Compositio Mathematica **119:** 41–52, 1999.

Remarks on Semistability of G-Bundles in Positive Characteristic

XIAOTAO SUN*
Institute of Mathematics, Academia Sinica, Beijing 100080, China
e-mail: xsun@math08.math.ac.cn

(Received: 14 August 1997; accepted in final form: 15 June 1998)

Abstract. We analyzes a notion of strong semistability of principal G-bundles by including reduction to nonreduced parabolic subgroup schemes. It turns out that strong semistability is equivalent to the Frobenius semistability of Ramanan and Rananathan. We also give a bound for nonstrongly semistability of a semistable $\mathrm{GL}(n)$-bundle improving a previous result of Shepherd-Barron.

Mathematics Subject Classifications (1991): 14Lxx, 14Mxx.

Key words: G-bundle, strong semistability, Frobenius, canonical reduction.

Introduction

Principal G-bundles on a compact Riemann surface of genus $g \geqslant 2$ were studied and the moduli space of semistable G-bundles was constructed by A. Ramanathan ([R1], [R2]). In the studying of G-bundles, the following result is important and essential. Let $\rho: G \to \mathrm{GL}(V)$ be an irreducible representation, then for any semistable G-bundle E the associated vector bundle $E_\rho(V)$ is semistable too. This theorem is no longer true in positive characteristics, so the construction of moduli space of G-bundles in positive characteristic remains open.

A principal G-bundle $E \to C$ is semistable if for every reduction $\sigma: C \to E/P$ to reduced parabolic subgroup schemes P of G, we have $\deg \sigma^*(T_{E/P}) \geqslant 0$ where $T_{E/P}$ is the tangent bundle along fibers of $E/P \to C$. In characteristic zero, all group schemes are reduced, thus the word *reduced* can be removed in the definition. In positive characteristic, nonreduced group schemes do occur. Thus it is natural to think that one may expect a new concept of semistability for a G-bundle E (see Definition 1.2) if the word *reduced* in above definition is removed. This new semistability, called strong semistability, behaves well under the extension of structure groups (Corollary 1.1). On the other hand, there is another notation of semistability, called F-semistability, if F_n^*E is semistable for any nth Frobenius $F_n: C \to C$. It turns out these two notations of semistability are equivalent to each

* Partly supported by the National Natural Science Foundation of China under grant No. NSFC 19671085.

other (Proposition 1.2). In Section 2 we prove that any semistable G-bundle on an elliptic curve is strongly semistable (see Theorem 2.1). It is well-known that a semistable G-bundle on a curve of genus $g \geqslant 2$ may not be strongly semistable. To compare semistability with strong semistability, we introduce some numerical invariants in Section 3. A natural problem is to bound these invariants. More precisely, to bound the instability of $\tilde{E} = F^*E$ for a semistable G-bundle E. We will treat the problem for vector bundles of arbitrary rank (see Theorem 3.1). This is an improvement of a result of N. I. Shepherd-Barron ([SB]) and, in the case of rank 2, coincides with that of H. Lange and U. Stuhler ([LS]).

0. Preliminaries

Let k be a field of characteristic $p > 0$ and $n > 0, \varphi: X \to \mathrm{Spec}(k)$ a scheme over k, the p^nth power map $\mathcal{O}_X \to \mathcal{O}_X$ given by $f \to f^{p^n}$ is a homomorphism and gives rise to a morphism $F_n: X \to X$ called the (absolute) Frobenius. Let $f_k: \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ be the morphism induced by $k \to k(a \to a^{p^n})$, and $X^{(1)} = f_k^* X$ such that

$$\begin{array}{ccccc} X & \xrightarrow{F_X} & X^{(1)} & \xrightarrow{A} & X \\ \downarrow & & \downarrow{\scriptstyle f_k^*\varphi} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Spec}(k) & = & \mathrm{Spec}(k) & \overset{f_k}{=} & \mathrm{Spec}(k). \end{array}$$

If k is a perfect field, f_k and A are isomorphisms. We call $F_X: X \to X^{(1)}$ the geometric Frobenius.

Let $\pi: E \to X$ be a G-bundle, pulling back by the Frobenius we get a G-bundle $F_n^*(E) \to X_F$ on X_F (where we take the k- structure on $F_n^*(E)$ to be the one defined by the composite $F_n^*(E) \to X_F \xrightarrow{f_k \cdot \varphi} \mathrm{Spec}(k)$). If k is a perfect field, we can change the k-structure of $F_n^*(E)$, X_F and G by composing their structure morphisms with $f_k^{-1}: \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ to get a bundle $F_n^*(E) \to X$ with structure group $f_k^*(G)$. Replacing X by G in (D), we see that A gives a k-isomorphism of $f_k^*(G)$ with G, the latter having the k-structure changed by f_k^{-1}. Let a group scheme G $\to \mathrm{Spec}(\mathbb{F}_q)(q = p^n)$ over $\mathbb{F}_q$ such that $G = \mathbf{G} \times_{\mathbb{F}_q} \mathrm{Spec}(k)$, then $f_k^*(G) = G$. So the $f_k^*(G)$-bundle $F_n^*(E) \to X$ gives a G-bundle.

