

ALGEBRAIC THREEFOLDS WITH TWO EXTREMAL MORPHISMS

ATANAS ILIEV

§0. Introduction

0.1. In [3] Mori gives a description of all extremal rays (extremal morphisms) arising on a smooth projective threefold with a numerically non-effective canonical bundle. Generally speaking, every smooth projective threefold V with a numerically non-effective canonical class K_V admits an extremal morphism $\pi : V \rightarrow Y$. The assumption that V admits a non-trivial pair of extremal morphisms

$$Y_1 \xleftarrow{\pi_1} V \xrightarrow{\pi_2} Y_2$$

imposes strong conditions on V . This is the essence of the Theorem 1.5 of the present work. In particular, we obtain a description of the threefolds which admit two biregular structures of conic bundles over non-singular surfaces $S_1 = Y_1$ and $S_2 = Y_2$. By the results of §3 the surfaces S_1 and S_2 must be either ruled surfaces with isomorphic basic curves, or $S_1 \simeq S_2 \simeq P^2$.

0.2. Remarks

0.2.1. In [5] E. Sato has obtained a description of the threefolds with two structures of P^1 -bundles; this description corresponds to the Case A.a of Theorem 1.5. The second basic result of [5] states that if $\dim V \geq 3$ and V admits two structures of projective space bundles over projective spaces $Y_1 = P^l$ and $Y_2 = P^m$, then: either V is a product $V = P^l \times P^m$, or $l = m$ and $V = P(T_\rho)$.

0.2.2. Every Fano threefold V with $\rho(V) \geq 2$ admits at least two extremal morphisms. However, in most of the cases V admits a ray of the type E_1 . Because of that, there are too many Fano threefolds with $\rho \geq 2$ in the list of Mori and Mukai in [4], in contrast to the list of Theorem 1.5 in which are classified only the strongly primitive ones.

Received July 26, 1989.

§1. Definitions and statement of the main theorem

1.1. Everywhere in the present article, we suppose that the threefold V is a smooth projective threefold over the field of complex numbers \mathbf{C} .

1.2. Definitions

1.2.1. $NV = \{1\text{-cycles on } V\} / \equiv \otimes \mathbf{R}$, where \equiv denotes the numerical equivalence of cycles. NV is a finite dimensional real vector space, which is dual to $NS(V) \otimes \mathbf{R}$, where $NS(V)$ is the Neron-Severi group of V .

1.2.2. $\rho(V) = \dim_{\mathbf{R}}(NV)$.

1.2.3. $\overline{NE}(V)$ is the closure of the convex cone $NE(V)$, generated by all the effective 1-cycles from NV (in the metrical topology of the vector space NV).

1.2.4. Let $Z \in \overline{NE}(V)$. The half-line $R = \mathbf{R}_+[z]$ is called an extremal ray, if: (a) $-K_V \cdot z > 0$; (b) for all $Z_1, Z_2 \in \overline{NE}(V)$, the assumption $Z_1 + Z_2 \in R$ implies $Z_1 \in R$ and $Z_2 \in R$, cf. [3].

1.2.5. Let R be an extremal ray on V . Then, there exists a unique, up to an isomorphism, morphism $\pi : V \rightarrow Y$ corresponding to R , such that: (a) $\pi_* \mathcal{O}_V = \mathcal{O}_Y$; (b) if $C \subset V$ is an irreducible curve, then $[C] \in R$ if and only if $\dim \pi(C) = 0$, cf. [3]. The morphism π is called a contraction of the extremal ray R , or an extremal morphism (corresponding to R).

1.3. Description of the extremal morphisms on V (cf. [3])

Let $\pi : V \rightarrow Y$ be an extremal morphism, and let $\rho(V) \geq 2$. Then π can be one of the following:

1.3.1. Type $E : \dim Y = 3$

The morphism π is a contraction of a divisor D on V , and π corresponds to one of the types E_1, E_2, E_3, E_4 , and E_5 . In the case E_1 the morphism π is a contraction of a ruled surface to a smooth curve, and the threefold Y is smooth. In the case E_2 the morphism π is a contraction of a divisor $D \simeq P^2$, with a normal bundle $\mathcal{O}_D(D) \simeq \mathcal{O}_{P^2}(-1)$, to a nonsingular point on Y . In the case E_3 the morphism π is a contraction of a quadric $D \simeq P^1 \times P^1$, with a normal bundle $\mathcal{O}(-1, -1)$, to an ordinary double point on Y . Moreover, the fibers $P^1 \times t$ and $s \times P^1$ are numerically equivalent on V , for $t, s \in P^1$. In the case E_4 the morphism π is a contraction of a quadratic cone $D \subset P^3$ to a double point on Y , and $\mathcal{O}_D(D) \simeq \mathcal{O}_D \otimes \mathcal{O}_{P^3}(-1)$. In the case E_5 the morphism π is a contraction of $D \simeq P^2$ to a quadruple point on Y , and $\mathcal{O}_D(D) \simeq \mathcal{O}_{P^2}(-2)$.

1.3.2. Type $C : \dim Y = 2$

The variety Y is a smooth projective surface, and π corresponds to one of the types C_1 or C_2 . In the case C_1 the morphism π defines a conic bundle $\pi : V \rightarrow Y$; in the case C_2 the morphism π defines a P^1 -bundle $\pi : V \rightarrow Y$.

1.3.3. Type D : $\dim Y = 1$

The variety Y is a smooth curve, $\rho(V) = \rho(Y) + 1 = 2$, and π corresponds to one of the types D_1 , D_2 , and D_3 . In the case D_1 the threefold V has a structure of a Del Pezzo bundle over the curve Y . In the case D_2 , V is isomorphic to a $P^1 \times P^1$ -bundle over the curve Y . In the case D_3 the threefold V is a P^2 -bundle over Y .

