has infinitely many zeros, for any positive integer* n*.*

The corollary follows immediately if one takes $k = 1$, and $Q(z) = z^{n+1}$. Moreover, when $Q(z) = z^n$, $n \geq 2$, one can extend a result of Hayman in [15, Theorem 2]. Note that Hayman considered the value distribution of $(f^n)^{(k)} - c$ for any nonzero complex number c, while we can take c to be a polynomial.

Corollary 2. *Let* f *be a transcendental meromorphic function and* p *be a nonzero polynomial. Then for* $n \geq 2, k \in \mathbb{N}$*,* $(f^n)^{(k)} - p$ *has infinitely many zeros.*

Remark 2. *Corollaries 1 and 2 are contained in the results of the paper by Bergweiler and Pang (see* [7, Theorem 1.1]*) who used the method of normal families and dynamic arguments, while we obtain these results from the other viewpoint mentioned before, which is an implication of Yamanoi's result.*

As a consequence, we also obtain the following corollary, which could be regarded as a precursor of Theorem 3.

Corollary 3. *Let* f *be a transcendental meromorphic function and* P *be a nonzero polynomial, then* $f' - Pf^n$ *has infinitely many zeros for any* $n \geq 3$*.*

Proof. Let $f = 1/g$, then

$$Pf^n - f' = \frac{P}{g^n} + \frac{g'}{g^2} = \frac{P + g'g^{n-2}}{g^n}.$$

Hence, the result follows from Corollary 1. □

Remark 3. *Indeed, if* P *is a nonzero constant, Corollary 3 was proved by Pang for* $n \geq 4$ *and by Chen and Fang* [9, Theorem 3] *for* $n = 3$ *from the point of view of the normal family.*

In view of the above results, we only consider the polynomial case, it is natural to ask what happens if one replaces the polynomial p in the above results with some small functions. This question is in general not easy to answer. However, using some ideas from Liao and Ye [22], the classical Logarithmic Derivative Lemma, and the Clunie Lemma, we give some partial results as follows.

To describe our result, we need to introduce some classes of meromorphic functions.

Let $T(r,f)$ be the Nevanlinna characteristic function of f. We denote by $S(r,f)$ any quantity which is of growth $o(T(r,f))$ as $r\to\infty$ outside a set $E\subset(0,\infty)$ of finite measure. A meromorphic function y is called a *small function* of f if it satisfies that $T(r,y)=S(r,f)$. The family of small functions of f is defined by $\mathcal{S}_f$. By $\mathcal{N}_0$ and $\mathcal{S}_0$ we mean that the family of meromorphic functions y with finitely many poles and $\mathcal{S}_0=\mathcal{S}_f\cap\mathcal{N}_0$, respectively. Clearly, the field $\mathbb{C}(z)$ of rational functions and the ring of entire functions are contained in $\mathcal{N}_0$. We say a differential polynomial in w is non-degenerate if it is not a polynomial in w.

Theorem 3. *Let*

$$P(z,w)=\sum_{j=m}^{n} b_j(z)w^j,\quad b_m\not\equiv 0$$

be a polynomial in w with coefficients $b_j(z)$ in the family $\mathcal{S}_0$, and

$$L(z,w)=\sum_{|I|=0}^{k} a_I(z)w^{i_0}w'^{i_1}\cdots(w^{(q)})^{i_q}$$

be a non-degenerate differential polynomial in w over $\mathcal{S}_0$, where $I=(i_0,\ldots,i_q)$ is a multi-index with length $|I|=i_0+\cdots+i_q$. If f is a transcendental meromorphic function in $\mathcal{N}_0$ such that $L(z,f)\not\equiv 0$, then $P(z,f)+L(z,f)$ has infinitely many zeros for any $m\geq k+2$.

