ESTIMATION OF THE LUNAR PHYSICAL LIBRATIONS

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Abstract. The recent long-term integration of JPL ephemeris DE403/LE403 yielded lunar physical librations covering 6000 years. A Fourier analysis of a 718-year subset of this span produced estimates of the component frequencies of the forced and free librations. A subsequent iterative least-squares estimation procedure provided precise values for phases and for time-varying amplitudes and frequencies. Two free libration modes were found; presence of a third is possible but close to the noise.

1. Introduction

Because of aspherical distribution of the lunar mass, the rotation of the Moon is not uniform. Departures from uniform rotation are called physical librations. There are two types of physical librations: *forced* librations, arising from time-varying torques on the lunar figure due to attraction by the Earth, Sun, and planets; and *free* librations, which are departures of the lunar angular position from an equilibrium state. In theory, the amplitudes, phases, and periods of the forced librations can be calculated from knowledge of the lunar figure, elastic deformation, and rotational dissipation. For the free librations only the periods can be calculated; the amplitudes and phases must be measured.

There are three modes of free libration, one in longitude and two in pole position. The longitude mode is a 2.9-year pendulum-like oscillation about the lunar polar axis. The two remaining modes describe the variation of the lunar pole position about that determined by the 18.6-year forced precession and the periodic forced librations. The first pole mode is called the *wobble mode* and is analogous to the Chandler wobble on the Earth. In a coordinate system rotating with the average sidereal rotation rate and orbital mean motion of the Moon, the polar axis traces out a prograde elliptical path with a 75-year period. The other pole-position mode is called

Celestial Mechanics and Dynamical Astronomy **66**: 21–30, 1997.

the *free precession* mode and corresponds to an 81-year motion of the pole in space.

Over time the free librations will damp out due to energy dissipation. If free librations exist, they result from geologically recent stimulation, such as impacts, passage through a resonance with a forced libration, or a possible core-mantle interaction.

A previous analysis of lunar laser ranging (LLR) data by Calame (1977) obtained the first detection of the free librations. Longer data spans and improved range accuracies now encourage a new analysis. Eckhardt (1981) published a comprehensive theory of the forced librations; results of this investigation are compared with that theory. Differences from Eckhardt's theory due to planets are partly accounted for by comparison with Moons (1984), where Eckhardt's theory is amended with a more extensive planetary treatment. A comprehensive treatment of the investigation summarized in this paper is given in Williams and Newhall (1997).

2. Mathematical Model

2.1. LIBRATION GEOMETRY

The angular position of the lunar rotation is specified by a set of Euler angles transforming between the mean equator and equinox system of J2000 and the selenographic principal-axis system. The three Euler angles are ϕ, the angle along the Earth's mean equator of J2000 from the equinox to the ascending node of the lunar equator; θ, the inclination of the lunar equator to the Earth's equator; and ψ, the angle along the lunar equator from the node to the selenographic prime meridian [see Figure 1(a)].

The computation of the these three angles is part of the integration of DE403/LE403. However, their second derivatives are not integrated directly; instead, their first derivatives are expressed in terms of the components of the body-fixed angular velocity vector $\boldsymbol{\omega}$:

$$\begin{aligned} \dot{\phi} &= (\omega_{x'} \sin\psi + \omega_{y'} \cos\psi)/\sin\theta \\ \dot{\theta} &= \omega_{x'} \cos\psi - \omega_{y'} \sin\psi \\ \dot{\psi} &= \omega_{z'} - \dot{\phi}\cos\theta \end{aligned} \tag{1}$$

where primes on the subscripts denote coordinates in the lunar principal-axis system. The formulation for the integration of $\boldsymbol{\omega}$ follows.

