# On the meaning of an equation in dual coordinates.

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If $L$, $M$, $N$ denote Prof. Study's Dual Coordinates of a straight line (see *Proc. Edinburgh Math. Soc.*, 44 (1926), 90-97), any (homogeneous) equation $F(L, M, N) = 0$ must define a certain system of lines. By the nature of dual numbers we must have

$$F(L, M, N) \equiv U + \epsilon V,$$

where $U$ and $V$ are functions of $l$, $m$, $n$, $\lambda$, $\mu$, $\nu$, the ordinary (Pluckerian) coordinates. Since $F = 0$ implies $U = 0$ and $V = 0$ the system of lines is a congruence. But it is a congruence of a very special kind, whose nature will now be considered.

*A*. When $F$ is an ordinary homogeneous function, *i.e.* when the constants contained in $F$ are ordinary, not dual, numbers, $F(x, y, z) = 0$ is the equation of a cone with the origin as vertex, and

$$F(L, M, N) \equiv F(l, m, n) + \epsilon\left(\lambda \frac{\partial F}{\partial l} + \mu \frac{\partial F}{\partial m} + \nu \frac{\partial F}{\partial n}\right).$$

Hence if $F(L, M, N)$ vanishes, we have both

$$F(l, m, n) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

which shows that the line is parallel to a generator of the cone $F(x, y, z) = 0$; and

$$\lambda \frac{\partial F}{\partial l} + \mu \frac{\partial F}{\partial m} + \nu \frac{\partial F}{\partial n} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

or

$$(yn - zm)\frac{\partial F}{\partial l} + (zl - xn)\frac{\partial F}{\partial m} + (xm - yl)\frac{\partial F}{\partial n} = 0 \quad \ldots\ldots (3)$$

if $(x, y, z)$ is any point of the line whose direction-ratios are $(l, m, n)$. But this is equation of the plane of normals to the cone $F(x, y, z) = 0$ at points of the generator. *The congruence of lines is composed of all the normals of the cone with vertex O reciprocal to the cone* $F(x, y, z) = 0$.

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The lines of the congruence all touch the cone enveloped by the planes (2). The focal surface of the congruence consists of this cone and the curve in which the cone $F(x, y, z) = 0$ meets the plane at infinity.

*B.* Dual numbers may be present not only in the current coordinates $L$, $M$, $N$, but in the function $F$. If $F$ is algebraic, each coefficient may be a dual number, and we must replace the $F$ of the preceding paragraph by

$$F_1 + \epsilon F_2$$

where $F_1$, $F_2$, are of the same order, and do not contain $\epsilon$. If $F$ is not algebraic it must also take this form, the whole being homogeneous in $L$, $M$, $N$: there is no other restriction on $F_1$ and $F_2$. The vanishing of $F$ now implies

$$F_1(l, m, n) = 0 \quad \dots\dots\dots\dots\dots\dots\dots (4)$$

$$\lambda \frac{\partial F_1}{\partial l} + \mu \frac{\partial F_1}{\partial m} + \nu \frac{\partial F_1}{\partial n} + F_2 = 0 \quad \dots\dots\dots\dots\dots (5)$$

The former shows that all the lines of the congruence are parallel to the generators of a cone $F_1(x, y, z) = 0$; and the latter shows that when values of $l$, $m$, $n$ satisfying (4) have been chosen, the lines having this direction lie in a plane (5) and touch the developable surface enveloped by these planes. The focal surface of the congruence consists of this developable and the curve at infinity defined by (4). A concise way of describing the lines of the system is to say that they are *tangents of a set of parallel geodesics on the developable.* When such a surface is developed into a plane, the geodesics become straight lines, and parallel geodesics are curves which become parallel lines. One such curve determines the complete congruence, for there is only one developable surface on which a given curve is geodesic, viz. that enveloped by planes which contain the tangent and binormal at each point. It will be seen also that when the coordinates of points on the given curve, and as a consequence the dual coordinates of its tangent lines, are known as functions of a parameter $t$, the dual coordinates of other lines of the system are obtained by substituting the dual number $t_1 + \epsilon t_2$ for $t$.