Remark 4. *In general, a meromorphic function that does not satisfy any nontrivial algebraic differential equation $L(z,w)=0$ is said to be hypertranscendental. Well-known examples of such meromorphic functions are the Euler gamma function and the Riemann zeta function. The study of hypertranscendental functions can be found in [1, 3, 18].*

In the special case that $P(z,w)=Q_1(z)w^n$ and $L(z,w)=Q_2(z)w^{(q)}-R(z)$, where $n,q\in\mathbb{N}$, $Q_1(z)$ and $Q_2(z)$ are nonzero rational functions, and $R(z)$ is a rational function, we obtain the following which is a generalization of [15, Theorem 8] and also partially answer a question proposed in [9, Proposition 1]:

Corollary 4. *Let f be a transcendental entire function and let R be a rational function. If $n\geq 3$, then $Q_1f^n+Q_2f^{(q)}-R$ has infinitely many zeros, and hence $Q_1f^n-Q_2f^{(q)}$ has infinitely many fixed points for all $q\in\mathbb{N}$.*

1.2 Organization of the paper. The rest of the paper is organized as follows. In Sect. 2 we state several results that will be used in our proofs. Then we prove our main results (Theorems 2 and 3) in Sect. 3.

2. Some Lemmas

We first recall some useful lemmas.

Lemma 1 ([19], Theorem 2.2.5). *Let f be a meromorphic function in $\mathbb{C}$ and $a \in \mathcal{S}_f$, then*

$$T(r, a, f) = T(r, f) + S(r, f)$$

and

$$T(r, af) = T(r, f) + S(r, f), \quad a \neq 0.$$

Lemma 2 ([13], Proposition 3). *Let g be a transcendental meromorphic function in $\mathbb{C}$, k be a positive integer, and $p(\not\equiv 0)$ be a polynomial. Then for any $\epsilon > 0$,*

$$(1-\epsilon)T(r, g)+(k-1)\overline{N}(r, g) \leq \overline{N}(r, \frac{1}{g})+N(r, \frac{1}{g^{(k)} - p})+S(r, f). \quad (2.1)$$

Lemma 3 ([19], Theorem 2.3.3). *Let f be a transcendental meromorphic function and $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

We also need the following Clunie lemma which plays an essential role in the proof of main results.

Lemma 4 ([16], Lemma 3.3). *Let f be a transcendental meromorphic function in the complex plane such that*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\lambda | \lambda \in \Lambda\}$, such that $T(r, a_\lambda) = S(r, f)$ for all $\lambda \in \Lambda$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivative is at most n, then

$$m(r, P(z, f)) = S(r, f) \quad \text{as} \quad r \to \infty.$$

3. Proof of Theorems 2 and 3

Proof of Theorem 2. Applying Lemma 2 to $g = Q(f)$, we have

$$\begin{aligned}
& (1-\epsilon)T(r,Q(f)) + (k-1)\overline{N}(r,Q(f)) \\
\leq\ & \overline{N}\left(r,\frac{1}{Q(f)}\right) + N\left(r,\frac{1}{Q(f)^{(k)}-p}\right) + S(r,f) \\
\leq\ & \sum_{i=1}^{l}\overline{N}\left(r,\frac{1}{f-a_i}\right) + \overline{N}\left(r,\frac{1}{b}\right) + N\left(r,\frac{1}{Q(f)^{(k)}-p}\right) + S(r,f) \\
\leq\ & lT(r,f) + N\left(r,\frac{1}{Q(f)^{(k)}-p}\right) + S(r,f),
\end{aligned}$$

and then from the Mohon'ko result [19, Theorem 2.2.5], it follows that

$$(q-l-\epsilon)T(r,f) + (k-1)\overline{N}(r,f) \leq N\left(r,\frac{1}{Q(f)^{(k)}-p}\right) + S(r,f)$$

for any $\epsilon > 0$. Therefore, $Q(f)^{(k)} = p$ has infinitely many solutions when $q \geq l+1$, which means that $Q(f)^{(k)} - p$ has infinitely many zeros. □

Proof of Theorem 3. Let $P(f) = P(z,f)$ and $L(f) = L(z,f)$. Suppose that $P(f)+L(f)$ takes zero finitely many. As f has finitely many poles and the coefficients of P and L are in $\mathcal{S}_0$, it follows that

$$\sum_{j=m}^{n} b_j f^j + L(f) = Ae^h \tag{3.1}$$

where A is a nonzero rational function and h is an entire function with $T(r,h) = S(r,f)$. By differentiating both sides of (3.1), we obtain that

$$\sum_{j=m}^{n} B_j f^{j-1} + L(f)' = A^* e^h \tag{3.2}$$

where $B_j = b_j' f + j b_j f'$ is a linear differential polynomial of f over $\mathcal{S}_0$ and $A^* = A' + Ah'$ with $T(r,A^*) = S(r,f)$. It follows from (3.1) and (3.2) that

