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WICKSELL'S PROBLEM IN LOCAL STEREOLOGY

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Abstract

Wicksell's classical corpuscle problem deals with the retrieval of the size distribution of spherical particles from planar sections. We discuss the problem in a local stereology framework. Each particle is assumed to contain a reference point and the individual particle is sampled with an isotropic random plane through this reference point. Both the size of the section profile and the position of the reference point inside the profile are recorded and used to recover the distribution of the corresponding particle parameters. Theoretical results concerning the relationship between the profile and particle parameters are discussed. We also discuss the unfolding of the arising integral equations, uniqueness issues, and the domain of attraction relations. We illustrate the approach by providing reconstructions from simulated data using numerical unfolding algorithms.

Keywords: Wicksell's corpuscle problem; local stereology; inverse problem; numerical unfolding; stereology of extremes

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1. Introduction and account of main results

In this work we discuss Wicksell's corpuscle problem in a local stereology framework, where the size distribution of spherical particles is recovered from plane sections through reference points. In order to describe the similarities and differences between the local and classical Wicksell problem, we start with a short outline of the latter.

Wicksell's classical corpuscle problem, described figuratively as the 'tomato salad problem' by Günter Bach, asks how to recover the size distribution of random balls in $\mathbb{R}^3$ from the observed size distribution of two-dimensional section profiles. Although the stereological literature often refers to ball-shaped particles as 'spheres', we decided to adopt terminology from pure mathematics, calling the solid particles 'balls', and reserving the word 'sphere' for the boundary of a ball. Assuming that the 'density of the centers and the distribution of the sizes being the same in all parts of the body [where the observation is taken]', Wicksell showed [35] that the density f_r of the profile radii r is given by

$$f_r(x) = \frac{x}{\mathbb{E}R} \int_x^\infty \frac{1}{\sqrt{y^2 - x^2}} \, \mathrm{d}F_R(y), \tag{1.1}$$

where F_R is the cumulative distribution function of the radius R of the balls in $\mathbb{R}^3$, and $\mathbb{E}R$ is the usual expectation of R. Stoyan and Mecke [19] made Wicksell's arguments rigorous,

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showing that (1.1) actually holds for all stationary particle processes of nonoverlapping balls, when R has a finite mean. The right-hand side of (1.1) is essentially an *Abel integral transform* of F_R. It can be inverted explicitly, and this shows in particular that F_R is determined by the cumulative distribution function F_r of r.

Relation (1.1) is the result of two mutually counteracting sampling effects. On the one hand, since the probability that a ball is hit by the plane is proportional to its radius, the radius distribution of the intersected balls is size weighted, preferring large balls. On the other hand, the profile radius is always smaller than the radius of the intersected ball, as the section plane almost surely (a.s.) misses the ball's center. Already Wicksell was aware of the fact that these two effects can annihilate each other. When R follows a Rayleigh distribution (the distribution of the length of a centered normally distributed two-dimensional vector), r also follows this distribution, with the same parameter. The Rayleigh distribution is the only *reproducing distribution* in this sense; see [8]. If R_w denotes the radius-weighted radii distribution with cumulative distribution function

$$F_{R_w}(x) = \frac{1}{\mathbb{E}R}\left(xF_R(x) - \int_0^x F_R(s)\,\mathrm{d}s\right)$$

for $x \geq 0$, the two sampling effects can also be expressed by the relation

$$r = \Lambda R_w, \tag{1.2}$$

where Λ is a random variable with Lebesgue density $s \mapsto \mathbf{1}_{[0,1]}(s)s/\sqrt{1-s^2}$ that is independent of R_w; see [4]. This also gives the well-known *moment relations*

$$\tilde{m}_k = c_{k+1}\frac{\tilde{M}_{k+1}}{\tilde{M}_1}, \qquad k = -1, 0, 1, 2, \ldots, \tag{1.3}$$

where $\tilde{m}_k$ and $\tilde{M}_k$ are the kth moments of r and R, respectively, and

