ON FIXED AND PERIODIC POINTS UNDER CERTAIN SETS OF MAPPINGS

R. D. Holmes*

(received May 9, 1969)

1. Introduction. Let (X, d) be a metric space and f a mapping of X into itself. D. F. Bailey [1] considered a class of mappings f satisfying the condition: $\forall x, y \in X, x \neq y$,

(1.1) $$\exists n(x, y) \in I^+ \text{ such that } d(f^n(x), f^n(y)) < d(x, y),$$

where I^+ denotes the set of positive integers. For X compact and f continuous, he proved that such mappings possess a unique fixed point. In considering a local version (i.e. (1.1) holds if $0 < d(x, y) < \epsilon$) he showed that f has a finite, nonempty set of periodic points.

In [3], V. M. Sehgal considered the special case when (1.1) is replaced by: $\forall x \in X$,

(1.2) $$\exists n(x) \in I^+ \text{ such that } d(f^n(x), f^n(y)) \leq \lambda d(x, y), \quad \forall y \in X$$

where $0 \leq \lambda < 1$, and proved that, if X is complete and f continuous, f has a unique fixed point.

In the present paper we consider both semigroups of mappings and single mappings satisfying conditions closely related to those studied in [1] and [3], namely: if F is a commutative semigroup of continuous mappings $f: X \to X$: $\forall x \in X$

*Izaac Walter Killam Scholar at Dalhousie University

Canad. Math. Bull. vol. 12, no. 6, 1969

(1.3) $\exists n(x) \in I^+$ and $f_x \in F$ such that $\forall y \in X$ we have

$$d(f_x^n(x), f_x^n(y)) \le \lambda d(x, y);$$

and the more general: $\forall x, y \in X$,

(1.4) $\exists n(x, y) \in I^+$ and $f_{x,y} \in F$ for which

$$d(f_{x,y}^n(x), f_{x,y}^n(y)) \le \lambda d(x, y).$$

As no member of the semigroup need satisfy either (1.1) or (1.2) it is quite clear that an extra hypothesis must be introduced if we wish to insure the existence of a common fixed point. This is especially true in the case when the space X is not assumed to be compact. Such a condition (cf. Theorem 1) is given by considering the "orbit" $F[x]$ $(= \{ f(x) : f \in F\})$ of a point $x \in X$ and requiring

(1.5) $\exists x \in X$ for which $f_{F[x]} = \{ f_y : y \in F[x]\}$ is finite.

In the case of a single mapping f (Section 3), we consider the stronger condition: $\forall x, y \in X$,

(1.6) $\exists N(x, y) \in I^+$ such that $d(f^{N+t}(x), f^{N+t}(y)) \le \lambda d(x, y)$,

$t = 0, 1, 2, \cdots$

to obtain results similar to [1] in non-compact spaces.

I should like to thank Professor M. Edelstein for his helpful advice in this research.

2. In this section let F (as above) denote a commutative semigroup of continuous self mappings of the space (X, d). The major result of this section is the following.

THEOREM 1. If (X, d) is complete and F is such that conditions (1.3) and (1.5) are satisfied, then there is a unique $z \in X$ such that $f(z) = z$ for all $f \in F$. Moreover, there is a sequence of functions $g_n \in F$ such that $g_n(y)$ converges to z for every $y \in X$.

https://doi.org/10.4153/CMB-1969-106-1 Published online by Cambridge University Press

COROLLARY. If (X,d) is complete and F is such that (1.3) is satisfied and either:

(i) X is bounded, or

(ii) f_X is finite

then the conclusions of Theorem 1 follow.

Remark 1. The Theorem of [3] is a special case of this Corollary for in this case f_X is the single mapping f and (ii) applies.

Remark 2. Examples are easily constructed in which no member of F satisfies e.g. (1.1) or (1.2). Thus, the consideration of families leads to strictly more general results.

Proof of Theorem 1 and Corollary. In order to simplify the notation in the proof we will denote the n'th iterate f_x^n of f_x by $f[n;x]$.