$$f^{m-1}H(z,f) = Q(z,f) \tag{3.3}$$

where

$$H(z,f) = \sum_{j=m}^{n}(AB_j f^{j-m} - A^* b_j f^{j-m+1})$$

is a differential polynomial in f with coefficients in $\mathcal{S}_f$ and

$$Q(z,f) = A^2\left(\frac{-L(f)}{A}\right)' + Ah'L(f) \tag{3.4}$$

is also an $\mathcal{S}_f$–differential polynomial with total degree at most $k(< m-1)$. We claim that $H(z,f) \not\equiv 0$. Otherwise, in view of (3.3), (3.4) and $L(f) \not\equiv 0$, one has $Q(z, f) \equiv 0$, and then $KL(f) = Ae^h$ with some constant K. Since f is a transcendental meromorphic function and $b_j \in \mathcal{S}_0$, (3.1) gives that $K \neq 1$ and

$$f^m \left(\sum_{j=m}^{n} b_j f^{j-m} \right) = (K-1)L(f)$$

with $\deg L = k < m$. By Clunie's lemma (Lemma 4), we have

$$m\left(r, \sum_{j=m}^{n} b_j f^{j-m}\right) = S(r, f)$$

Thus, by $N(r, f) = O(\log r)$ and $b_j \in \mathcal{S}_0$, we have

$$T\left(r, \sum_{j=m}^{n} b_j f^{j-m}\right) = S(r, f)$$

yielding a contradiction. Hence $H(z, f) \not\equiv 0$.

Applying Clunie's lemma (Lemma 4) to (3.3), we have

$$m(r, H(z, f)) = S(r, f).$$

As $A^*, A \in \mathcal{S}_f$, B_j are differential polynomials of f with coefficients in $\mathcal{S}_0$ and $b_j \in \mathcal{S}_0$, for $j = m, \dots, n$, it is not hard to see that

$$N(r, H(z, f)) = S(r, f) \quad \text{and hence} \quad T(r, H(z, f)) = S(r, f).$$

Therefore, by the Mohon'ko result [19, Theorem 2.2.5] and $T(r, H(z, f)) = S(r, f)$, we have

$$T(r, Q(z, f)) = T(r, H(z, f)f^{m-1}) = (m-1)T(r, f) + S(r, f).$$

From the form of $Q(z, f)$, one can rewrite it as follows

$$Q(z, f) = \sum_{l=0}^{k} C_l(z) f(z)^l$$

where $C_l(z)$'s are differential polynomials in f'/f and its derivatives with meromorphic coefficients in $\mathcal{S}_f$, and hence by the LDL, $m(r, C_l) = S(r, f)$ for all l . Since

$$\begin{aligned} m\left(r, \sum_{l=0}^{k} C_l f^l\right) &\leq m\left(r, f \sum_{l=1}^{k} C_l f^{l-1}\right) + m(r, C_0) + O(1) \\ &\leq m(r, f) + m\left(r, \sum_{l=1}^{k} C_l f^{l-1}\right) + m(r, C_0) + O(1), \end{aligned}$$

an immediate inductive argument implies that

$$m\left(r, \sum_{l=0}^{k} C_l f^l\right) \leq km(r, f) + \sum_{l=0}^{k} m(r, C_l) + S(r, f) = km(r, f) + S(r, f).$$

https://doi.org/10.4153/S000843952400095X Published online by Cambridge University Press

From the form of $Q(z,f)$ in (3.4) and $N(r,f)=O(\log r)$, it follows that

$$N(r,Q(z,f))=S(r,f),$$

thus one can obtain that

$$T(r,Q(z,f))=m(r,Q(z,f))+N(r,Q(z,f))\le km(r,f)+S(r,f)$$

and hence

$$(m-1)T(r,f)+S(r,f)\le km(r,f)+S(r,f)\le kT(r,f)+S(r,f),$$

which is impossible, as $m\ge k+2$. □

Acknowledgments

The authors would like to thank Fang Mingliang for introducing his recent related work and pointing out the counterexample in Remark 1. The authors also wish to express gratitude to the anonymous referees for a number of useful remarks that have been made for the improvement of this paper.

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[1]School of Mathematical Sciences, Shenzhen University, Guangdong, 518060, P. R. China

[2]Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, China

[†]Corresponding author
E-mail address: wangyf@math.ac.cn