$$c_k = \frac{\sqrt{\pi}}{2}\frac{\Gamma((k+1)/2)}{\Gamma(k/2+1)}. \tag{1.4}$$

The inversion of the integral equation (1.1) is ill-posed, which can informally be described as 'small deviations of the data can lead to arbitrarily large deviations of the solution'. There exist several methods, both distribution free (nonparametric) and parametric, for numerically solving Wicksell's classical problem. Examples of distribution-free methods are finite difference methods, spectral differentiation, product integration methods, and kernel methods. Parametric methods can be divided into maximum likelihood methods and methods that only use the moment relations. All methods have their advantages and disadvantages and none appears to be generally best. References for the respective methods and an overview of existing methods are given in, for example, [1], [2], [6], [7], and [26, Section 11.4.1].

Stereology of extremes has received much interest due to applications in material science; see [31] for an application to metallurgy. Studies of the maximum size of balls in Wicksell's classical problem are presented in [8], [28], [29], and [30]. These studies also show that F_R and F_r belong to the same type of extreme value distribution. Extremes of the size and shape parameters of spheroidal particles are studied in [5], [11], [12], and [13].

More detailed reviews of the classical Wicksell problem can be found in [7], [22, Chapter 6], [26, Section 11.4], and earlier contributions listed in these sources.

In the above description of Wicksell's problem, we adopted the common model-based approach, where the particle system is random, and the probe can be taken with an arbitrary (deterministic) plane due to stationarity. Jensen [14] proved that (1.1) also holds in a design-based setting, where the particle system is deterministic—possibly inhomogeneous—but the plane is randomized. In order to obtain a representative sample, it is enough to choose a fixed orientation uniform random (FUR) plane. Inspired by local stereology, we discuss here a design-based sampling scheme, where each particle contains a reference point and the individual particle is sampled with an isotropic plane through that reference point. This design is tailor-made for applications, e.g. in biology, where cells are often sampled using a confocal microscope by focusing on the plane through the nucleus or the nucleolus of a cell; see the monograph [16] on local stereology. To our knowledge, the first explicit mention of Wicksell's problem in a local setup is [15], where it is remarked that the problem is trivial whenever the reference point coincides with the ball's center, as the size distributions of balls and section profiles are then identical. This being obvious, we want to show here that the local Wicksell problem is far from trivial if this condition is violated.

Although biological application suggests restricting consideration to $\mathbb{R}^3$, we will consider particles in n-dimensional space intersected by hyperplanes, as the general theory does not pose any essential extra difficulties. As in the classical setting, we assume that the particles are (approximate) balls, and in order to incorporate the natural fluctuation, we assume that both the radius of the ball and the position of the reference point in the ball are random. This way, a ball-shaped particle is described by two quantities: its random radius R, the size of the ball, and the distance of the reference point from the center of the ball. It turns out to be favorable to work with the *relative distance* $Q \in [0, 1]$ instead, meaning that QR is the distance of the reference point to the ball's center. The variable Q can be considered as a *shape* descriptor. We do not take into account the direction of the reference point relative to the ball's center, as this direction appears to be of minor interest. In addition, it cannot be determined from isotropic sections, unless we would also register the orientation of the section plane (which we will not do here). If a ball with a given reference point is intersected by an isotropic hyperplane through the reference point (independent of the ball), an $(n-1)$-dimensional ball is obtained. We let r be its radius and q be the relative distance of the reference point to its center.