2.2. EQUATIONS OF MOTION

Central to the libration formulation is the assumption of a non-rigid, dissipative moon. In such a situation the lunar moment-of-inertia tensor is

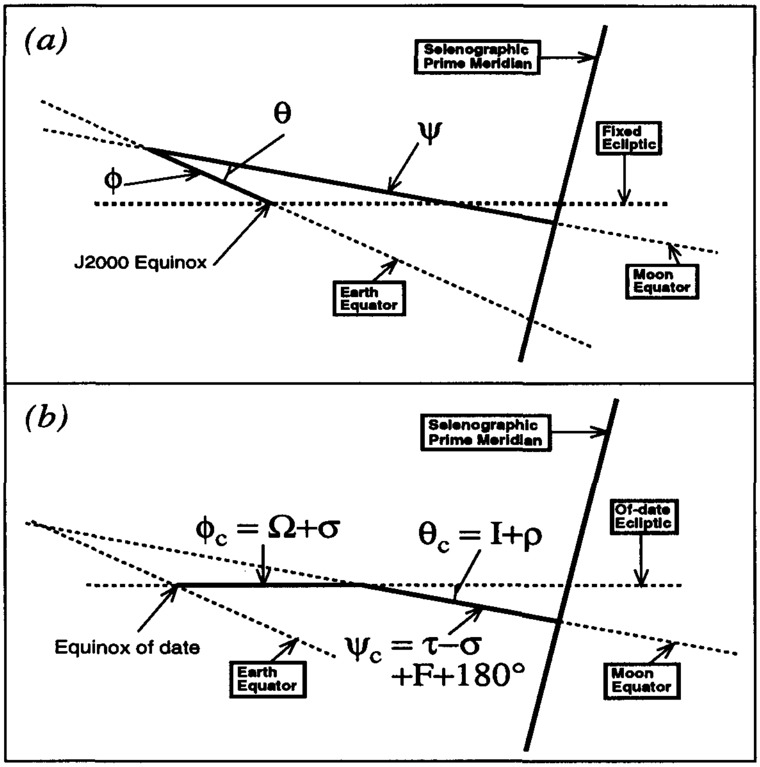

Figure 1. Part (a) shows the equatorial system in which libration Euler angles are integrated. [The value of the angle ϕ shown in this diagram is negative.] Part (b) defines the ecliptic system angles to which the equatorial angles were transformed and analyzed.

time-varying and is a function of the lunar state and rotation at a retarded or displaced time $t - \tau_m$, where τ_m is estimated from LLR data to be ~4 hours.

In space-fixed coordinates, the equation relating angular momentum $\mathbf{L}$ and torque $\mathbf{N}$ is $\dot{\mathbf{L}} = \mathbf{N}$. The angular velocity $\boldsymbol{\omega}$ is connected to $\mathbf{L}$ by $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, where $\mathbf{I}$ is the lunar moment-of-inertia tensor. Because $\mathbf{I}$ is expressed in selenographic (body-fixed) coordinates, we must express all vectors and derivatives in that system. The relation between derivatives in the two systems is

$$\left.\frac{d}{dt}\right|_{\text{selenographic}} = \left.\frac{d}{dt}\right|_{\text{space}} - \boldsymbol{\omega} \times$$

The torque equation in selenographic coordinates becomes

$$\frac{d}{dt}(\mathbf{I}\boldsymbol{\omega}) = \mathbf{N} - \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega}$$

https://doi.org/10.1017/S0252921100046339 Published online by Cambridge University Press

from which we get

$$\mathbf{I}\dot{\boldsymbol{\omega}} = -\dot{\mathbf{I}}\boldsymbol{\omega} - \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} + \mathbf{N} \tag{2}$$

2.3. DEFINITION OF THE INERTIA TENSOR

The lunar inertia tensor is modeled as consisting of a diagonal portion corresponding to the rigid-body contribution and two time-varying components describing the effects of elasticity and dissipation. The complete expression for the inertia tensor is:

$$\begin{aligned}
\mathbf{I} &= G\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \\
&- \frac{k_{2m}\mu_e R_m^5}{r_{me}^5}\begin{bmatrix} x^2 - \frac{1}{3}r_{me}^2 & xy & xz \\ xy & y^2 - \frac{1}{3}r_{me}^2 & yz \\ xz & yz & z^2 - \frac{1}{3}r_{me}^2 \end{bmatrix} \\
&+ \frac{k_{2m}R_m^5}{3}\begin{bmatrix} \omega_x^2 - \frac{1}{3}(\omega^2 - n^2) & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & \omega_y^2 - \frac{1}{3}(\omega^2 - n^2) & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & \omega_z^2 - \frac{1}{3}(\omega^2 + 2n^2) \end{bmatrix}
\end{aligned}$$