1.4. DEFINITION. *The threefold V is called strongly primitive if there are no extremal rays of type E_1 on V .*

1.5. The Main Theorem

THEOREM. *Let V be a (smooth, projective) strongly primitive threefold which admits two extremal morphisms $\pi_1 : V \rightarrow Y_1$ and $\pi_2 : V \rightarrow Y_2$. Then, the following cases are possible:*

Case 1. *The morphisms π_1 and π_2 correspond to the type C . Then $2 \leq \rho(V) \leq 3$ and:*

(1.A) *If $\rho(V) = 3$, then*

either: (A.a). $V \simeq S_1 \times_C S_2$, where S_1 and S_2 are ruled surfaces over a curve C ,

or: (A.b). V is a two-sheeted covering of $S_1 \times_C S_2$, where S_1 , S_2 , and C are as in (A.a).

(1.B) *If $\rho(V) = 2$, then V is a Fano threefold (see Corollary 2.6.2).*

Case 2. *The morphism π_1 corresponds to the type C , and the morphism π_2 corresponds to one of the types D or E . Then V is a Fano threefold (see Corollary 4.2, Corollaries 5.3 and 5.4).*

Case 3. *Let the extremal morphisms π_1, π_2, \dots on V be of the type E . Then the corresponding divisors D_1, D_2, \dots are mutually disjoint (see §7).*

Remark. The rest of the paper is devoted to the proof of Theorem 1.5. Especially, Case 1 is discussed in §2 and §3. It follows from the considerations in §3 that the double covering $\pi : V \rightarrow S_1 \times_C S_2$, in case (A.b), has the following properties:

Let \mathcal{E}_i be a normalized locally free sheaf of rank 2, over the base curve C , such that $S_i = P_C(\mathcal{E}_i)$, $i = 1, 2$ (see [1, ch. V, §2]). Let $e_i = -\deg(\det \mathcal{E}_i)$, let φ_i be the general fibre of $S_i \rightarrow C$, and let b_i be the section of S_i such that $\mathcal{L}(b_i) = \mathcal{O}_{P(\mathcal{E}_i)}(1)$, $i = 1, 2$. Let $p_i : S_1 \times_C S_2 \rightarrow S_i$ be the natural projections, and let $C_i = p_i^*(b_i)$, $F_i = p_i^*(\varphi_i)$, $i = 1, 2$. Then F_1 and F_2 are numerically equivalent, i.e. $F_1 \equiv F_2 \equiv F$ for some $F \in p_1^*(\text{Pic } S_1) \cap p_2^*(\text{Pic } S_2)$. The branch divisor $B \subset S_1 \times_C S_2$ of π is smooth, and B is numerically equivalent to $2.C_1 + 2.C_2 + 2q.F$ for some $q > 0$. Moreover, the threefold V is a standard conic bundle over S_i with a

discriminant curve $\Delta_i \equiv 4.b_i + (4q - 2e_i). \varphi_i$, where $\{i, j\} = \{1, 2\}$.

§2. The case (C, C)

2.1. Let π_1 and π_2 be of type C. Let $\pi_1 : V \rightarrow Y_1$ and $\pi_2 : V \rightarrow Y_2$ be the corresponding extremal morphisms. In particular, $S_1 = Y_1$ and $S_2 = Y_2$ are smooth surfaces (see 1.3.2). Denote by f_k the general fiber of the morphism π_k , $k = 1, 2$.

2.2. PROPOSITION. *If $\rho(V) \geq 3$, then $\rho(V) = 3$, and S_1 and S_2 are ruled surfaces.*

Proof.

2.2.1. Let H be a very ample divisor on S_2 , and let $C \in |H|$ be a smooth curve. Then $(\pi_2^* C, \pi_2^* C)_V = m.f_2$, where $m = (C, C)_{S_2} > 0$. Therefore, $\pi_2^* C \notin \pi_1^*(\text{Pic } S_1)$; hence, the mapping $\pi_1 : \pi_2^* C \rightarrow S_1$ is surjective. Since $\chi(\pi_2^* C) = -\infty$, then $\chi(S_1) = -\infty$ (here $\chi(X)$ is the Kodaira dimension of X). Similarly $\chi(S_2) = -\infty$. Consequently, there exist morphisms $h_k : S_k \rightarrow S_{k,o}$, where $S_{k,o}$ are ruled surfaces or P^2 . As $\rho(V) \geq 3$, then $\rho(S_k) \geq 2$.

Let, for example, $S_{1,o} = P^2$. Then the surface S_1 is rational, and the morphism $h_1 : S_1 \rightarrow S_{1,o} = P^2$ is non-trivial; in the opposite case $\rho(V) = \rho(P^2) + 1 = 2$, which contradicts the assumption $\rho(V) \geq 3$. Consequently, there exists a morphism $h'_1 : S_1 \rightarrow F_1$, such that $h_1 = h'_1.\sigma$, where $\sigma : F_1 \rightarrow P^2$ is a blowing-up of a point in P^2 . Therefore, we can always assume that $S_{1,o}$ and $S_{2,o}$ are ruled surfaces (rational or non-rational).

Let $S_{k,o} = P(\mathcal{E}_k)$, let $\mathcal{L}(b_{k,o}) = \mathcal{O}_{P(\mathcal{E}_k)}(1)$, and let $\varphi_{k,o}$ be the general fiber of $S_{k,o}$, $k = 1, 2$ (see the Remark after Theorem 1.5). Let

$$\text{Num } S_k = \mathbf{Z}b_k \oplus \mathbf{Z}\varphi_k \oplus \bigoplus_{i=1}^{m_k} \mathbf{Z}\varepsilon_{k,i},$$

where $\varepsilon_{k,i}$ are the exceptional curves of h_k , and b_k and φ_k are the preimages of $b_{k,o}$ and $\varphi_{k,o}$ on S_k , $k = 1, 2$. Let $m = \rho(V) - 1$. Obviously $\rho(S_k) = m = m_k + 2$, $k = 1, 2$.