Our first result, Theorem 3.1, shows that the joint distributions of (r, q) and (R, Q) are connected by an explicit integral transform. Here and in the following we assume that all balls have a positive radius, and that the reference point a.s. does not coincide with the center of a ball, that is, we agree on $\mathbb{P}(Q = 0) = 0$. The last assumption is not essential, and most of our results can be extended to the case $\mathbb{P}(Q = 0) > 0$, as outlined in Remark 3.1. As in the classical Wicksell problem we show that the marginals of r and q always have probability densities f_r and f_q, respectively, and we determine their explicit forms in Corollary 3.2. However, the joint distribution of (r, q) need not have a density. Corollary 3.2 also shows that f_q depends only on the distribution of Q and not on R, which explains why we are working with *relative* distances. In Proposition 3.1 we show that

$$r = \Gamma R$$

with a random variable Γ, whose density can be given explicitly. This is in analogy to (1.2) in the classical case. However, there is no size weighting in our local stereological design, and the variable Γ is now depending on the distribution of Q. Thus, when R and Q are independent, so are R and Γ, and moment relations in analogy to (1.3) are readily obtained: if m_k and M_k are the kth moments of r and R, respectively, then

$$m_k = c_k(Q) M_k, \qquad k = 0, 1, 2, \ldots. \tag{1.5}$$

The constants $c_k(Q)$ depend on the distribution of Q and are given in Remark 3.2. However, (1.5) cannot be applied directly to obtain (estimates of) the moments M_k, since $c_k(Q)$ depends on the shape of the full particle. Corollary 5.1 shows that $c_k(Q)$ can be expressed by the distribution of q, making it possible to estimate both m_k and $c_k(Q)$ from the section profiles and thus to access M_k. Simulation studies showed that this estimation procedure is quite stable, as described after Corollary 5.1.

We then turn to uniqueness in Section 4. That the distribution of Q is uniquely determined by the distribution of q follows from the fact that the two distributions are connected by an Abel-type integral equation. It can be inverted explicitly; see Proposition 5.1 for $n = 3$. Theorem 4.1 shows that even the joint distribution of (R, Q) is uniquely determined by the distribution of the profile quantities (r, q), but only under the assumption that R and Q are stochastically independent. The two marginals of (r, q) do not uniquely determine (R, Q) without this extra assumption, as shown by an example after Theorem 4.1. It is an open problem whether the *joint distribution of* (r, q) determines the joint distribution of (R, Q) without the independence assumption.

To reconstruct F_R and F_Q from simulated data, when R and Q are independent, we chose to use distribution-free methods. Maximum likelihood methods and the method of moments can also be used though, as for the classical Wicksell problem. In Section 6 we describe the implementation of a Scheil–Schwartz–Saltykov-type method [24]. Following [7], the method can be classified as a finite difference method, more specifically a 'successive subtraction algorithm'. The data is grouped and the distributions F_Q and F_R discretized. Then F_q written in terms of F_Q, and F_r written in terms of $F_{R,Q}$ become systems of linear equations, which can be solved. We carried out a number of simulation studies which illustrate the feasibility of the approach. An example of a reconstruction is reported in Figure 2.

In Section 7 we discuss practical examples and then turn to stereology of extremes in Section 8. Similar results as in the classical case are obtained. Proposition 8.1 shows that if the particle parameters R and Q are independent then F_R and F_r belong to the same type of extreme value distribution. An analogous result holds for the shape parameters. To our knowledge, stereology of extremes in a local setting has only been treated in [23]. There the shape and size parameters of spheroids are studied but the isotropic section plane is always taken through the center of the spheroid.

As mentioned earlier, a number of results and the implemented reconstruction algorithm depend on the assumption that the particle parameters R and Q are independent. We comment on this assumption in the concluding Section 9.

2. Preliminaries

Throughout, we let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and O its origin. The Euclidean scalar product is denoted by $\langle\cdot,\cdot\rangle$ and the Euclidean norm by $\|\cdot\|$. We let e_i be the vector in $\mathbb{R}^n$ with 1 in the ith place and 0s elsewhere. For a set $Y \subseteq \mathbb{R}^n$, we define

$$Y + x = \{y + x \mid y \in Y\}, \quad x \in \mathbb{R}^n, \qquad \alpha Y = \{\alpha y \mid y \in Y\}, \quad \alpha > 0. \tag{2.1}$$