A smooth 1-foliation $\mathcal{F}$ on a smooth variety X is a subbundle of the tangent bundle $\mathcal{T}_X$ of X, closed under the Lie bracket and pth powers of derivations. Given a smooth 1-foliation $\mathcal{F}$ on X, we have a smooth variety $X/\mathcal{F}$ and a k-morphism $\rho: X \to X/\mathcal{F}$ such taht $\mathcal{O}_{X/\mathcal{F}}$ is the algebra of functions annihilated by $\mathcal{F}.\rho$ is purely inseparable of degree $p^{rk(\mathcal{F})}$ factoring a kth geometric Frobenius $F_X: X \to X^{(1)}$ as $F_X = \sigma \circ \rho$ for some $\sigma: X/\mathcal{F} \to X^{(1)}$, where $X^{(1)} = X \times_k k$ is the base change of X by the p^kth power map of k. Given a factorization $X \xrightarrow{\rho} Y \xrightarrow{\sigma} X^{(1)}$, we have a smooth 1-foliation $\mathcal{F} := \mathrm{Ker}(d\rho)$ on X such that $Y = X/\mathcal{F}$. This way gives a one-to-one correspondence between the factorizations $X \xrightarrow{\rho} Y \xrightarrow{\sigma} X^{(1)}$ of geometric Frobenius morphisms and the smooth 1-foliations $\mathcal{F} = \mathrm{Ker}(d\rho)$ on X (see [Ek] for the proof).

1. Semistable G-Torsors

Let G be a group scheme over an algebraically closed field k, which is flat and locally of finite-type, but not necessarily smooth. Let X be a scheme, and X_{fl} denote the big flat site $(LFT/X)_{fl}$. For any open set U of X_{fl}, G defines a sheaf $G(U)$ on U_{fl}. Let S be a sheaf of sets on X_{fl}, on which G acts.

DEFINITION 1.1. S is called a G-torsor if there is a covering $(U_i \to X)$ for the flat topology on X such that $S|_{U_i}$ is isomorphic with its G-action to $G(U_i)$.

A G-torsor S is representable by a X-scheme if G is affine, or G is smooth and separated over X and X has dimension at most one. It is known that if G is smooth, respectively étale, respectively proper, then so also is any G-torsor.

Let $\mathcal{U} = (U_i \to X)_{i\in I}$ be a covering of X_{fl}. A 1-cocyle for $\mathcal{U}$ with values in G is a family $(g_{ij}\colon U_{ij} \to G)_{I\times I}$ such that $(g_{ij}|_{U_{ijk}})(g_{jk}|_{U_{ijk}}) = (g_{ik}|_{U_{ijk}})$. Two cocycles g and g' are cohomologous if there is family $(h_i\colon U_i \to G)_{i\in I}$ such that $g'_{ij} = (h_i|_{U_{ij}})g_{ij}(h_j|_{U_{ij}})^{-1}$. This is an equivalence relation, the set of the equivalent classes (i.e. the cohomology classes) is denoted by $\check{H}^1(\mathcal{U}/X, G)$. It is known that there is a one to one corresponding between isomorphism classes of G-torsors that become trivial on a given covering $\mathcal{U}$ and elements of $\check{H}^1(\mathcal{U}/X, G)$ (See [Mi]). Thus if $\rho\colon G \to H$ is a morphism of groph schemes and E is a G-torsor, we can associate a unique H-torsor $E_\rho(H)$ since ρ induces a map

$$\check{H}^1(\mathcal{U}/X, G) \to \check{H}^1(\mathcal{U}/X, H).$$

Let E be a G-torsor over X and $\widetilde{P}\subset G$ a subgroup scheme, we define $E/\widetilde{P}$ to be the sheaf on X_{fl} such that for any $U \to X$, $E/\widetilde{P}(U) := E(U)/\widetilde{P}$. If G is affine, $E/\widetilde{P}$ is representable by a X-scheme. It is not difficult to verify that $E \to E/\widetilde{P}$ is a $\widetilde{P}$-torsor over $E/\widetilde{P}$ if $G \to G/\widetilde{P}$ is a $\widetilde{P}$-torsor over $G/\widetilde{P}$. Let $\widetilde{P}$ be a subgroup scheme of G, $\tilde{\mathfrak{p}}$ and $\mathfrak{g}$ the Lie algebras of P and G, then the adjoint representation of G induces a representation $\rho\colon \widetilde{P} \to \mathrm{GL}(V)$ of $\widetilde{P}$ on $V = \mathfrak{g}/\tilde{\mathfrak{p}}$. Thus for any $\widetilde{P}$-torsor $E \to X$, we can associate a vector bundle $E_\rho(V)$ on X. In particular, the conjugation of G induces an action of $\widetilde{P}$ on the tangent bundle $T_{G/\widetilde{P}}$ of $G/\widetilde{P}$ at $\bar{e}$, thus we have an associated vector bundle $T_{E/\widetilde{P}}$ of $E \to E/\widetilde{P}$ on $E/\widetilde{P}$, which is nothing but the tangent bundel along fibres of $E/\widetilde{P} \to X$. A subgroup scheme $\widetilde{P}\subset G$ is called a parabolic subgroup scheme of G if $G/\widetilde{P}$ is a projective scheme over k. Now I would like to make the following definition.

https://doi.org/10.1023/A:1001512029096 Published online by Cambridge University Press

DEFINITION 1.2. A G-torsor E on a nonsingular projective curve X is called semistable if for every parabolic subgroup scheme $\widetilde{P}\subset G$ and every section $\sigma\colon X\to E/\widetilde{P}$, we have $\deg(\sigma^*T_{E/\widetilde{P}})\geqslant 0$.

We are interested in the case when G is a reduced reductive algebra group. In this case, a G-torsor E is also called a G-bundle. It is clear that E is a semistable G-bundle if E is a semistable G-torsor. Thus, we will call E a strongly semistable G-bundle if E is semistable as a G-torsor. From now on, we always assume that G is a reductive algebraic group, for a reduced subgroup scheme of G, we simply call it a subgroup of G. A subgroup scheme $\widetilde{P}$ of G is parabolic if and only if $\widetilde{P}_{\mathrm{red}}(:= P)$ is a parabolic subgroup of G (i.e. containing a Borel subgroup [W1]). For any group scheme $\widetilde{P}$, the group of characters of $\widetilde{P}$ is defined to be $\mathrm{Hom}(\widetilde{P}, G_m)$, the group of group scheme theoretic homomorphisms, which was determined by [HL] as the following.