Let $C_1 = \pi_1^* b_1$, $C_2 = \pi_1^* \varphi_1$, $C_{i+2} = \pi_1^* \varepsilon_{1,i}$, $D_1 = \pi_2^* b_2$, $D_2 = \pi_2^* \varphi_2$, $D_{i+2} = \pi_2^* \varepsilon_{2,i}$ ($i = 1, 2, \dots, m - 2$). If $\pi_k : V \rightarrow S_k$ is a conic bundle, then $\text{Pic } V = \pi_2^* \text{Pic } S_k + \mathbf{Z}K_V$; if $\pi_k : V \rightarrow S_k$ is a P^1 -bundle, then $\text{Pic } V = \pi_k^* \text{Pic } S_k + \mathbf{Z}L_k$, where L_k corresponds to a section of π_k . In both cases

$$2. \text{ Pic } V \subseteq \pi_k^* \text{Pic } S_k + \mathbf{Z}K_V,$$

i.e. the divisors D_i (resp. C_i) are linear combinations, with integer or half-integer

coefficients, of the divisors C_i (resp. D_i) and K_V (in the numerical sense). Therefore, there exists a system of equations of the form:

$$2.2.2. \quad \begin{aligned} C_i + \sum_j d_{ij} D_j &\equiv -d_i K_V \\ \sum_j c_{ij} C_j + D_i &\equiv -c_i K_V, \end{aligned}$$

where the numbers $2d_{ij}$, $2d_i$, $2c_{ij}$, and $2c_i$ are integer.

Let $D = (d_{ij})$, $C = (c_{ij})$, $d = (d_1, \dots, d_m)^t$, $c = (c_1, \dots, c_m)^t$, and let E be the unit matrix of rank m . By the adjunction formula $K_V \cdot f_k = -2$, $k = 1, 2$; and from 2.2.2 we conclude that $C_i \cdot f_2 = 2d_i$, $D_i \cdot f_1 = 2c_i$. The integers $2d_i$ and $2c_i$ are non-negative; they can be interpreted as follows:

If $d_i = 0$, then $C_i \in \pi_2^* \text{Pic } S_2$;

if $d_i > 0$, then d_i = the degree of the covering $\pi_2 : C_i \rightarrow S_2$;

(similarly – for c_i). Further, from 2.2.2 we derive:

$(-c_i + \sum_l c_{il} d_l) \cdot K_V \equiv D_i - \sum_{l,j} c_{il} d_{lj} D_j$, $i = 1, 2, \dots, m$. Therefore, from the formula connecting $\text{Pic } V$ and $\text{Pic } S_2$, we obtain that the both sides of the last equation are equal to zero, in the numerical sense. Hence, $C \cdot D = E$, and $C d = c$. These matrix equations will be used in the proof of Proposition 2.3.

2.2.3. Let $C_i C_j = \gamma_{ij} f_1$, $D_i D_j = \delta_{ij} f_2$, $k_{ij} = K_V C_i D_j$. After multiplying the first m equations from 2.2.2 by $C_i D_j$ we obtain the following system:

$$2.2.4. \quad R_{ijk} = 2d_i \sum_l d_{kl} \delta_{lj} + d_k k_{ij} + \gamma_{ki} \cdot 2c_j = 0.$$

By the choice of the curves b_k , φ_k , ε_{ki} , the numbers γ_{ki} and δ_{ij} satisfy the following conditions:

- 2.2.5. (a) $\gamma_{ii} = -p_i < 0$, $\delta_{ii} = -q_i < 0$, $i \geq 3$;
- (b) $\gamma_{1i} = \gamma_{2i} = \delta_{1i} = \delta_{2i} = 0$, $i \geq 3$;
- (c) $\gamma_{22} = \delta_{22} = 0$, $\gamma_{11} = -e_1$, $\delta_{11} = -e_2$,
 where $-e_k = (b_k, b_k)_{S_k} = (b_{k,o}, b_{k,o})_{S_{k,o}}$, $k = 1, 2$.

2.2.6. LEMMA. If $d_2 = 0$, then $d_3 = \dots = d_m = 0$ (similarly – for c_i).

Proof. Every $\varepsilon_{1,i}$ is a component of some degenerating fiber $\varphi_{1,i} \equiv \varphi_1$ of h_1 , $\varphi_{1,i}$ being a linear combination with integer coefficients of exceptional curves and the preimage of some fiber of $S_{1,o}$. Let, for example, $\varphi_1 \equiv \sum_{n \geq o} \lambda_n \varepsilon_{1,n}$, where $\lambda_n \geq 0$ and $\varepsilon_{1,o}$ is the proper preimage of some fiber of $S_{1,o}$, over which we take blowing-ups. Then

$$2.2.7. \quad 0 = 2d_2 = C_2 f_2 = \lambda_o \cdot \pi_1^* \varepsilon_{1,o} f_2 + \sum_{n \geq 1} \lambda_n \cdot 2d_{n+2}.$$

Hence, $2d_{i+2} = C_{i+2} f_2 = \pi_1^* \varepsilon_{1,i} f_2 = 0$.

2.2.8. LEMMA. *If $m = \rho(V) - 1 \geq 3$, then $\prod_{i=3}^m c_i d_i = 0$.*

Let us look at the equations $R_{13k} = 0, k \geq 2$, and $R_{23k} = 0, k \geq 2$ (see 2.2.4). We shall give a proof of 2.2.8 on an example, which is not different from the general case.

EXAMPLE. $m = 4$; i.e. from 2.2.5 we have $\delta_{33} \neq 0, \delta_{43} \neq 0, \delta_{13} = \delta_{23} = 0$. For definiteness, we may assume that $\delta_{33} = -2$ and $\delta_{43} = 1$; therefore $\delta_{44} = -1$. The surface S_2 is obtained from $S_{2,\rho}$ after blowing-up a point not lying on the base section, and a second blowing-up with a centre lying on the first exceptional divisor. The equations $R_{13k} = 0$ and $R_{23k} = 0, k \geq 2$, take the form:

$$\begin{aligned} 2.2.9. \quad R_{132} &= -2c_3 + (-2d_{23} + d_{24}).2d_1 = -d_2k_{13} \\ R_{13k} &= (-2d_{k3} + d_{k4}).2d_1 = -d_kk_{13}, \quad k = 3,4 \\ R_{23k} &= (-2d_{k3} + d_{k4}).2d_2 = -d_kk_{23}, \quad k = 2,3,4 \end{aligned}$$