We use ∂Y for the boundary and $\mathbf{1}_Y$ for the indicator function of Y. The unit ball in $\mathbb{R}^n$ is $B_n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$ and its boundary is the unit sphere (in $\mathbb{R}^n$) $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. A ball in $\mathbb{R}^n$ of radius R centered at O is denoted by RB_n, in accordance to (2.1). We write σ_n for the surface area of the unit ball in $\mathbb{R}^n$, i.e. $\sigma_n = \mathcal{H}_n^{n-1}(S^{n-1})$, where $\mathcal{H}_n^d$ is the d-dimensional *Hausdorff measure* in $\mathbb{R}^n$. When n is clear from the context, we abbreviate $\mathcal{H}_n^d(\mathrm{d}u)$ by $\mathrm{d}u^d$.

For $p = 0, 1, \dots, n$, let

$$\mathcal{L}^n_{p[O]} = \{L^n_{p[O]} \subseteq \mathbb{R}^n : L^n_{p[O]} \text{ is a } p\text{-dimensional linear subspace}\},$$
$$\mathcal{L}^n_p = \{L^n_p \subseteq \mathbb{R}^n : L^n_p \text{ is a } p\text{-dimensional affine subspace}\},$$

be the families of all p-dimensional linear and affine subspaces of $\mathbb{R}^n$, respectively. These spaces are equipped with their standard topologies (see [25]) and we denote their Borel σ-algebras by $\mathcal{B}(\mathcal{L}^n_{p[O]})$ and $\mathcal{B}(\mathcal{L}^n_p)$, respectively. The spaces are furthermore endowed with their natural invariant measures, and we write $\mathrm{d}L^n_{p[O]}$ and $\mathrm{d}L^n_p$, respectively, when integrating with respect to these measures. We use the same normalization as in [25], i.e.

$$\int_{\mathcal{L}^n_{p[O]}} \mathrm{d}L^n_{p[O]} = 1.$$

A random subspace $L^n_{p[O]}$ is called *isotropic random* (IR) if and only if its distribution is given by

$$\mathbb{P}_{L^n_{p[O]}}(A) = \int_{\mathcal{L}^n_{p[O]}} \mathbf{1}_A \, \mathrm{d}L^n_{p[O]}, \qquad A \in \mathcal{B}(\mathcal{L}^n_{p[O]}).$$

Similarly, a random flat $L^n_p \in \mathcal{L}^n_p$ is called *isotropic uniform random* (IUR) *hitting a compact object* Y if and only if its distribution is given by

$$\mathbb{P}_{L^n_p}(A) = c \int_{\mathcal{L}^n_p} \mathbf{1}_{A \cap \{L^n_p \in \mathcal{L}^n_p :\, L^n_p \cap Y \neq \varnothing\}} \, \mathrm{d}L^n_p, \qquad A \in \mathcal{B}(\mathcal{L}^n_p),$$

where c is a normalizing constant. We let $x|L^n_{p[O]}$ be the orthogonal projection of $x \in \mathbb{R}^n$ onto $L^n_{p[O]}$. We furthermore adopt the convention of writing $v^\perp$ for the hyperplane with unit normal $v \in S^{n-1}$. We use $B(z; a, b)$ to denote the *incomplete beta function*, given by

$$B(z; a, b) = \int_0^z t^{a-1}(1-t)^{b-1} \, \mathrm{d}t, \qquad 0 \le z \le 1,\ a, b > 0.$$

When $z = 1$, we write $B(a, b)$. Note in particular that $B(\frac{1}{2}, (n-1)/2) = \sigma_n/\sigma_{n-1}$. For an arbitrary function f, we let $f^+(x) = \max\{f(x), 0\}$ be its positive part. Given a random variable X, its *characteristic function* (or Fourier transform of its distribution) is defined by

$$\varphi_X(t) = \mathbb{E}\mathrm{e}^{\mathrm{i}tX}, \qquad t \in \mathbb{R}.$$

3. The direct problem

Consider a random ball in $\mathbb{R}^n$ with positive radius, centered at O' and containing the origin. Let R and Q denote the random variables giving the radius of the ball, and the relative distance of the center of the ball from O, respectively. Intersect the ball with an IR hyperplane, $L^n_{n-1[O]}$, independent of the ball. Then an $(n-1)$-dimensional ball is obtained. Let r be its radius and $q = (1/r)\|O'|L^n_{n-1[O]}\|$ the relative distance of its center from O; see Figure 1. Note that r is a.s. positive.