PROPOSITION 1.1 ([HL]). *Let $\widetilde{P}$ be the parabolic subgroup scheme corresponding to the W-function f. Then the group of characters of $\widetilde{P}$ is the group, $\mathbb{X}(\widetilde{P})=\coprod_{\alpha\in\Delta}\mathbb{Z}p^{f(\alpha)}\omega_\alpha$, where ω_α are the fundamental dominant weights corresponding to the simple roots $\alpha\in\Delta$, p^∞ is understood to be 0.*

Let $T\subset P\subset\widetilde{P}$ be a torus of G, $\mathbb{X}(\widetilde{P})\subset\mathbb{X}(P)\subset\mathbb{X}(T)$. A character χ of $\widetilde{P}$ is dominant iff it is dominant as a character of P. Let $X=G/\widetilde{P}$ and $\chi=\Sigma_{\alpha\in\Delta}m_\alpha p^{f(\alpha)}\omega_\alpha\in\mathbb{X}(\widetilde{P})$, then there is an induced line bundle $L_{\widetilde{P}}(\chi)$ on X. In this way, one has an identification $\mathrm{Pic}(X)=\mathbb{X}(\widetilde{P})$ and $L_{\widetilde{P}}(\chi)$ is ample if and only if $m_\alpha>0$ for each $\alpha\in\Delta$ (see Corollary 7 of [HL]). The line bundle $L_{\widetilde{P}}^{-1}(\chi)$ on $G/\widetilde{P}$ gives naturally a line bundle $E(\chi)$ on $E(G/\widetilde{P})=E/\widetilde{P}$. The following proposition gives some equivalent description of a strongly semistable G-bundle.

PROPOSITION 1.2. *Let G be a reductive algebraic group, and E a G-bundle over a nonsingular projective curve C. Then the following are equivalent*

(1) *E is strongly semi-stable.*
(2) *$F_n^*(E)$ are semi-stable for any Frobenius $F_n\colon C\to C$.*
(3) *For any parabolic subgroup scheme $\widetilde{P}$ and section $\sigma\colon C\to E/\widetilde{P}$, we have $\deg\sigma^*E(\chi)\leqslant 0$ for any nontrivial dominant character χ on $\widetilde{P}$ where $E(\chi)$ is the natural line bundle on $E/\widetilde{P}$ given by $L_{\widetilde{P}}^{-1}(\chi)$ on each fibre $G/\widetilde{P}$.*

Proof. Suppose that E is strongly semistable and $\mathcal{U}=(U_i\to X)_{i\in I}$ the covering (actually in Zariski's topology) of C such that E is determined by 1-cocycle $(g_{ij}\colon U_{ij}\to G)_{I\times I}$. One can see that the 1-cocycle of F_n^*E is $(g_{ij}^{p^n}\colon U_{ij}\to G)_{I\times I}$, and F_n^*E is nothing but the associated G-bundle $E':=E_{F_G}(G)$ of E by the geometric Frobenius $F_G\colon G\to G$. If E' is not semistable, we consider the canonical reduction (P,σ) (see [Be], [KN]), where the parabolic subgroup $P\subset G$ and section $\sigma\colon C\to E'/P$ are unique. The uniqueness of (P,σ) implies that $P=f_k^*P$, and thus defined over $\mathbb{F}_{p^n}$. Let $\widetilde{P}=F_G^{-1}(P)$, then F_G induces an

https://doi.org/10.1023/A:1001512029096 Published online by Cambridge University Press

isomorphism $G/\widetilde{P} \cong G/P$ (note that G/P is defined over $\mathbb{F}_{p^n}$). Hence, we have $\deg(\sigma^* T_{E/\widetilde{P}}) = \deg(\sigma^* T_{E'/P}) < 0$ contradicting (1). This shows that (1) implies (2) (one remarks that $F_n^*(E)$ is semistable for any n if and only if it is semistable for sufficiently big n).

Suppose that $F_n^* E$ are semi-stable for any Frobenius $F_n^*: C \to C$, we need to prove (3). The ample line bundle $L_{\widetilde{P}}(\chi)$ on $X = G/\widetilde{P}$ is generated by global sections and determines a morphism of X to a projective space such that $L_{\widetilde{P}}(\chi)$ is the pullback $\mathcal{O}(1)$ of the hyperplane line bundle of the projective space (see [La]). By definition, $E(\chi)$ is the dual of $E(\mathcal{O}(1))$ on $E/\widetilde{P}$. Thus we only need to show that $\deg(\sigma) := \deg(\sigma^* E\mathcal{O}(1))) \geqslant 0$. Let x_0 be the generic point of C, then, if $\sigma(x_0)$ is a semistable point of G in the sense of geometric invariant theory, we have $\deg(\sigma) \geqslant 0$ (see Proposition 3.10 of [RR]). If $\sigma(x_0)$ is nonsemistable, after a Frobenius base change $F_n^*: C \to C$, we can assume that E is semistable and $\sigma(x_0)$ has an instability 1-PS defined over the function field of C, then we have that $\deg(\sigma) \geqslant 0$ (see Proposition 3.13 of [RR]). This shows that (2) implies (3).

To prove that (3) implies (1), we need a result of [HL], which says that there is a non-trivial dominant character χ on $\widetilde{P}$ such that $\det(T_{G/\widetilde{P}}) = L_{\widetilde{P}}(\chi)$ (see Proposition 7 of [HL]). Thus we have that

$$\det(T_{E/\widetilde{P}}) = E(\det(T_{G/\widetilde{P}}) = E(\chi)^{-1},$$

namely $\deg(\sigma^* T_{E/\widetilde{P}}) = -\deg \sigma^* E(\chi) \geqslant 0$, which shows that (3) implies (1).