From 2.2.9 we easily derive that *either $d_2 = 0$ (and hence, according to Lemma 2.2.6, $d_3 = \dots = d_m = 0$), or the assumption $d_3 \neq 0$ implies $c_3 = 0$.*

2.2.10. LEMMA. *If $\rho(V) \geq 3$, then $\rho(V) = 3$.*

Proof. According to Lemma 2.2.8, if $m \geq 3$, then $\prod_{i \geq 3} c_i d_i = 0$. Let, for example, $c_3 = 0$. Then $D_3 \in \pi_1^* \text{Pic } S_1$. Hence $D_3 = \pi_1^* C$, where $C \in \text{Pic } S_1$ and $(C, C)_{S_1} = r \in \mathbf{Z}$. Then $-q_3 f_2 = (D_3, D_3)_V = (\pi_1^* C, \pi_1^* C)_V = r f_1$, where $q_3 > 0$ (i.e. $q_3 \neq 0$) – a contradiction. Therefore $m = \rho(V) - 1 = 2$, and the Proposition 2.2 is proved.

2.3. PROPOSITION. *Let $\rho(V) = 3$. Then $\pi_1^* \varphi_1 \in \pi_2^* \text{Pic } S_2$ and $\pi_2^* \varphi_2 \in \pi_1^* \text{Pic } S_1$, where φ_k is the general fiber of the ruled surface $S_k, k = 1, 2$.*

Proof. For convenience, we shall change the notation. As $m = 2$, the system 2.2.2 takes the form:

$$\begin{aligned} 2.3.1. \quad -C_1 & & + g_1 C_2 + d_1 F_2 & \equiv r_1 K_V \\ & - F_1 + b_1 C_2 + a_1 F_2 & \equiv c_1 K_V \\ g_2 C_1 + d_2 F_1 - C_2 & & & \equiv r_2 K_V \\ b_2 C_1 + a_2 F_1 & & - F_2 & \equiv c_2 K_V, \end{aligned}$$

where $F_k = \pi_k^* \varphi_k, C_k = \pi_k^* b_k, k = 1, 2$, and all the coefficients are either integers, or half-integers.

The equality $C.D = E$ (see 2.2.2) implies $g_2 = \varepsilon a_1, d_2 = -\varepsilon d_1, b_2 = -\varepsilon b_1$,

and $a_2 = \varepsilon g_1$, where $\varepsilon = (g_1 a_1 - b_1 d_1)^{-1}$. From $Cd = c$ and $Dc = d$ (ibid.) we obtain:

$$\begin{aligned}
 2.3.2. \quad & c_2 + a_2 c_1 + b_2 r_1 = 0 \\
 & r_2 + d_2 c_1 + g_2 r_1 = 0 \\
 & c_1 + a_1 c_2 + b_1 r_2 = 0 \\
 & r_1 + d_1 c_2 + g_1 r_2 = 0.
 \end{aligned}$$

After multiplying both sides of the equalities 2.3.1 by f_1 and f_2 we obtain $F_1 f_2 = 2c_1$, $F_2 f_1 = 2c_2$, $C_1 f_2 = 2r_1$, and $C_2 f_1 = 2r_2$. The system 2.2.5 for γ_{ij} and δ_{ij} takes the form:

$$\begin{aligned}
 C_1^2 &= -e_1 f_1, \quad C_2^2 = -e_2 f_2, \quad C_1 F_1 = f_1, \\
 C_2 F_2 &= f_2, \quad F_1^2 = F_2^2 = 0.
 \end{aligned}$$

We divide the proof in several cases:

Case 1. S_1 and S_2 are rational.

CLAIM. *In Case 1, the equality $c_1 c_2 r_1 r_2 = 0$ is fulfilled.*

Proof of the Claim. Assume that $c_1 c_2 r_1 r_2 \neq 0$; and let $\varepsilon < 0$. From the equation $c_1 K_V F_1 C_2 = 2a_1 c_1 - 2b_1 e_2 c_1$, we get $K_V F_1 C_2 = 2a_1 - 2b_1 e_2$; therefore $c_2(2a_1 - 2b_1 e_2) = c_2 K_V F_1 C_2 = -2c_1 + 2b_2 r_2$. By 2.3.2, $2c_1 + 2a_1 c_2 = -2b_1 r_2$, where $b_2 = -\varepsilon b_1$. Hence:

$$2.3.3. \quad (2\varepsilon - 2) b_1 r_2 = 2b_1 e_2 c_2.$$

From $\varepsilon < 0$, $r_2 > 0$, $c_2 > 0$, and $e_2 \geq 0$, we get that $b_1 = 0$; in particular $b_2 = -\varepsilon b_1 = 0$. Thus, from $c_1 K_V F_1 F_2 = 2b_1 c_1$ and $c_1 > 0$, we obtain that $K_V F_1 F_2 = 0$. Then, from $r_2 K_V F_1 F_2 = 0$ and $r_1 K_V F_1 F_2 = 0$, we conclude that $-2c_1 + 2g_2 c_2 = 0$ and $-2c_2 + 2g_1 c_1 = 0$. Therefore $g_1 > 0$, $g_2 > 0$, and $g_1 \cdot g_2 = 1$.

From $K_V F_1 C_2 = 2a_1$, and from the equations of the type $R_{ijk} = 0$ for $r_2 K_V F_1 C_2$ we obtain:

$$2.3.4. \quad (a_1 - g_2) r_2 = 2e_2 c_1,$$

where $a_1 = \varepsilon g_1$, $g_2 = \varepsilon a_1$, $g_1 > 0$, $g_2 > 0$, and $\varepsilon < 0$. In that case, the equation 2.3.4 contradicts the assumption that $c_1 > 0$ and $r_2 > 0$.

Let $\varepsilon < 0$ and $c_1 c_2 r_1 r_2 = 0$. In particular, if $r_1 = 1$ then $2e_1 c_2 = r_1 K_V F_2 C_1 = 0$. Therefore, either $c_2 = 0$, or $e_1 = 0$. If $e_1 = 0$, then $S_1 \simeq P^1 \times P^1$, and we can assume that $b_1 \subset S_1$ is a fiber (cf. 2.2.1).