When $Q = 0$, the ball is centered at the origin and all hyperplanes give equivalent $(n-1)$-dimensional balls of radius R. We exclude this throughout, i.e. we assume that

$$\mathbb{P}(Q > 0) = 1.$$

This assumption can easily be relaxed; see Remark 3.1. The cumulative distribution function $F_{(r,q)}$ of (r, q) is given in the following theorem.

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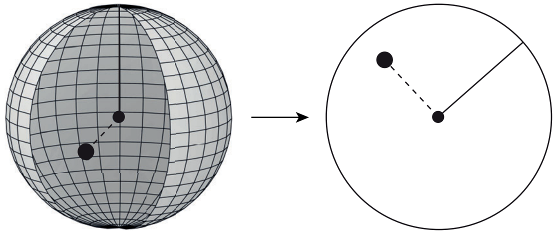

FIGURE 1: *Left*: $RB_3 + O'$ with reference point (large filled circle). The solid line segment has length R and the dashed line segment has length RQ. *Right*: section plane with profile. The solid line and dashed line segments have lengths r and rq, respectively.

Theorem 3.1. *Let $RB_n + O'$ be a random ball in $\mathbb{R}^n$ containing O with $\|O'\| = RQ$, and let*

$$Z = Z(R, Q, x, y) = \frac{1}{Q^2} \max\left\{ \frac{(R^2 - x^2)^+}{R^2}, \frac{(Q^2 - y^2)^+}{1 - y^2} \right\} \tag{3.1}$$

for $x \in [0, \infty)$ and $y \in [0, 1)$. If $L^n_{n-1[O]}$ is an IR hyperplane, independent of $RB_n + O'$, we have

$$F_{(r,q)}(x, y) = 1 - \frac{\sigma_{n-1}}{\sigma_n} \mathbb{E}\left[B\left(1 - (1 - Z)^+; \frac{1}{2}, \frac{n-1}{2}\right)\right] \tag{3.2}$$

for $x \geq 0$ and $0 \leq y < 1$. For $y = 1$, we obtain the marginal distribution function of r by

$$F_r(x) = F_{(r,q)}(x, 1) = 1 - \frac{\sigma_{n-1}}{\sigma_n} \mathbb{E}\left[B\left(1 - \left(1 - \frac{1}{Q^2 R^2}(R^2 - x^2)^+\right)^+; \frac{1}{2}, \frac{n-1}{2}\right)\right], \tag{3.3}$$

where $x \geq 0$.

Proof. To avoid confusion, we adopt the notation $\mathbb{E}_X$ and $\mathbb{E}_{X,Y}$ for the expectation with respect to the random variable X and the pair of random variables (X, Y), respectively. Assume without loss of generality that $O' = RQe_n$. (Otherwise, both $RB_n + O'$ and the section plane can be appropriately rotated. As the rotation is independent of the section plane, the rotated plane is still IR.) Let v be an isotropic vector on S^{n-1} representing the unit normal direction of $L^n_{n-1[O]}$. Applying Pythagoras' theorem, we obtain

$$r = R\sqrt{1 - Q^2 \langle e_n, v\rangle^2} \quad \text{and} \quad q = Q\sqrt{\frac{1 - \langle e_n, v\rangle^2}{1 - Q^2\langle e_n, v\rangle^2}}. \tag{3.4}$$