Remark 1.1. It was pointed out by the referee that the Frobenius semistability of Ramanan and Ramanathan corresponds to reduction to the special class of non-reduced parabolic subgroup schemes $G_n P$ (where G_n denotes the nth Frobenius kernel of G). Thus it is clear that (1) implies (2) in Proposition 1.2.

COROLLARY 1.1. *Let $f: G \to H$ be a homomorphism of reductive algebraic groups, which maps the centre of G into that of H. Then if E is a strongly semistable G-bundle then the extended H-bundle $E(H)$ is strongly semistable.*

Proof. To prove that $E' = e(H)$ is strongly semistable, let $\widetilde{P}$ be a parabolic subgroup scheme of H corresponding to the W-function f, χ a dominant character on $\widetilde{P}$ and $\sigma: C \to E'/\widetilde{P}$ a section, we need to show that $\deg \sigma^* E'(\chi) \leqslant 0$ by the Proposition 1.2. From the proof of Proposition 1.2, we know that χ induces an ample line bundle L on $X = H/\widetilde{P}$ such that $E'(\chi) = E'(L)^{-1}$ on $E'(X) = E'/\widetilde{P}$. Now the group G acts through f on the projective scheme X linearly with respect to L, thus the group scheme $E(G)$ acts on $E'(L)(= E(L))$ and $E'(X)(= E(X))$ compatibility over the curve C. If $\sigma(x_0)$ is a semistable point under the action of $E(G_0)$, then the Proposition 3.10 of [RR] implies that $\deg \sigma^* E'(\chi) \leqslant 0$. If $\sigma(x_0)$ is a nonsemistable point, then, after a Frobenius base change of C, we can assume that $\sigma(x_0)$ has an instability 1-PS defined over $K(C)$, thus we have $\deg \sigma^* E'(\chi) \leqslant 0$ (see Proposition 3.13 of [RR]).

2. *G*-Bundles on an Elliptic Curve

Let $\pi: E \to C$ be a G-bundle over a smooth projective curve C of genus $g \geqslant 0$, namely, G operates on E on the right and π is G-variant and locally trivial in the étale topology. We know that there do exist Frobenius semistable (thus strongly semistable) bundles (see [RR] p.289). On the other hand, it is not difficult to construct nonstrongly semistable but semistable bundles on a smooth projective curve of genus $g > 1$ (see Proposition 4.4 of [RR]). In this section, we will prove the following result.

THEOREM 2.1. *Every semi-stable G-bundle on an elliptic curve is strongly semi-stable.*

Before the proof, let us recall some notations and facts which we need. If F is a quasi-projective scheme on which G operates (on the left), the associated bundle $E(F)$ is the quotient of $E \times F$ under the action of G given by $g(e, f) = (e \cdot g, g^{-1} \cdot f), e \in E, f \in F, g \in G$ ([Se]). Let G act on itself by inner conjugation, then the associated bundle $E(G) \to C$ is naturally a group scheme over C and acts naturally on the C-scheme $E(F)$.

Let $\mathrm{Par}(E(G)/C)$ be the scheme consisting of the parabolic subgroups of $E(G)$, which is a smooth projective C-scheme. It is easy to see that $\mathrm{Par}(E(G)/C)$ is naturally isomorphic to $E(\mathrm{Par}(G/k))$, where G acts on $\mathrm{Par}(G/k)$ by inner conjugation. If P is a parabolic subgroup of G, then the map $G/P \to \mathrm{Par}(G/k)$ given by $gP \to gPg^{-1}$ is a G-equivariant isomorphism of G/P onto the connected component $\mathrm{Par}_P(G/k)$ of $\mathrm{Par}(G/k)$ consisting of parabolic subgroups conjugate to P. Therefore we have a natural isomorphism $E(G/P) \cong E(\mathrm{Par}_P(G/k)) \subset \mathrm{Par}(E(G)/C)$ and the sections of $E(G/P) \to C$ are in bijective correspondence with parabolic subgroup schemes of $E(G)$ of type P (i.e., each geometric fibre is conjugate to P).

Proof of Theorem 2.1. Let $\pi: E \to C$ be a semistable G-bundle on an elliptic curve C, and $F: C \to C$ be the Frobenius map of degree p. It is enough, by Proposition 1.2, to show that $\widetilde{E} := F^*E$ is semistable.

If $\widetilde{E}$ is not semistable, then there is a unique canonical smooth parabolic subgroup scheme $\mathcal{P} \subset \widetilde{E}(G)$ such that $\deg(\mathcal{P}) > 0$ (see Theorem 7.3 of [Be]). Let $\mathfrak{G}$ and $\mathfrak{P}$ be the Lie algebra bundles of $\widetilde{E}(G)$ and $\mathcal{P}$ respectively, then $\deg(\mathfrak{G}/\mathfrak{P}) < 0$ and $H^0(\mathfrak{G}/\mathfrak{P}) = 0$ since $g(C) = 1$ (see the Conjecture 7.6 of [Be] and the remarks there). By the remarks at the beginning of this section, it is equivalent to say that there is a unique parabolic subgroup P of G and a unique section $\sigma: C \to \widetilde{E}(G/P) := \widetilde{X}$, such that $N_{C_0/\widetilde{X}} = \mathfrak{G}/\mathfrak{P}$, where $C_0 = \sigma(C)$, thus

$$H^0(N_{C_0/\widetilde{X}}) = 0 \quad \deg(N_{C_0/\widetilde{X}}) < 0.$$

On the other hand, let F_k be the pth Frobenius of $\mathrm{Spec}(k)$ and $C^{(1)} = C \times_k k$, then $E^{(1)} := F_k^*(E)$ is a semistable $G^{(1)} := F_k^*(G)$-bundle. If we identify $F_k^*(G)$

with G and write $X^{(1)} = E^{(1)}(G/P)$, then we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{F}_C} & X^{(1)} \\ \Big\downarrow{\scriptstyle \tilde{f}} & & \Big\downarrow{\scriptstyle f} \\ C & \xrightarrow{F_C} & C^{(1)}, \end{array}$$

since $\widetilde{X} = \widetilde{E}(G/P)$ is the same as $X^{(1)} \times_{C^{(1)}} C$.