Let $c_1 = 0$, but $c_2 > 0$. Then 2.3.2 implies that $c_2 + b_2 r_1 = 0$, i.e. $b_2 < 0$. But,

from the equations $0 = c_1 K_V F_2 C_1 = -2c_2 + 2b_1 r_1$ and $b_2 = -\varepsilon b_1$, $\varepsilon < 0$, we obtain that $b_2 > 0$, which is impossible.

COROLLARY. *If $\varepsilon < 0$, and S_1 and S_2 are rational, then $c_1 = c_2 = 0$.*

If $\varepsilon > 0$, we come to a contradiction in the same way. The Claim is proved. Proceeding in a similar way, from the above Claim and from 2.3.2, we obtain the following

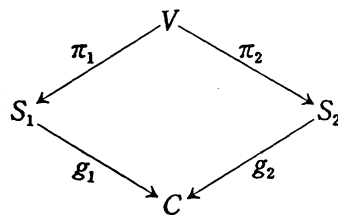
COROLLARY. *In the Case 1 we have $c_1 = c_2 = 0$.*

Then, from 2.3.1, we obtain that $F_1 \in \pi_2^ \text{Pic } S_2$ and $F_2 \in \pi_1^* \text{Pic } S_1$.*

Case 2. S_1 or S_2 is non-rational.

Let, for example, S_1 be an irregular ruled surface and let $g_1 : S_1 \rightarrow C$, $g_2 : S_2 \rightarrow C'$ be the corresponding representations of the surfaces S_1 and S_2 as P^1 -bundles over the curves C and C' , where $g(C) = g \geq 1$. Then the general fibers of $|L_1| = g_1 \pi_1 : V \rightarrow C$ and $|L_2| = g_2 \pi_2 : V \rightarrow C'$ are rational surfaces.

Let $|L_1| \neq |L_2|$. Then $f = L_2|_{L_1}$ is a curve on L_1 and $(f, f)_{L_1} = L_2 L_2 L_1 = 0$. Hence, the restriction $|L_2|_{L_1} : L_1 \rightarrow C$ defines, on the rational surface L_1 , a structure of bundle with rational curves as fibers and with a non-rational base C , which is impossible. Therefore $C' \simeq C$, and the diagram



where $g(C) = g \geq 1$, is commutative. Evidently, in this case $\pi_1^* \varphi_1 \in \pi_2^* \text{Pic } S_2$ and $\pi_2^* \varphi_2 \in \pi_1^* \text{Pic } S_1$. The Proposition 2.3 is proved.

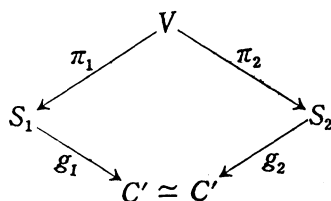
2.4. COROLLARY. *If $\rho(V) = 3$, then the equation 2.3.2 take the form:*

$$rK_V \equiv -C_1 - C_2 + dF,$$

where $F_1 \equiv F_2 \equiv F \in \pi_1^* \text{Pic } S_1 \cap \pi_2^* \text{Pic } S_2$, the numbers $2r$ and $2d$ are integer, and $r > 0$.

The Corollary is a direct consequence from Proposition 2.3, and from the first and the third equations of 2.3.1. Note that from the two other equations of 2.3.1 we obtain that $b_1 = b_2 = 0$ and $a_1 = a_2 = 1$. Then, the former two equations give $\varepsilon = -1$.

2.5. COROLLARY. *If $\rho(V) = 3$, then there exists a curve C such that the diagram*



is commutative.

Proof. For S_1 and S_2 – non-rational, the Corollary is proved in 2.3, Case 2.

Let S_1 and S_2 be rational ruled surfaces. By Proposition 2.3, we have $\pi_1^* \varphi_1 \in \pi_2^* \text{Pic } S_2$ and $\pi_2^* \varphi_2 \in \pi_1^* \text{Pic } S_1$. Consequently, there are correctly defined morphisms $\lambda = g_1 \pi_1 \pi_2^{-1} g_2^{-1} : C' \rightarrow C$ and $\lambda^{-1} = g_2 \pi_2 \pi_1^{-1} g_1^{-1} : C \rightarrow C'$, where $g_1 : S_1 \rightarrow C \simeq P^1$ and $g_2 : S_2 \rightarrow C' \simeq P^1$ define structures of ruled surfaces on S_1 and S_2 . Therefore $\lambda \in \text{Aut } P^1$; and if we replace g_2 by $\lambda \cdot g_2$, we shall obtain the commutative diagram from above.

2.6. Case $\rho(V) = 2$

Let us consider the case $\rho(V) < 3$. Then $\rho(V) = 2$, and there are on V two extremal rays R_1 and R_2 of type C . As $\rho(V) = \dim_{\mathbf{R}}(NV) = 2$, then R_1 and R_2 form a base of the two-dimensional real vector space NV . Let $R_1 = \mathbf{R}_+[l_1]$ and $R_2 = \mathbf{R}_+[l_2]$. Since R_1 and R_2 are extremal rays in the two-dimensional cone $\overline{NE}(V) \subset NV$, and since $K_V \cdot l_1 < 0$, $K_V \cdot l_2 < 0$, then $K_V \cdot Z < 0$ for any $Z \in \overline{NE}(V)$. By the Kleiman’s criterion we derive that $-K_V$ is ample, i.e. V is a Fano threefold.