Let x_1 be the point whose existence is guaranteed by (1.5), and set $n_1 = n(x_1)$, $f_1 = f_{x_1}$ and in general $n_r = n(f_{r-1}^{n_{r-1}} f_{r-2}^{n_{r-2}} \dots f_1^{n_1}(x_1))$ and $f_r = f[1; f_{r-1}^{n_{r-1}} \dots f_1^{n_1}(x_1)]$, $r = 2, 3, \dots$. Denote $f_{r-1}^{n_{r-1}} \dots f_1^{n_1}$ by g_r, and $g_r(x_1)$ by x_r. Then, by (1.3), we have, for all y,

$$(2.1) \qquad d(x_r, g_r(y)) \le \lambda d(x_{r-1}, g_{r-1}(y)) \le \cdots \le \lambda^{r-1} d(x_1, y),$$

and in particular, for $y = f_r^{n_r}(x_1)$, we have

$$d(x_r, x_{r+1}) \le \lambda^{r-1} d(x_1, f_r^{n_r}(x_1)).$$

Now, if the $d(x_1, f_r^{n_r}(x_1))$, $r = 1, 2, \cdots$, were bounded (as they are in the corollary (i)), the sequence $\{x_r\}$ would be a Cauchy sequence and would thus converge to some $z \in X$. But, for any r and $h \in F$, we would have, by (2.1),

$$d(x_r, h(x_r)) = d(x_r, g_r(h(x_1))) \leq \lambda^{r-1} d(x_1, h(x_1))$$

and, letting $r \to \infty$, we get $d(z, h(z)) = 0$, or $h(z) = z$.

If we also have $h(w) = w$ for all $h \in F$, then

$$d(z, w) = d(f_z^{n(z)}(z), f_z^{n(z)}(w)) \leq \lambda d(z, w).$$

Hence $d(z, w) = 0$ and z is unique. Finally, letting $r \to \infty$ in (2.1), we have $\{g_r(y)\} \to z$ for all $y \in X$.

To show that the $d(x_1, f_r^{n_r}(x_1))$ are bounded, let $h_1, h_2, \cdots, h_k$ be the set (finite by (1.5), or by (ii) in the Corollary) of distinct f_r's. Let the first occurence of h_i be as f_{r_i} and set

$$B = \text{maximum } \{d(x_1, x_{r_i+1}) : i = 1, 2, \cdots, k\}$$

$$D = \text{maximum } \{d(x_{r_i}, x_{r_i+1}) : i = 1, 2, \cdots, k\} \quad \text{and}$$

$$C = \max\{\max\{d(x_{r_i}, f[j; r_i](x_1)) : i = 0, 1, \cdots, n_{r_i}\} : i = 1, 2, \cdots, k\}.$$

Now, consider $d(x_1, f_r^{n_r}(x_1))$. For some j, $f_r = h_j$ and we can set $n_r = sn_{r_j} + t$ with $0 \leq t \leq n_{r_j}$ and we have

$$d(x_1, f_r^{n_r}(x_1)) \leq d(x_1, x_{r_j+1}) + d(x_{r_j+1}, f_r^{n_r}(x_1))$$

$$\leq B + d(h[n_{r_j};j](x_{r_j}), h[sn_{r_j} + t;j](x_1))$$

$$\leq B + \lambda d(x_{r_j}, h[(s-1)n_{r_j} + t;j](x_1)) \quad \text{by (1.3)}$$

$$\leq B + \lambda d(x_{r_j}, x_{r_j+1}) + \lambda d(x_{r_j+1}, h[(s-1)n_{r_j} + t;j](x_1))$$

$$\leq B + \lambda D + \lambda^2 d(x_{r_j}, h[(s-2)n_{r_j} + t;j](x_1))$$

$$\leq \cdots$$

$$\leq B + (\lambda + \lambda^2 + \cdots + \lambda^{s-1}) D + \lambda^s d(x_{r_j}, h[t;j](x_1))$$

$$\leq B + \frac{\lambda}{1-\lambda} D + C.$$

Hence the $d(x_1, f_r^{n_r}(x_1))$ are bounded and the theorem is proven.

Locally, we have the following:

THEOREM 2. *Let* (X, d) *be compact and suppose that the following local version of* (1.4) *is satisfied:*

(2.2) $\forall x, y \in X$, $d(x, y) < \epsilon$ *imply that* $\exists n(x, y) \in I^+$ *and an* $f_{x,y} \in F$ *for which* $d(f^n(x), f^n(y)) \leq \lambda d(x, y)$.

Then, each finite collection $\{f_1, f_2, \cdots, f_r\} \subseteq F$ *has at least one common periodic point.*

Proof. For a fixed x and y, $d(x, y) < \epsilon$, define $n_1 = n(x, y)$, $f_1^{n_1} = f_{x,y}^{n_1}$, $f_r^{n_r} = f_{g_r(x), g_r(y)}^{n_r}$, $g_r = f_{r-1} f_{r-2} \cdots f_1$ and

$n_r = n(g_r(x), g_r(y))$, $r = 2, 3, \dots$. Then

$$(2.3) \qquad d(g_r(x), g_r(y)) = d(f_{r-1}(g_{r-1}(x)), f_{r-1}(g_{r-1}(y)))$$
$$\le \lambda d(g_{r-1}(x), g_{r-1}(y)) \le \cdots$$
$$\le \lambda^{r-1} d(x, y).$$

By compactness, there is a subsequence $\{g_{r_i}(x)\}$ of $\{g_r(x)\}$ which converges to some point $z \in X$. By (2.3), $\{g_{r_i}(y)\}$ also converges to z. Note that g_r and the sequence $\{r_i\}$ depend on both x and y.