The first term is the rigid-body component, with the customary principal moments of inertia A, B, and C. The second term expresses the effect of the Earth's tidal deformation of the lunar moment of inertia, and the third matrix accounts for the deformation due to lunar rotation. Here, G is the gravitational constant [it appears explicitly or implicitly as a factor in every term], k_{2m} is the lunar Love number; μ_e is G times the mass of the Earth; R_m is the equatorial radius of the Moon; r_{me} is the Earth-Moon distance; x, y, and z are the components of the Earth-Moon vector expressed in the selenographic system; ω_x, ω_y, and ω_z are the components of $\boldsymbol{\omega}$ in the selenographic system; and n is the average lunar mean motion. It must be stressed that all variable quantities in $\mathbf{I}$ are evaluated at time $t - \tau_m$, and that the torque $\mathbf{N}$ depends on $\mathbf{I}$.

3. Integration and Transformation of Librations

3.1. NUMERICAL INTEGRATION

The lunar physical librations ϕ, θ, and ψ defined above were integrated simultaneously with the lunar and planetary equations of motion during the creation of DE403/LE403. During the integration the derivatives $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ were computed from Equations (1); the derivatives of the components of $\boldsymbol{\omega}$ were obtained by solving Equation (2). The starting epoch for the inte-

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gration was JED 2440400.5 [June 28, 1969]; the integration was extended forward to the year 3000 and backwards to the year −3000.

3.2. TRANSFORMATION OF THE LIBRATIONS

The object of this effort was to estimate periodic components of the librations. The quantities used by Eckhardt are denoted τ, ρ, and $I\sigma$ and are based on the of-date ecliptic system. The equator-system Euler angles ϕ, θ, and ψ were transformed to the ecliptic system, giving new Euler angles ϕ_c, the angle from the equinox of date to the *descending* node of the lunar equator on the ecliptic; θ_c, the inclination of the lunar equator to the ecliptic; and ψ_c, the angle along the lunar equator from its descending node on the ecliptic to the selenographic prime meridian [see Figure 1(b)].

3.2.1. *The Parameter* ρ

One of the parameters analyzed is derived from θ_c. The angle θ_c is modeled as $\theta_c = I + \rho$, where I is a linear polynomial and ρ is defined to be the periodic part of the solution.

3.2.2. *The Parameter* $I\sigma$

Because the inclination θ_c is small ($I \approx 5553\overset{''}{.}6 \approx 1\overset{\circ}{.}542 \approx 0.0269$ rad, and $|\rho| \lesssim 130''$, if axis offsets are ignored), the angles ϕ_c and ψ_c are poorly defined, so direct analysis of these two angles is subject to error. To select parameters immune to small-angle problems, we note that θ_c is the angular displacement (colatitude) of the instantaneous selenographic north pole from the ecliptic pole, and that the instantaneous longitude of the selenographic pole is $\phi_c + 90°$. Uncertainties in the value of ϕ_c are reduced by the factor $\sin\theta_c$ when mapped to uncertainties in pole position. Therefore, because $\sin\theta_c \approx \sin I \approx I$, the quantity analyzed is $I\phi_c$.

The ascending node of the lunar orbit on the ecliptic precesses retrograde with a period of 18.6 years. The descending node of the selenographic equator on the ecliptic is locked to within a maximum of about $1\overset{\circ}{.}3$ of the ascending orbit node and oscillates around it. We assume $\phi_c = \Omega + \sigma$, where Ω is the longitude of the orbit node and σ is the instantaneous difference between the two nodes. We model $I\phi_c = I\Omega + I\sigma$ as consisting of a cubic polynomial plus periodic terms and define $I\sigma$ to consist of a constant and the periodic part of the solution.

3.2.3. *The Parameter* τ

The third parameter estimated is derived from ψ_c. The angle ψ_c suffers the same small-angle indeterminacy as ϕ_c, but with the opposite sign on σ: $\psi_c = \tau - \sigma + F + 180°$, where F is the lunar argument of latitude. We note that σ is absent from the sum $\phi_c + \psi_c = \tau + \Omega + F + 180°$. As with $I\phi_c$,

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we model $\phi_c + \psi_c$ as a cubic plus periodic terms and define τ to consist of a constant and the periodic part of the result.