2.6.1. COROLLARY. *If $\rho(V) = 2$ and (π_1, π_2) is of type (C, C) , then V is a Fano threefold.*

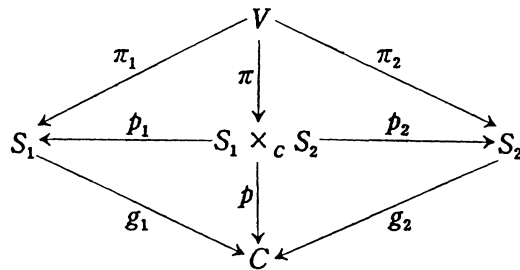
2.6.2. COROLLARY (see [4]). *In the conditions of 2.6.1, the threefold V is one of the following:*

- (1) a divisor of bidegree (2,2) in $P^2 \times P^2$;
- (2) a divisor of bidegree (2,1) in $P^2 \times P^2$;
- (3) a divisor of bidegree (1,1) in $P^2 \times P^2$;
- (4) a two-sheeted covering of a divisor D of bidegree (1,1) in $P^2 \times P^2$, with a branch divisor $B \in |-K_D|$.

§3. Construction of threefolds of type (C, C) with $\rho = 3$,

Let V be of type (C, C) , and let $\pi_1 : V \rightarrow S_1, \pi_2 : V \rightarrow S_2, C$, etc., be as in §2. It follows from the considerations in §2 that there exists a commutative diagram of natural morphisms:

3.1.



where p_1 and p_2 are the natural projections, and $\deg \pi = 1$ or $\deg \pi = 2$ (see 2.5).

We shall consider the case $\deg \pi = 2$. In this setting, we shall obtain numerical formulae for the branch divisor of the double covering π .

3.2. Let f_k be the (general) fiber of π_k , and let σ_k be the (general) fiber of R_k , $k = 1, 2$. Evidently $\sigma_k \simeq P^1$ for any $\sigma_k, k = 1, 2$. Let $\mathcal{E}_k, b_k, \varphi_k, C_k, k = 1, 2$, and $F_1 \equiv F_2 \equiv F$ be as in the Remark after Theorem 1.5. Let $\varepsilon_k = \det(\mathcal{E}_k), e_k = -\deg(\varepsilon_k)$; and let $C_{1V} = \pi^*C_1, C_{2V} = \pi^*C_2, F_{1V} \equiv F_{2V} \equiv F_V$ be the divisors on V , in the sense indicated in Corollary 2.4, i.e. $F_V \equiv \pi^*p^*(x) \equiv \pi^*F$ (where $F \equiv p^*(x), x \in C$). It is easy to see that:

3.3. $K_{S_1 \times_c S_2} = -2C_1 - 2C_2 + (\varepsilon_1 + \varepsilon_2 + k)F$, where $k = K_C$ is the canonical divisor of C . The branch divisor B of π has the form:

3.4. $B = 2q_1C_1 + 2q_2C_2 + 2qF$, where q_1 and q_2 are integers, and $2qF$ is used in the sense that $2qF = p^*(q)$ for the divisor q on C .

We claim that $q_1 = q_2 = 1$.

In fact, as $p_1 : S_1 \times_c S_2 \rightarrow S_1$ is a P^1 -bundle, then $\text{Pic}(S_1 \times_c S_2) = p_1^* \text{Pic } S_1 \oplus \mathbf{Z}C_2$ (since C_2 is an 1-section of p_1). Therefore $\text{Pic}(S_1 \times_c S_2) = \mathbf{Z}C_1 \oplus g_1^*$

(Pic C) $\oplus \mathbf{Z}C_2$, in sense that $g_1^*(\text{Pic } C) \cdot F = p_1^* g_1^*(\text{Pic } C) = p^*(\text{Pic } C)$. As $\pi : f_1 \rightarrow \sigma_1$ is a two-sheeted covering for the general $f_1 \simeq P^1$ and $\pi(f_1) = \sigma_1 \simeq P^1$, then it has two branch points. Therefore $\text{deg}(B|_{\sigma_1}) = \text{deg}(B|_{\sigma_2}) = 2$, i.e. $2 = \text{deg}(B|_{\sigma_1}) = (2q_1C_1 + 2q_2C_2 + 2qF) \cdot \sigma_1 = 2q_2C_2\sigma_1 = 2q_2$; similarly – for q_1 . As V is smooth, then B is smooth, and we derive:

COROLLARY. *The (smooth) branch divisor B of π has the form*

$$B = 2C_1 + 2C_2 + 2qF$$

for some divisor q on C , where $2qF = p^*(2q)$.

3.5. In the context of the situation, we shall derive some necessary numerical conditions for B .

The general surface $C_{1V} = \pi^*C_1$ is smooth, and it is a two-sheeted covering of C_1 with a branch divisor $B|_{C_1} = (2C_1 + 2C_2 + 2qF)|_{C_1} = 2C_{12} + 2(\varepsilon + q)\sigma_1$, where $C_{12} = C_1 \cdot C_2$, and $(C_{12}, C_{12})_{C_1} = C_2C_2C_1 = -e_2\sigma_2C_1 = -e_2$. Therefore, for the existence of a (smooth) effective divisor $C_{12} \subset C_1$, one must have $2(q - e_1) \geq 2e_2$ if $e_2 \geq 0$, or $2(q - e_1) \geq \frac{1}{2} \cdot 2e_2 = e_2$ if $e_2 < 0$ (see [1, Ch. V, §2]); here $q = \text{deg}(q)$. The same restrictions are available for C_2 and e_1 , and we derive:

COROLLARY. *Let B , e_1 , and e_2 be as in 3.2–3.4. Then for $q = \text{deg}(q)$ we have:*

- (a) $q \geq e_1 + e_2$ if $e_1 \geq 0, e_2 \geq 0$;
- (b) $2q \geq 2e_1 + e_2$ if $e_1 \geq 0, e_2 < 0$;
- (c) $2q \geq e_1 + 2e_2$ if $e_1 < 0, e_2 \geq 0$;
- (d) $2q \geq \max\{2e_1 + e_2, e_1 + 2e_2\}$ if $e_1 < 0, e_2 < 0$.