The natural morphism $\widetilde{F}_C: \widetilde{X} \to X^{(1)}$ gives a factorization of $F_{\widetilde{X}}$, and thus determines a unique smooth 1-foliation $\mathcal{F} = \mathrm{Ker}(\mathcal{T}_{\widetilde{X}} \xrightarrow{d\widetilde{F}_C} \widetilde{F}_C^*\mathcal{T}_{X^{(1)}})$ on $\widetilde{X}$. It is easy to see that $\mathcal{F} = \tilde{f}^*\mathcal{T}_C$; in fact,

$$\mathcal{F} = \mathcal{T}_{\widetilde{X}/X^{(1)}} = \tilde{f}\mathcal{T}_{C/C^{(1)}} = \mathcal{T}_C = \mathcal{O}_C.$$

The $\mathcal{F}|_{C_0}(= \mathcal{O}_{C_0})$ is a subbundle of $\mathcal{T}_{\widetilde{X}}|_{C_0}$, by the following exact sequence

$$0 \to \mathcal{T}_{C_0} \to \mathcal{T}_{\widetilde{X}}|_{C_0} \xrightarrow{\delta} N_{C_0/\widetilde{X}} \to 0$$

and the fact $H^0(N_{C_0/\widetilde{X}}) = 0$, we have $\delta(\mathcal{F}|_{C_0}) = 0$ and thus $\mathcal{F}|_{C_0} \hookrightarrow \mathcal{T}_{C_0}$ is a smooth 1-foliation on C_0 since $\mathcal{F}|_{C_0}$ is closed under the Lie bracket and pth powers of derivations. Hence $\widetilde{F}_C$ induces an inseparable morphism $C_0 \to \widetilde{F}_C(C_0)$ of degree p, we get a section $C_1 := \widetilde{F}_C(C_0)$ of $f: X^{(1)} \to C^{(1)}$ such that

$$K_{X^{(1)}/C^{(1)}} \cdot C_1 = \frac{1}{p}\widetilde{F}_{C_*}(\widetilde{F}_C^* K_{X^{(1)}/C^{(1)}} \cdot C_0) = \frac{1}{p}K_{\widetilde{X}/C} \cdot C_0,$$

where $\mathcal{O}_{X^{(1)}}(K_{X^{(1)}/C^{(1)}}) = \det(\mathcal{T}^{\vee}_{X^{(1)}/C^{(1)}})$ and $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/C}) = \det(\mathcal{T}^{\vee}_{\widetilde{X}/C})$. Therefore

$$\deg(\sigma_1^* T_{E^{(1)}/P}) = -K_{X^{(1)}/C^{(1)}} \cdot C_1 = -\frac{1}{p}K_{\widetilde{X}/C} \cdot C_0 = \frac{1}{p}\deg(N_{C_0/\widetilde{X}}) < 0,$$

where $\sigma_1: C^{(1)} \to X^{(1)}$ determined by C_1, which contradicts the semistability of E, thus proved the theorem.

Remark 2.1. If F is a nonsemistable G-bundle over a smooth projective curve C of genus g and (P, F_P) the canonical reduction of F, then $F_P(\mathfrak{g}/\mathfrak{p})$ is a vector bundle on C, where $\mathfrak{g}$ and $\mathfrak{p}$ are the Lie algebras of G and P. Conjecture 7.6 of [Be] is that $H^0(C, F_P(\mathfrak{g}/\mathfrak{p})) = 0$.

In the case of characteristic zero, this conjecture was proved by S. Kummar and M. S. Narasimhan (see [KN], Lemma 3.6). In the case of positive characteristic, some partial results were known (see [Be]), for example, it is true when $g \leqslant 1$.

https://doi.org/10.1023/A:1001512029096 Published online by Cambridge University Press

If $F = \widetilde{E}$ is the pullback of a semistable G-bundle E by a Frobenius of degree p, then we actually proved that there is a nontrivial morphism $\mathcal{T}_C \to \widetilde{E}_P(\mathfrak{g}/\mathfrak{p})$ on any smooth projective curve C. Thus

$$H^0(C, \widetilde{E}_P(\mathfrak{g}/\mathfrak{p}) \otimes \omega^C) \neq 0,$$

which means that the instability of $\widetilde{E}$ cannot be too 'large' if one believes the conjecture.

It is known that a nonsemistable G-bundle E on a smooth projective curve has a canonical reduction to a unique parabolic subgroup P of G with unipotent radical U and a Levi component $L \subset P$ such that $P = L \cdot U$. The following result should be known to experts, at least in characteristic zero. Since there is no publishing proof, we give a proof here in arbitrary characteristic.

THEOREM 2.2. *Every nonsemistable G-bundle E on an elliptic curve C admits a semistable reduction to the Levi component L.*

Proof. Let (P, E_P) be the canonical reduction of E and $P = L \cdot U$, where E_P is the P-bundle obtained from E by the canonical reduction. We will show that E_P admits a reduction to L, which is obviously a semistable reduction. It is equivalent to show that $E_P(P/L) \to C$ has a section. Since P acts on U by inner conjugation (U is a normal subgroup of P), we can associate a group scheme $E_P(U) \to C$. It is not difficult to see that $E_P(P/L) \to C$ is a principal homogeneous space under the group scheme $E_P(U) \to C$, hence we are reduced to prove that the non Abelian cohomology group $H^1(C, E_P(U))$ is trivial (see [Mi], Section 4 of Chapter 3).