4. Estimation of the Libration Parameters

4.1. FOURIER ANALYSIS

To estimate the components of the parameters $I\phi_c$, θ_c, and $\phi_c+\psi_c$, a general model was needed. If we let f represent any of those three quantities, the general form for f was taken to be

$$f = \sum_{j=0}^{m} a_j t^j + \sum_{i=1}^{n} (1 + \epsilon_i t)[C_i \cos(\omega_i t + \nu_i t^2) + S_i \sin(\omega_i t + \nu_i t^2)] \qquad (3)$$

where $m = 1$ for θ_c and $m = 3$ for $I\phi_c$ and $\phi_c + \psi_c$. The coefficients ϵ_i are introduced to allow for a time-varying amplitude, and the ν_i allow the estimation of variable frequencies. For each quantity $I\phi_c$, θ_c, and $\phi_c+\psi_c$ separately we estimate a set of a_j, C_i, S_i, and ω_i, deferring the determination of any ϵ_i and ν_i until later.

The estimation procedure went as follows: First, a set of 131,072 (2^{17}) values of $I\phi_c$, θ_c, and $\phi_c + \psi_c$ was obtained by reading and transforming the equator-system librations from the ephemeris file at two-day intervals, giving a total span of 262,144 days (~718 years). Next, treating each quantity separately, the polynomial coefficients a_j were least-squares estimated. Then an iterative loop was entered:

1. The polynomial coefficients a_j and whatever Fourier amplitudes C_i and S_i and frequencies ω_i had been determined in the previous iteration were used to evalute Equation (3) at each of the 131,072 points in the set of derived libration quantities. These values were subtracted from the original values, giving the residuals $(O - C)$.
2. A Fast-Fourier Transform was done on the residuals, and the approximate frequencies ω_i corresponding to the six largest amplitudes were obtained from examination of the spectra.
3. Using these frequencies as input, partial derivatives of the quantity being analyzed [Equation (3)] with respect to the Fourier coefficients C_i and S_i were generated, and the C_i and S_i were estimated simultaleously with new values of the polynomial terms and of any previous C_i, S_i, and ω_i.
4. Partial derivatives for Equation (3) with respect to ω_i were generated, and the new set of ω_i was estimated simultaneously with all other parameters obtained up to this point.
5. Control returned to Step 1 of this loop, and iteration continued until a sufficient number of parameters had been estimated to cover all amplitudes of interest.

4.2. VARIABLE AMPLITUDES AND FREQUENCIES

In a few cases, the post-fit Fourier spectra exhibited noticeable residual amplitudes near frequencies that had been estimated. The theory of lunar motion and librations suggests that in some cases the Fourier amplitudes are variable, and in others the frequency variation is large enough to estimate.

Libration theory requires that the frequencies for $I\sigma$ and ρ be identical; therefore, frequencies common to both sets of ω_i were constrained during estimation to be equal. The fourth largest term in both sets has a period of 6797 days (18.61 years). The period of node Ω of the lunar orbit (and of the lunar equator) on the ecliptic is 6798.4 days. There are two unresolved lines, and their relative motion causes the amplitude of the blended pair to exhibit a secular change. The quantity ϵ_4 was estimated for both $I\sigma$ and ρ, with the constraint that they be equal. The result was $\epsilon_4 = -.00752$/cy. The Fourier post-fit residuals for this frequency disappeared when ϵ_4 was estimated.

The three largest amplitudes also exhibited significant post-fit Fourier spectra at their estimated frequencies. Theory shows that these terms arise from l, the lunar mean anomaly, and F, the lunar argument of latitude (see Table 1).

TABLE 1. Terms with estimated variable frequencies.

$I\sigma$ Amp [″]	ρ Amp [″]	Period [days]	Source	ν_i [″/cy^2]
101.33	99.02	27.55	l	32.01
78.91	78.95	27.21	F	−12.61
24.58	24.65	26.88	$2F-l$	−57.22

The estimated quadratic frequencies given in the last column of Table 1 are in close agreement with the expected quadratic components of the corresponding quantities in column 3.