3.6. The canonical divisor K_V , and the surfaces C_{1V} and C_{2V}

3.6.1. It follows from the preceding that

$K_V = -C_{1V} - C_{2V} + (\varepsilon_1 + \varepsilon_2 + \mathfrak{k} + q)F_V$. Therefore, by the adjunction formula

$K_{C_{1V}} = -C_{12V} - (\varepsilon_1 + \varepsilon_2 + \mathfrak{k} + q)f_1$, where $C_{12V} = C_{1V} \cdot C_{2V}$. Evidently, the self-intersection number of C_{12V} in C_{1V} is equal to $-2e_2$, and $C_{12V} \cdot f_1 = 2$. Therefore $K_{C_{1V}} \cdot K_{C_{1V}} = (8 - 8g) - (4q - 4e_1 - 2e_2)$, where $q = \text{deg}(q)$ and $g = g(C)$.

3.6.2. From the Corollary in 3.5, we obtain $4q - 2e_1 - 2e_2 \geq 0$; similarly for C_{2V} . From $K_{C_{1V}} \equiv -C_{2V}|_{C_{1V}} + (2g - 2 - e_1 - e_2 + q) \cdot f_1$ we conclude that

$h_1: C_{1V} \rightarrow (C_{1V})_{\min}$ is a composition of σ -processes with centers lying on the curve $h_1(C_{12V})$; here $(C_{1V})_{\min}$ is some (relatively) minimal model of C_{1V} .

3.6.3. For $F_V = \pi^* p^*(x)$, $x \in C$, we have similarly: $K_{F_V} = (-C_{1V} - C_{2V}) \cdot F_V$ and $K_{F_V} \cdot K_{F_V} = 4$. The surface F_V is obtained from P^2 after blowing-up of five points.

3.7. Examples of Fano threefolds of type (C, C)

3.7.1. Let $\deg \pi = 2$. Then V is of type (C_1, C_1) , and $K_V = -C_{1V} - C_{2V} + (2g - 2 + q - e_1 - e_2) \cdot F_V$. Let, moreover, V be a Fano threefold. Then $K_V C_{1V} C_{2V} = 4g + 2q - 4 < 0$; in particular $g = 0$ and $q \leq 1$. Therefore (see the Corollary in 3.5) $1 \geq q \geq e_1 + e_2$ (since $e_1 \geq 0, e_2 \geq 0$).

Let $e_1 \geq e_2$, and let $e_1 = 1, e_2 = 0$. Then $B|_{C_1} = 2C_2|_{C_1}$ is not a reducible divisor, which is impossible. Therefore $e_1 = e_2 = 0$ and $K_V \equiv -C_{1V} - C_{2V} - F_V$. The manifold V is a two-sheeted covering of $S_1 \times_C S_2 = (P^1 \times_{P^0}) \times_{P^0} (P^1 \times_{P^0} P^1) \simeq P^1 \times P^1 \times P^1$ with a branch divisor $B = 2C_1 + 2C_2 + 2F$ of multidegree $(2, 2, 2)$.

3.7.2. Let $\deg \pi = 1$. Then $V = S_1 \times_C S_2$, and $K_V \equiv -2C_1 - 2C_2 + (2g - 2 - e_1 - e_2) \cdot F$, where $K_V C_1 C_2 = 2g - 2 + e_1 + e_2$ and $K_V^3 = 24 \cdot (2g - 2)$. If V is a Fano threefold, then $K_V^3 < 0$, hence $g = 0$. Therefore $K_V C_1 C_2 = e_1 + e_2 - 2 < 0$, where $e_1 \geq 0, e_2 \geq 0$.

Let $e_1 \geq e_2$, and let, for example, $e_1 = 1, e_2 = 0$. Then $V = \mathbf{F}_1 \times_{P^1} (P^1 \times P^1)$, and $K_V \equiv -2C_1 - 2C_2 - 3F, K_V C_1 C_2 = -1, K_V^3 = -48, K_V \sigma_1 = K_V \sigma_2 = -2$. On the other hand, $-K_V|_F = 2\sigma_1 + 2\sigma_2, -K_V|_{C_1} = 2C_2|_{C_1} + \varphi_1$, and $-K_V|_{C_2} = 2C_1|_{C_2} + 3\varphi_2$ are ample divisors on the surfaces $F \simeq P^1 \times P^1, C_1 \simeq P^1 \times P^1$, and $C_2 \simeq \mathbf{F}_1$. Since $K_V \cdot C_{12} = -1$, and $C_{12} = C_1 \cap C_2$ is a rational curve, then there exists an extremal ray $R_3 = \mathbf{R}_+[C_{12}]$ (see [3]). As $(C_2|_{C_1}, C_2|_{C_1}) = 0$, then $C_2|_{C_1}$ moves in C_1 as one of the rulings of the quadric $D = C_1 \simeq P^1 \times P^1$. The restriction of the normal bundle $N_{C_1|V}$ on C_{12} has a degree -1 . In fact $N_{C_1|V}|_{C_{12}} \simeq \mathcal{O}_{C_1}(C_1) \otimes \mathcal{O}_{C_{12}} \simeq \mathcal{O}_{C_1}(-\sigma_1) \otimes \mathcal{O}_{C_{12}} \simeq \mathcal{O}_{C_{12}}(-1)$, where $C_{12} = P^1$. Therefore, we can contract C_1 along C_{12} (see [2, p.1020], or [3]); that is, there is an extremal ray of type E_1 on V , i.e. V is not strongly primitive (see 1.4 and 0.2.2).

Let $e_1 = e_2 = 0$. Then $V \simeq (P^1 \times_{P^0}) \times_{P^0} (P^1 \times P^1) \simeq P^1 \times P^1 \times P^1$, and $-K_V = 2C_1 + 2C_2 + 2F$ is a divisor of multidegree $(2, 2, 2)$ in $P^1 \times P^1 \times P^1$.

3.8. The discriminant curves for π_1 and π_2

Let Δ_1 and Δ_2 be the discriminant curves of π_1 and π_2 . Then $\Delta_1 \equiv 4b_1 + (4q - 2e_2)\varphi_1$ on S_1 , and $\Delta_2 \equiv 4b_2 + (4q - 2e_1)\varphi_2$ on S_2 . These numerical equalities follow immediately from the formula $-4 \cdot K_S \equiv \pi_*(-K_V)^2 + \Delta$, connecting the discriminant curve Δ of a conic bundle $\pi: V \rightarrow S$ with the canonical bundles

of V and S .