To show that $H^1(C, E_P(U)) = 1$, we consider the filtration

$$U = U_0 \supset U_1 \supset \cdots \supset U_n = \{e\}$$

of U defined in Proposition 2.1 of [SGA3], which satisfies

(1) U_i are P-invariant normal subgroups of P and the commutator $[U_i, U_j] \subseteq U_{i+j+1}$.
(2) $U_i/U_{i+1} = W_i$ are vector groups and the inner conjugation of P acts on W_i linearly.

The exact sequence $0 \to E_P(U_{i+1}) \to E_P(U_i) \to E_P(W_i) \to 0$, induce exact sequences of pointed sets

$$H^1(C, E_P(U_{i+1})) \to H^1(C, E_P(U_i)) \to H^1(C, E_P(W_i)).$$

Therefore the theorem will be proved if we can show for all W_i that $H^1(C, E_P(W_i)) = 0$. By the definition of canonical reduction, $\mathcal{W}_i := E_P(W_i)$ are vector bundles on C of $\deg(\mathcal{W}_i) > 0$. Thus $H^1(C, E_P(W_i))$ coincides with the cohomology $H^1(\mathcal{W}_i)$ of coherent sheaves by the definitions of the non Abelian cohomology and the ćech cohomology of sheaves. It is clear that $H^1(\mathcal{W}_i) \cong H^0(\mathcal{W}_i^\vee)$ will be trivial if $\mathcal{W}_i$ are semistable vector bundles.

https://doi.org/10.1023/A:1001512029096 Published online by Cambridge University Press

From the property of the above filtration, U acts on W_i trivially since $(U_0, U_i) \subseteq U_{i+1}$, thus $P \to \mathrm{GL}(W_i)$ factors through $P/U \to \mathrm{GL}(W_i)$ and the centre of P/U goes to that of $\mathrm{GL}(W_i)$. Let $E_{P/U}$ be the associated P/U-bundle of E_p through $P \to P/U$, then $E_{P/U}$ is a semistable P/U-bundle on an elliptic curve C by the definition of canonical reduction of E (see [Be], Definition 5.6), which is strongly semistable by our Theorem 2.1. Therefore the associated vector bundles $\mathcal{W}_i = E_P(W_i) = E_{P/U}(W_i)$ are also strongly semistable by the Corollary 1.1. We are done.

3. Instability of G-Bundles

It is well-known that a semistable G-bundle on a curve of genus $g \geqslant 2$ may not be strongly semistable. To measure the nonstrongly semistability of a semistable G-bundle, we can introduce some numerical invariants in a geometric way. A natural problem is, of course, to bound these invariants. The philosophy here is that the Frobenius pull back of a semistable G-bundle should not be too unstable, the instability of it should be bounded. This is done in this section for $G = \mathrm{GL}(V)$.

Let $\pi: E \to C$ be a G-bundle on a smooth projective curve of genus $g \geqslant 2$. For any parabolic subgroups P of G and $\pi_P: X_P = E/P = E(G/P) \to C$, we define the divisor $K_{X_P/C}$ on X_P to be the relative canonical divisor of X_P/C. For any irreducible horizontal curve D of $\pi_P: X_P \to C$, the map $\pi_P|_D: D \to C$ a finite morphism. Let $p^{i(D)}$ be the pure inseparable degree of $\pi_P|_D$, we call D the curve of type $i(D)$. Write

$$\mu_i(P) = \sup\{K_{X_P/C} \cdot D | D \subset X_P \text{ of type } i(D) \leqslant i\},$$

$$\mu_i(E) = \sup\{\mu_i(P) | P \in \mathrm{Par}(G/k)\}.$$

Then it is clear that we have

$$\mu_0(E) \leqslant \mu_1(E) \leqslant \mu_2(E) \leqslant \cdots \leqslant \mu_i(E) \leqslant \cdots.$$

The semistability of E means that for any $P \in \mathrm{Par}(G/P)$ and any section C_0 of $\pi_P: X_P \to C$, we have that $K_{X_P/C} \cdot C_0 \leqslant 0$. In particular, $\mu_0(E) \leqslant 0$ implies that E is semistable. Actually, we have

PROPOSITION 3.1. *E is a semistable G-bundle if and only if $\mu_0(E) \leqslant 0$.*

Let D be a curve of type 0 on X_P and $\widetilde{C} \to D$ the normalization of D in the function field of D. Let $f: \widetilde{C} \to C$ be the finite morphism $\widetilde{C} \to D \xrightarrow{\pi_P|_D} C$,

https://doi.org/10.1023/A:1001512029096 Published online by Cambridge University Press

which is separable by the definition of D, and $\widetilde{\pi}: \widetilde{E} = E \times_C \widetilde{C} \to \widetilde{C}$ the pull back of E, then we have commutative diagram

$$\begin{array}{ccc} \widetilde{E}/P = \widetilde{X}_P & \xrightarrow{\tilde{f}} & X_P \\ \downarrow & & \downarrow \\ \widetilde{C} & \xrightarrow{f} & C. \end{array}$$

Thus there is a section $\widetilde{C}_0$ of $\widetilde{X}_P \to \widetilde{C}$ such that $\tilde{f}_* \widetilde{C}_0 = D$, which implies that

$$K_{X_P/C} \cdot D = \tilde{f}_*(\widetilde{C}_0 \cdot \tilde{f}^* K_{X_P/C}) = \tilde{f}_*(K_{\widetilde{X}_P/\widetilde{C}} \cdot \widetilde{C}_0).$$

Thus the proposition is equivalent to the following lemma.