Next, we show that each $h \in F$ has periodic points. Let $p \in X$, $h \in F$. Then, by compactness, $\exists s, t \in I^+$ such that $d(h^s(p), h^{s+t}(p)) < \epsilon$. Let $x = h^s(p)$ and $y = h^{s+t}(p) = h^t(x)$ and apply the preceding paragraph. Hence $\exists$ a sequence $\{g_u\}$ and a $z \in X$ such that $\lim_{u\to\infty} g_u(x) = z$ and $z = \lim_{u\to\infty} g_u(y) = \lim_{u\to\infty} g_u h^t(x) = h^t(\lim_{u\to\infty} g_u(x)) = h^t(z)$ and z is a period point of h.

Now, to proceed by induction, suppose $h_1, h_2, \dots, h_n$ have a common period point w, and let h be an arbitrary element of F. Then, by the preceding, there is a $\{g_u\}$, a z, and an $s \in I^+$, such that $\lim_{u\to\infty} g_u h^s(w) = z$ and z is a periodic point of h. If $h_j^t(w) = w$, then

$$h_j^t(z) = h_j^t(\lim_{u\to\infty} g_u h^s(w)) = \lim_{u\to\infty} g_u h^s h_j^t(w)$$
$$= \lim_{u\to\infty} g_u h^s(w) = z$$

and z is a common periodic point of $h, h_1, \dots, h_n$.

https://doi.org/10.4153/CMB-1969-106-1 Published online by Cambridge University Press

COROLLARY. *Any continuous mapping of a compact metric space into itself, which commutes with an* ϵ*-local contraction, has periodic points.*

Remark. Theorem 2 generalizes [1, Corollary 2 of Theorem 2].

Example. Let X be the interval $[-1, 1]$ with the usual metric and define

$$f(x) = \begin{cases} \frac{1+x}{2} & \text{if } x \geqq 0, \\ \frac{1+x-2x^2}{2} & \text{if } -\frac{1}{2} \leqq x \leqq 0, \\ 2x+1 & \text{if } -1 \leqq x \leqq -\frac{1}{2}. \end{cases}$$

and set $g = f^{-1}$. f and g are clearly continuous and commuting. If $\lambda > \frac{1}{2}$ and $\epsilon \leqq \lambda - \frac{1}{2}$ then (2.2) is satisfied by f for $x \geqq 0$ and by g for $x < 0$. f and g have common periodic points 1 and -1 although neither is locally contractive.

3. In this section we consider a more restrictive contraction condition (1.6) on a single mapping. This allows for some relaxation of the conditions on the space.

THEOREM 3. *Let* (X,d) *be a metric space and* $f: X \to X$ *a continuous mapping such that* (1.6) *is satisfied and*

(3.1) $\exists x \in X$ *such that* $\{f^n(x)\}$ *contains a subsequence* $\{f^{n_i}(x)\}$ *converging to some point* $z \in X$.

Then z *is the unique fixed point and, for all* $y \in X$, *the sequence* $\{f^n(y)\}$ *converges to* z.

Proof. Let x be as in (3.1) and set $N_1 = N(x, f(x))$, and $N_{k+1} = N_k + N(f^{N_k}(x), f^{N_k+1}(x))$, $k = 1, 2, 3, \ldots$. Thus, $d(f^{N_k+t}(x), f^{N_k+t+1}(x)) \leq \lambda^k d(x, f(x))$. Let i_r be the smallest integer such that $n_{i_r} \geq N_r$ and $i_r > i_{r-1}$. Then $f^{n_{i_r}}(x) \to z$ and

https://doi.org/10.4153/CMB-1969-106-1 Published online by Cambridge University Press

$d(f^{n_{i_r}}(x), f^{n_{i_r}+1}(x)) \leqq \lambda^r d(x, f(x))$. Hence

$$f(z) = \lim_{r\to\infty} f^{n_{i_r}+1}(x) = \lim_{r\to\infty} f^{n_{i_r}}(x) = z$$

and z is fixed. If $y \in X$ is arbitrary, then

$$d(f^{N(y,z)+t}(y), z) = d(f^{N(y,z)+t}(y), f^{N(y,z)+t}(z)) \leqq \lambda d(y, z).$$

Repeating this argument with $f^{N(y,z)}(y)$ in place of y and continuing inductively, we get that $\{f^n(y)\}$ converges to z.