Using the conditional expectation, we have, for $x \in [0, \infty)$ and $0 \leq y \leq 1$,

$$\begin{aligned} &F_{(r,q)}(x, y) \\ &\quad = \mathbb{E}_{R,Q}\mathbb{E}_v\left[\mathbf{1}\left\{ R\sqrt{1 - Q^2\langle e_n, v\rangle^2} \leq x,\ Q\sqrt{\frac{1 - \langle e_n, v\rangle^2}{1 - Q^2\langle e_n, v\rangle^2}} \leq y \right\} \,\middle|\, R, Q\right] \\ &\quad = \mathbb{E}_{R,Q} \frac{1}{\sigma_n} \int_{S^{n-1}} \mathbf{1}\left\{ R\sqrt{1 - Q^2\langle e_n, v\rangle^2} \leq x,\ \frac{1 - \langle e_n, v\rangle^2}{1 - Q^2\langle e_n, v\rangle^2} \leq \frac{y^2}{Q^2} \right\} \mathrm{d}v^{n-1}. \end{aligned}$$

We use cylindrical coordinates [20, p. 1], writing $v = te_n + \sqrt{1 - t^2}\omega$, with $\omega \in S^{n-1} \cap e_n^\perp$ and $t \in [-1, 1]$. Using $\langle e_n, v\rangle = t$ and $\mathcal{H}^{n-2}_{n-1}(S^{n-1} \cap e_n^\perp) = \sigma_{n-1}$, the cumulative distribution

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function becomes

$$F_{(r,q)}(x,y)=\mathbb{E}_{R,Q}\frac{\sigma_{n-1}}{\sigma_n}\int_{-1}^{1}\mathbf{1}\bigg\{R\sqrt{1-Q^2t^2}\le x,\ \frac{1-t^2}{1-Q^2t^2}\le\frac{y^2}{Q^2}\bigg\}(1-t^2)^{(n-3)/2}\,\mathrm{d}t.$$

The integrand is an even function of t. This and a rearrangement of the indicator functions gives

$$\begin{aligned}F_{(r,q)}(x,y)=\frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}_{R,Q}\int_0^1\bigg(\mathbf{1}\bigg\{t\ge\frac{1}{QR}\sqrt{(R^2-x^2)^+},\ y=1\bigg\}&\\ +\mathbf{1}\{t\ge\sqrt{Z},\ y<1\}\bigg)(1-t^2)^{(n-3)/2}\,\mathrm{d}t,&\end{aligned}$$

where Z is given by (3.1). Using the substitution $s=t^2$, the cumulative distribution function of (r,q), under the assumption $0\le y<1$, becomes

$$\begin{aligned}F_{(r,q)}(x,y)&=\frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}_{R,Q}\int_{\min\{1,Z\}}^{1}s^{1/2-1}(1-s)^{(n-1)/2-1}\,\mathrm{d}s\\&=1-\frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}_{R,Q}\bigg[B\bigg(1-(1-Z)^+;\frac{1}{2},\frac{n-1}{2}\bigg)\bigg].\end{aligned}$$

Using similar calculations, we obtain (3.3) for the marginal distribution of r.

Remark 3.1. If $\mathbb{P}(Q>0)<1$ then the result of Theorem 3.1 and, similarly, results in the subsequent sections can be generalized by conditioning on the event $Q>0$. When $Q=0$, we have $r=R$, and, hence,

$$\begin{aligned}F_{(r,q)}(x,y)=\mathbb{P}(Q>0)\bigg(1-\frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}\bigg[B\bigg(1-(1-Z)^+;\frac{1}{2},\frac{n-1}{2}\bigg)\ \bigg|\ Q>0\bigg]\bigg)&\\+\mathbb{P}(R\le x,\ Q=0),&\end{aligned}$$

when $x\ge 0$ and $0\le y<1$. A similar modification allows us to generalize (3.3).

The distribution functions given by (3.2) and (3.3) simplify considerably when $n=3$.