§4. The Case (C, D)

4.1. Let the extremal morphism π_1 be of the type C , and let π_2 be of the type D . In particular $\rho(V) = 2$ (see 1.3.3). In just the same way as in 2.6 we obtain that V is a Fano threefold.

4.2. COROLLARY. *Let the pair (π_1, π_2) be of the type (C, D) . Then V is one of the following (see [4]):*

- (1) $V = P^2 \times P^1$;
- (2) a two-sheeted covering $\pi : V \rightarrow P^2 \times P^1$ with a branch divisor $B \subset P^2 \times P^1$ of bidegree $(4,2)$;
- (3) a two-sheeted covering $\pi : V \rightarrow P^2 \times P^1$ with a branch divisor $B \subset P^2 \times P^1$ of bidegree $(2,2)$.

§5. The Case (C, E)

Let π_1 be of type C , and π_2 be of type E . We have to prove that if V is strongly primitive, then V must be a Fano threefold. We shall consider separately the cases E_2, E_3, E_4 , and E_5 (see 1.3.1 and 1.4).

5.1. The cases (C, E_2) and (C, E_5)

Let π_2 belongs to one of the types E_2 or E_5 . In particular, the morphism π_2 is a contraction of a divisor $D \simeq P^2$ in V to a point (see 1.3.1). The morphism π_1 maps $D \simeq P^2$ onto S_1 . Actually, in the opposite case π_1 contracts D , because π_2 (= a contracting of D) is an extremal morphism. But π_1 is also an extremal morphism, hence π_1 coincides with π_2 – a contradiction. Therefore $\pi_2(D) = S_1$ and $S_1 \simeq P^2$.

5.2. The cases (C, E_3) and (C, E_4)

Let π_2 belongs to one of the types E_3 or E_4 . Just as above, the fact that π_1 and π_2 are different extremal morphisms, implies that the morphism π_1 maps the quadric $D \subset V$, corresponding to π_2 (see 1.3.1), onto the surface S_1 . As S_1 is smooth, it must be either P^2 (in the cases E_3 and E_4) or $P^1 \times P^1$ (in the case E_3).

Let $S_1 \simeq P^1 \times P^1$. Let ϕ_1 and ϕ_2 be the rulings of S_1 , and let $\varphi_1 = s \times P^1 \equiv P^1 \times t = \varphi_2$ be the rulings of $D \simeq P^1 \times P^1$. Since $(\pi_1|_D^* \phi_i, \pi_1|_D^* \phi_i)_D = 0, i = 1,2$, then $\pi_1|_D^* \phi_i = m_i \varphi_i$, where m_i is a positive integer. Therefore $(\pi_1|_D^* \phi_1, \pi_1|_D^* \phi_2)_D = (m_1 \varphi_1, m_2 \varphi_2)_D = m_1 m_2$. On the other hand, the last equals to $m = \deg(\pi|_D)$. But φ_1 and φ_2 are numerically equivalent on V ; therefore $0 = \varphi_1 \cdot \pi_1^* \phi_1$

$= \varphi_2 \cdot \pi_1^* \phi_1 = m_2$. In particular, $\deg \pi = m = m_1 m_2 = 0$, which is impossible. Consequently, in the cases (C, E_3) and (C, E_4) the surface S_1 is isomorphic to P^2 .

5.3. COROLLARY. *Let π_1 be of the type C and π_2 be of the type E (E_2, E_3, E_4 , or E_5). Then V is a Fano threefold.*

Proof. In fact, we obtained that in all cases $S_1 \simeq P^2$ (see 5.1 and 5.2). Therefore $\rho(V) = \rho(P^2) + 1 = 2$, and V admits two different extremal morphisms. It follows that V is a Fano threefold (see 2.6).

5.4. COROLLARY (see [4]). *Let V, π_1, π_2 , etc., be as in 5.3. Then V is one of the following:*

- (1) $V = P(\mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(1))$, in the case (C, E_2) ;
- (2) $V = P(\mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(2))$, in the case (C, E_5) ;
- (3) V is a two-sheeted covering of $Y = P(\mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(1))$ with a branch divisor $B \in |-K_Y|$, in the cases (C, E_3) and (C, E_4) .

§6. The case (D, D)

6.1. Let π_1 and π_2 be both of the type D (see 1.3.3). Let $S_1 = \pi_1^*(x)$, $x \in C_1$, and $S_2 = \pi_2^*(x)$, $x \in C_2$, where $\pi_k : V \rightarrow C_k$, $k = 1, 2$, are the corresponding extremal morphisms. As $\rho(V) = \rho(C_k) + 1 = 2$, then S_1 is represented in the form $S_1 \equiv a.S_2 + b.K_V$, for some rational a, b . In particular, $K_{S_1} = (K_V + S_1)|_{S_1} = (1/b).(-a.S_2 + (b + 1).S_1)|_{S_1}$. Hence

$$K_{S_1}.K_{S_1} = (1/b^2).((b + 1).S_1 - a.S_2)^2.S_1 = 0,$$

since $S_1.S_1 = S_2.S_2 = 0$. On the other hand, the divisor $-K_{S_1}$ must be ample, since S_1 is a Del Pezzo surface, P^2 , or $P^1 \times P^1$ (see 1.3.3). We come to a contradiction.

6.2. COROLLARY. *There are no manifolds for which π_1 and π_2 are both of type D .*

§7. The case (E, E, \dots, E)

Let V admits morphisms $\pi_1, \pi_2, \dots, \pi_n$ of the type E , and let V be strongly primitive. Let D_1, D_2, \dots, D_n be the corresponding divisors on V , which $\pi_1, \pi_2, \dots, \pi_n$

contract (see 1.3.1). Then, by [4, p. 124 (8.1)], the divisors D_i are mutually disjoint. Consequently, the contractions π_i carry out independently.

Theorem 1.5 is proved.

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*Institute of Mathematics
Bulgarian Academy of Sciences
ul. Acad. G. Bonchev, bl.8
1113 Sofia, Bulgaria*