LEMMA 3.1. *If $\pi: E \to C$ is a semi-stable G-bundle, then for any separable finite morphism $f: \widetilde{C} \to C$, the pull back f^*E is also semistable.*

Proof. We can assume that $f: \widetilde{C} \to C$ is a Galois cover with Galois group $\mathcal{G}$. If $\widetilde{E} = f^*E$ is not semistable, then there is a unique $P \in \mathrm{Par}(G/k)$ and a unique section $\widetilde{C}_0$ of $\widetilde{X}_P = \widetilde{E}(G/P) \to \widetilde{C}$ such that $K_{\widetilde{X}_P/\widetilde{C}} \cdot \widetilde{C}_0 > 0$. The $\mathcal{G}$ acts naturally on $\widetilde{X}_P$, and $\widetilde{C}_0$ is invariant under the $\mathcal{G}$ action, which gives a section C_0 of $X_P \to C$ such that $K_{X_P/C} \cdot C_0 > 0$ contradicts the semistability of E.

COROLLARY 3.1. *Let $F_n: C \to C$ denote the nth Frobenius and E a G-bundle, them the following are equivalent*

(1) *$F_n^* E$ is semistable*
(2) $\mu_n(E) \leqslant 0$.

In particular, E is strongly semistable if and only if $\lim_{n\to\infty} \mu_n(E) \leqslant 0$.

Proof. F_n^*E is semistable if and only if F_k^*E are semistable bundles for all $k \leqslant n$. Thus we are reduced to show that $K_{X_p/C} \cdot D \leqslant 0$ for any irreducible curve D on X_p such that $\pi_P|_D: D \to C$ is a pure inseparable cover of degree p^n, which must be the nth Frobenius. Therefore the corollary follows the same argument as above.

In the rest of this section, we restrict ourselves to the special case that $G = \mathrm{GL}_r(k)$, the $\mathcal{E}$ denotes the associated vector bundle of a $\mathrm{GL}_r(k)$-bundle on C by the standard representation k^r of $\mathrm{GL}_r(k)$. For any vector bundle $\mathcal{E}$ on C, we write $\mu(\mathcal{E}) = \deg(\mathcal{E})/rk(\mathcal{E})$. The Harder–Narasimhan filtration of $\mathcal{E}$ is

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E},$$

such that $\mathcal{G}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ are semistable vector bundles and $\mu(\mathcal{G}_1) > \cdots > \mu(\mathcal{G}_n)$. Write $\mu(\mathcal{G}_1) = \mu_{\max}(\mathcal{E})$ and $\mu(\mathcal{G}_n) = \mu_{\min}(\mathcal{E})$, then we have the following fact.

https://doi.org/10.1023/A:1001512029096 Published online by Cambridge University Press

LEMMA 3.2. *It* $\mathcal{A}$ *and* $\mathcal{B}$ *are vector bundles on* C *and* $\mu_{\min}(\mathcal{A}) > \mu_{\max}(\mathcal{B})$, *then* $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = 0$.

THEOREM 3.1. *Let* $\mathcal{E}$ *be a semistable vector bundle of* rank r *on a smooth projective curve* C *of genus* $g \geqslant 2$ *and* $F: C \to C$ *the Frobenius. Write* $F^*\mathcal{E} = \widetilde{\mathcal{E}}$, *then* $\mu_{\max}(\widetilde{\mathcal{E}}) - \mu_{\min}(\widetilde{\mathcal{E}}) \leqslant (r-1)(2g-2)$.

Proof. Let $C^{(1)} = C \times_k k$ be the base change of C by the pth power map of k and $\mathcal{E}^{(1)}$ the pullback of $\mathcal{E}$, then $\mathcal{E}^{(1)}$ is semistable and $\widetilde{\mathcal{E}} = F_C^*\mathcal{E}^{(1)}$ for the geometric Frobenius $F_C: C \to C^{(1)}$. Let $0 = \widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{E}}_1 \subset \cdots \subset \widetilde{\mathcal{E}}_n = \widetilde{\mathcal{E}}$ be the Harder–Narasimhan filtration of $\widetilde{\mathcal{E}}$ and consider the exact sequence $0 \to \widetilde{\mathcal{E}}_i \to \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i \to 0$. Put $\mathbb{P}_1 = \mathbb{P}(\widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i)$, $\widetilde{\mathbb{P}} = \mathbb{P}(\widetilde{\mathcal{E}})$ and $\mathbb{P} = \mathbb{P}(\mathcal{E}^{(1)})$, then we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{P}_1 & \xrightarrow{\iota} & \widetilde{\mathbb{P}} & \xrightarrow{\widetilde{F}_C} & \mathbb{P} \\ & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ & & C & \xrightarrow{F_C} & C^{(1)}. \end{array}$$

The map $\widetilde{F}_C$ determines a line subbundle $\mathcal{F}$ of $\mathcal{T}_{\widetilde{\mathbb{P}}}$ such that $\mathbb{P} = \widetilde{\mathbb{P}}/\mathcal{F}$ and $\mathcal{F} = \tilde{f}^*\mathcal{T}_C$. By the exact sequence $0 \to \mathcal{T}_{\mathbb{P}_1} \to \iota^*\mathcal{T}_{\widetilde{\mathbb{P}}} \to N_{\mathbb{P}_1/\widetilde{\mathbb{P}}} \to 0$, one gets a nontrivial map $\iota^*\mathcal{F} \to N_{\mathbb{P}_1/\widetilde{\mathbb{P}}}$. Otherwise, $\iota^*\mathcal{F}$ gives a line subbundle of $\mathcal{T}_{\mathbb{P}_1}$, which is a smooth 1-foliation on $\mathbb{P}_1$, and $\mathbb{P}_1$ maps p-to-1 to its image in $\mathbb{P}_1$ giving a subscroll of $\mathbb{P}$ that destabilizes $\mathcal{E}^{(1)}$ (see [SB]). Thus we have $\tilde{f}_1^*\mathcal{T}_C \hookrightarrow \mathcal{O}_{\mathbb{P}_1}(1) \otimes \tilde{f}_1^*(\widetilde{\mathcal{E}}_i^{\vee})$, where $\tilde{f}_1 = \tilde{f} \cdot \iota$ is the natural projection $\mathbb{P}(\widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i) \to C$, namely $\mathcal{T}_C \hookrightarrow \widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i \otimes \widetilde{\mathcal{E}}_i^{\vee}$, which implies that $\mathrm{Hom}(\mathcal{T}_C \otimes \widetilde{\mathcal{E}}_i, \widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i) \neq 0$. Thus Lemma 3.2 gives us the following inequality $\mu_{\min}(\mathcal{T}_C \otimes \widetilde{\mathcal{E}}_i) - \mu_{\max}(\widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i) \leqslant 0$.