A similar analysis for τ yields a variable amplitude of -0.002511/cy at the period 365.2596 days, due to varying eccentricity of the Earth's orbit, and another variable amplitude of -0.002055/cy at the period 6798.37 days arising from a 31000-year beat period between two closely spaced lines. A variable frequency large enough to be reliably estimated occurred for the term with period 1095.175 days. Its value is $-80\overset{''}{.}006/\text{cy}^2$; it arises from the lunar term $(2F-2l)$.

5. Estimated Librations

The iterative procedure described above was followed to extract the periodic components of the three parameters $I\sigma$, ρ, and τ. The number of terms and the minimum amplitude estimated are shown in Table 2.

https://doi.org/10.1017/S0252921100046339 Published online by Cambridge University Press

TABLE 2. Libration terms estimated.

Parameter	Smallest Amplitude Estimated	Number of Terms
$I\sigma$	$0\rlap{.}''022$	31
ρ	$0\rlap{.}''022$	30
τ	$0\rlap{.}''155$	29

5.1. FORCED LIBRATIONS

For both of $I\sigma$ and ρ, all of the estimated forced-libration terms arise from Earth-Moon-Sun effects, where theory identifies the only sources as various combinations of the angles from lunar theory l, l', F, D, and Ω. For τ, eight of the estimated terms are identified as having contributions from planetary effects.

5.1.1. *Terms with Lunar-theory Argument*

There was disagreement between only one of the estimated lunar-argument forced terms and the corresponding terms from Eckhardt's (1981) theory. The effects on the lunar orbit of the Earth's J_2, the rotation of the ecliptic, and Earth-Moon figure-figure interaction lead to a term actually composed of two periods approximately four days apart whose combination is 6797.025 days. The beat period between these two terms is close to 31,000 years. This case is where a variable amplitude was estimated for all three libration parameters. The differences between the amplitudes estimated here and those in Eckhardt's (1982) theory are $0\rlap{.}''25$ for $I\sigma$, $0\rlap{.}''37$ for ρ, and $0\rlap{.}''47$ for τ. [It should be noted that, because of its nature, the variable amplitude cannot account for the these amplitude differences.]

5.1.2. *Planetary Terms*

The effects of the orbits of Venus (V), Earth (T), Mars (M), and Jupiter (J) arise in some of the estimated forced libration terms for the variable τ. Most of the estimated amplitudes of those terms are somewhat smaller than Eckhardt's (1982) theory predicts, though there is one severely discrepant term with argument $2T - 2J + 2D - 2l$ whose estimated amplitude $0\rlap{.}''257$ is 62% larger than the $0\rlap{.}''159$ value from the theory. The remaining term involving Jupiter has the argument $3J - 2T - 2D + 2l$; the estimated amplitude $0\rlap{.}''287$ is 10% smaller than the theory value of $0\rlap{.}''320$. Moons' (1984) independent theory (amplitudes $0\rlap{.}''253$ and $0\rlap{.}''285$, respectively) has better agreement with our estimated terms and shows that the discrepancies involving Jupiter's effects is due to the fact that Eckhardt did not apply perturbations by Jupiter on the radial coordinate of the lunar orbit.

5.2. FREE LIBRATIONS

Free librations are implicit in the equations of motion for the rotation of the Moon. The estimation process produced nine frequencies that have no counterpart in the results from theory and hence that are possible components of free librations.

5.2.1. *Wobble Mode*

For each of the quantities $I\sigma$ and ρ, there are three terms that are combined to produce the elliptical path of the principal body axis rotating at the sidereal rate. The semi-axes of the ellipse are $8\overset{''}{.}19$ and $3\overset{''}{.}31$, with the major axis parallel to the lunar principal y-axis of inertia. The wobble period is 74.63 years. There are no nearby forced terms.

5.2.2. *Precession Mode*

Detection of a precession-mode free libration is less certain. Fits to $I\sigma$ and ρ separately produce a term in each with amplitude $0\overset{''}{.}022$, which is above the noise of the spectra but close to that of the LLR data. When these terms are combined, the result in a coordinate frame rotating with the retrograde node rate is that the actual pole describes a prograde circular motion about its mean position with radius $0\overset{''}{.}022$ (about 18 cm at the lunar surface) and a period of 24.16 years. In a space-fixed frame, the motion is retrograde with a period of 80.77 years.