If we localize condition (1.6), we can conclude that f must at least have periodic points.

THEOREM 4. Let (X, d) be a metric space and $f: X \to X$ a continuous mapping such that (3.1) holds and

(3.2) $\exists \epsilon > 0, \lambda, 0 \leq \lambda < 1,$ such that $d(x, y) < \epsilon$ implies that $\exists N(x, y)$ for which $d(f^{N+t}(x), f^{N+t}(y)) \leqq \lambda d(x, y)$, $t = 0, 1, \cdots$

then z (of (3.1)) is a periodic point.

Proof. By (3.1), there is a point x with $\lim_{i\to\infty} f^{m_i}(x) = z$ and $d(x, f^{m_1}(x)) < \epsilon$. Continuing as in Theorem 3, set $N_1 = N(x, f^{m_1}(x))$ and $N_k = N(f^{N_{k-1}}(x), f^{N_{k-1}+m_1}(x)) + N_{k-1}$, $k = 2, 3, \cdots$. Thus

$$d(f^{N_k+t}(x), f^{N_k+t+m_1}(x)) \leqq \lambda^k d(x, f^{m_1}(x)).$$

Let i_r be the smallest integer such that $m_{i_r} \geqq N_r$ and $i_r > i_{r-1}$. Then $\{f^{m_{i_r}}(x)\} \to z$, and, as

https://doi.org/10.4153/CMB-1969-106-1 Published online by Cambridge University Press

$d(f^{m_{i_r}}(x), f^{m_{i_r}+m_1}(x)) \leqq \lambda^r d(x, f^{m_1}(x))$, we have

$$f^{m_1}(z) = \lim_{r\to\infty} f^{m_{i_r}+m_1}(x) = \lim_{r\to\infty} f^{m_{i_r}}(x) = z$$

and z is periodic.

Finally, to ensure a fixed point in the local case, it is sufficient to assume that X is ϵ-chainable.

THEOREM 5. _Let_ X _and_ f _be as in Theorem 4 and in addition suppose that_ X _is_ ϵ_-chainable. Then_ z _is a unique fixed point and, for every_ $x \in X$, _the sequence_ $\{f^n(x)\}$ _converges to_ z.

Proof. Define a metric D on X by setting $D(x,y)$ equal to the infimum of the lengths of all ϵ-chains from x to y. This is easily shown to be a metric equivalent to d (cf. e.g. [2]).

Let $x, y \in X$ be artibrary, but fixed, and let $0 < \rho \leqq \frac{1-\lambda}{2} D(x,y)$. Now, there is an ϵ-chain $\{x = x_0, x_1, \cdots, x_k = y\}$ from x to y such that $\lambda D(x,y) + \rho \geqq \Sigma_{i=1}^{k} \lambda d(x_i, x_{i-1})$. For each i we have $d(x_i, x_{i-1}) < \epsilon$ and, thus, by (3.2), there is an N_i for which $d(f^{N_i+t}(x_i), f^{N_i+t}(x_{i-1})) \leqq \lambda d(x_i, x_{i-1}) < \epsilon$. Therefore, setting $N = \text{Max}\{N_i\}$ we have

$$\Sigma_{i=1}^{k} \lambda d(x_i, x_{i-1}) \geqq \Sigma_{i=1}^{k} d(f^{N+t}(x_i), f^{N+t}(x_{i-1}))$$

$$\geqq D(f^{N+t}(x), f^{N+t}(y))$$

and, setting $\tilde{\lambda} = \frac{1+\lambda}{2} < 1$,

$$\tilde{\lambda} D(x,y) = \lambda D(x,y) + \frac{1-\lambda}{2} D(x,y) \geqq \lambda D(x,y) + \rho \geqq D(f^{N+t}(x), f^{N+t}(y)).$$

As this construction can be carried out for all pairs x, y, $x \neq y$, (X, D), f, and $\tilde{\lambda}$ satisfy the conditions of Theorem 3, and the conclusions follow.

REFERENCES

1. D. F. Bailey, Some theorems on contractive mappings. J. Lond. Math. Soc. 41 (1966) 101-106.

2. P.R. Meyers, On the converse to the contraction mapping principle. (Ph.D. Thesis, U. of Maryland, 1966).

3. V.M. Sehgal, A fixed point theorem for local contraction mappings. Amer. Math. Soc. Notices 12 (1965) 461.

University of Alberta
Edmonton, Alberta

https://doi.org/10.4153/CMB-1969-106-1 Published online by Cambridge University Press