On the other hand, it is easy to see that

$$\mu_{\min}(\mathcal{T}_C \otimes \widetilde{\mathcal{E}}_i) = \mu(\widetilde{\mathcal{E}}_i/\widetilde{\mathcal{E}}_{i-1}) + 2 - 2g, \qquad \mu_{\max}(\widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i) = \mu(\widetilde{\mathcal{E}}_{i+1}/\widetilde{\mathcal{E}}_i),$$

if one notes that $0 = \widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{E}}_1 \subset \cdots \widetilde{\mathcal{E}}_{i-1} \subset \widetilde{\mathcal{E}}_i$,

$$0 = \widetilde{\mathcal{E}}_i/\widetilde{\mathcal{E}}_i \subset \widetilde{\mathcal{E}}_{i+1}/\widetilde{\mathcal{E}}_i \subset \cdots \subset \widetilde{\mathcal{E}}_n/\widetilde{\mathcal{E}}_i = \widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i$$

are the Harder–Narasimhan filtrations of $\widetilde{\mathcal{E}}_i$ and $\widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i$. Write $\mu(\widetilde{\mathcal{E}}_i/\widetilde{\mathcal{E}}_{i-1}) = \mu_i$ and we have

$$\mu_{\max}(\widetilde{\mathcal{E}}) - \mu_{\min}(\widetilde{\mathcal{E}}) = \mu_1 - \mu_n = \sum_{i=1}^{n}(\mu_i - \mu_{i+1}) \leqslant (n-1)(2g-2),$$

which proves the theorem since $n \leqslant rk(\mathcal{E})$.

Remark 3.1. Our bound is an improvement of a result of N.I. Shepherd–Barron and the proof is a modification of [SB]. When $g = 1$, this gives another proof of

https://doi.org/10.1023/A:1001512029096 Published online by Cambridge University Press

our Theorem 2.1 in the case of vector bundles. In the case of rank two, the bound coincides with that of H. Lange and U. Stuhler (see [LS], Satz 2.4).

Acknowledgements

I would like to express my thanks to Professor M. S. Narasimhan, who suggested this problem and spent much time explaining the mathematics, and Professor M. S. Raghunathan, who recommended some useful references about nonreduced parabolic group schemes and had some helpful discussions with me about Section 1. This work was done during my stay at the Mathematics Section of ICTP as a visiting fellow. I thank the ICTP for its hospitality and financial support. Finally, I thank the referee very much for his (or her) helpful comments.

References

[Be] Behrend, K. A.: Semi-stability of reductive group schemes over curves, *Math. Ann.* **301** (1995), 281–305.

[Ek] Ekedahl, T.: Foiliations and inseparable morphisms, *Proc. Sympos. Pure Math.* **46** (1987), 139–149.

[HL] Haboush, W. and Lauritzen, N.: Varieties of unseparated flags, *Contemp. Math*. **153** (1993), 35–57.

[KN] Kumar, S. and Narasimhan, M. S.: Picard group of the moduli spaces of G-bundles, *Math. Ann.* **308** (1997), 155–173.

[La] Lauritzen, N.: Embeddings of homogeneous spaces in prime characteristics, *Amer. J. Math.* **118** (1996), 377–387.

[LS] Lange, H. and Stuhler, U.: Vectorbündel auf kurven and darstellungen der algebraischer fundamentalgruppe, *Math. Z.* **156** (1977), 73–83.

[Mi] Milne, J. S.: *Étale Cohomology*, Princeton University Press, 1980.

[Ri] Ramanathan, A.: Stable principal bundles on a compact Riemann surface, *Math. Ann.* **213** (1975), 129–152.

[R2] Ramanathan, A.: Moduli for principal bundles over algebraic curves (I), *Proc. Indian Acad. Sci. (Math. Sci)* **106**(3) (1996), 301–328; (II), *Proc. Indian Acad. Sci. (Math. Sci.)* **106**(4) (1996), 421–449.

[RR] Ramanan, S. and Ramanathan, A.: Some remarks on the instability flag, *Tohoku Math. J.* **36** (1984), 269–291.

[SB] Shepherd-Barron, N. I.: Miyaoka's theorems on the generic seminegativity of T_X and on the Kofaira dimension of minimal regular threefolds, *Astérisque* **211** (1992), 103–124.

[Se] Serre, J. P.: *Espace Fibrés Algébriques*, Anneaux de Chow et applications, Séminairs C. Chevalley E.N.S. (1958), 1–37.

[SGA3] Demazure, M. and Grothendieck, A.: *Schémas en groupes*, Lecture Notes in Math. 151, 152, 153, Springer, Berlin, 1970.

[W1] Wenzel, C.: Classification of all parabolic subgroup schemes of a reductive linear algebraic group over an algebraic closed field, *Trans. Amer. Math. Soc*. **337** (1993), 211–218.

[W2] Wenzel, C.: Rationality of G/P for a nonreduced parabolic subgroup scheme P, *Proc. Amer. Math. Soc*. **117** (1993), 899–904.