5.2.3. *Longitude Mode*

Results for the free libration in longitude are less transparent. The theoretical value for the longitude period is approximately 1056.1 days. However, there are two forced libration terms that arise from the perturbation of the lunar orbit by Venus; their periods are 1056.342 days and 1056.415 days, too close to separate with the 718-year span of the analysis.

The term in τ exhibiting a possible blend of free and forced librations has period 1056.197 days, amplitude $1\overset{''}{.}807$, and phase 224° at epoch J2000. To estimate a minimum amplitude for the free-libration component, a sequence of free-libration periods between 1055.80 days and 1056.30 days was established, and the corresponding amplitudes and phases of the two Venus forced terms were derived using Eckhardt's (1982) theory. In each case the derived forced quantity was vectorially subtracted from the blend, leaving the free libration amplitude and phase resulting from that assumed period.

In no case could the theory accept an assumed free period and yield values of forced amplitude and phase that completely account for the estimated blend term. By matching the apparent period, the best estimate of the value of the free value gives an amplitude of $1\overset{''}{.}4$ at period 1056.12 days.

5.3. IMPLICATIONS OF FREE LIBRATIONS

The damping times for librations in pole position are typically 10^5–10^6 years or more; that for the librations in longitude is $\sim 10^4$–10^5 years, implying the necessity for some sort of excitation mechanism to account for the findings of this paper. Proposed mechanisms include:

Impact by a Large Body. Large impacts are statistically unlikely, all the more so when constrained to have conditions that would produce a small precession-mode libration.

Passage Through a Resonance. Eckhardt (1993) proposed that over long times the slowly varying frequency of a Venus forced term could assume the natural frequency of the longitude libration and serve as a resonant longitude stimulation [two such terms were found in this study]. However, there are no candidate terms for the pole-position librations.

Existence of a Liquid Lunar Core. Yoder (1981) has proposed that the Moon has a liquid core, with attendant turbulent coupling with the mantle at the core-mantle boundary [see also Dickey *et al.* (1994)]. The core rotation axis would not in general be aligned with the 18.6-year precession of the mantle; the offset might stimulate the pole-position librations. Therefore, it appears that a combination of two excitation sources are needed to account for the observed free librations.

Acknowledgements. The research described in this paper was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

References

Calame, O.: 1977, "Free librations of the Moon from Lunar Laser Ranging", in: *Scientific Applications of Lunar Laser Ranging* (J. D. Mulholland, ed.), Reidel, Dordrecht/Boston, pp. 53–63.

Dickey, J. O., Bender, P. L., Faller, J. E., Newhall, X X, Ricklefs, R. L., Ries, J. G., Shelus, P. J., Veillet, C., Whipple, A. L., Wiant, J. R., Williams, J. G., and Yoder, C. F.: 1994, "Lunar Laser Ranging: A Continuing Legacy of the Apollo Program", *Science* **265**, 482–490.

Eckhardt, D. H.: 1981, "Theory of Libration of the Moon", *The Moon and the Planets* **25**, 3–49.

Eckhardt, D. H.: 1982, "Planetary and Earth-Figure Perturbations in the Librations of the Moon", in *High-Precision Earth Rotation and Earth-Moon Dynamics* (O. Calame, ed.), Reidel, Dordrecht/Boston, pp. 193–198.

Eckhardt, D. H.: 1993, "Passing Through Resonance: The Excitation and Dissipation of the Lunar Free Libration in Longitude", *Celest. Mech. & Dyn. Astron.* **25**, 3–49.

Moons, M.: 1984, "Planetary Perturbations on the Libration of the Moon", *Celest. Mech.* **34**, 263–273.

Williams, J. G. and Newhall, X X: 1997, "The Lunar Free Librations: Why Isn't the Rotation Quiescent?", to appear.

Yoder, C. F.: 1981, "The free librations of a dissipative Moon", *Phil. Trans. R. Soc. Lond.* Series A **303**, 327–338.