Corollary 3.1. *When $n=3$,*

$$F_{(r,q)}(x,y)=\mathbb{E}(1-\sqrt{Z})^+,\qquad 0\le x,\ 0\le y<1,$$

and

$$F_r(x)=\mathbb{E}\bigg[\bigg(1-\frac{1}{RQ}\sqrt{(R^2-x^2)^+}\bigg)^+\bigg],\qquad x\ge 0. \tag{3.5}$$

From (3.2) we immediately infer that the marginal distribution of q is given by

$$F_q(y)=1-\frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}\bigg[B\bigg(\frac{(Q^2-y^2)^+}{Q^2(1-y^2)};\frac{1}{2},\frac{n-1}{2}\bigg)\bigg],\qquad 0\le y<1, \tag{3.6}$$

which does not depend on the distribution of R. This is an important fact, which we will use later. It follows from (3.4) that one of the variables r and q determines the other whenever Q and R are given. This implies that a joint probability density function of (r,q) need not exist. However, it is elementary to show that the marginal probability density functions exist. Let $\phi(\cdot)$ be any smooth function having a compact support in $(0,\infty)$. Then $\mathbb{E}\phi(X)=\int_0^\infty\phi'(x)(1-F_X(x))\,\mathrm{d}x$, where X is equal to r or q, respectively. Using Fubini and Tonelli arguments, integration by parts, and Leibnitz's rule, we obtain the following corollary.

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Corollary 3.2. *Adopt the setup in Theorem 3.1. The probability density functions of r and q exist. The function*

$$f_r(x) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\left[\frac{\mathbf{1}\{R\sqrt{1-Q^2} \le x < R\}x}{QR\sqrt{R^2-x^2}}\left(1-\frac{R^2-x^2}{Q^2R^2}\right)^{(n-3)/2}\right] \tag{3.7}$$

for $x \ge 0$ is a density function of r and

$$f_q(y) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\left[\mathbf{1}\{0 < y < Q\}\left(\frac{y}{Q}\right)^{n-2}\frac{(1-Q^2)^{(n-1)/2}}{\sqrt{Q^2-y^2}(1-y^2)^{n/2}}\right] \tag{3.8}$$

for $0 \le y < 1$ is a density function of q.

We remark that, when R and Q are independent and $n = 3$, (3.7) simplifies to

$$f_r(x) = x\mathbb{E}_R\left[\frac{\mathbf{1}\{x < R\}}{R\sqrt{R^2-x^2}}\left(\mathbb{E}_Q\frac{\mathbf{1}\{R\sqrt{1-Q^2} \le x\}}{Q}\right)\right], \qquad x \ge 0. \tag{3.9}$$

According to (1.2) the radius of the section profile can be written as a multiple of the size-weighted radius R_w of the intersected ball in the classical Wicksell problem. A similar result holds in the local Wicksell problem but here the radii of the intersected balls are not size weighted.

Proposition 3.1. *If the assumptions of Theorem 3.1 hold then*

$$r = \Gamma R,$$

where the random variable Γ has density

$$f_\Gamma(z) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\left[\mathbf{1}\{\sqrt{1-Q^2} \le z < 1\}\frac{z}{Q^{n-2}\sqrt{1-z^2}}(Q^2-1+z^2)^{(n-3)/2}\right] \tag{3.10}$$

for $z \ge 0$. If R and Q are independent then R and Γ are independent.

Proof. Let $h(x|R = R_0)$, $x \ge 0$, be the conditional density of r given that a ball of radius R_0 is cut by the section plane. By (3.7),

$$h(x|R = R_0) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\left[\frac{\mathbf{1}\{R_0\sqrt{1-Q^2} \le x < R_0\}x}{QR_0\sqrt{R_0^2-x^2}}\left(1-\frac{R_0^2-x^2}{Q^2R_0^2}\right)^{(n-3)/2}\,\middle|\, R = R_0\right]$$

for $x \ge 0$ is a version of this density. We note that h satisfies the scaling property

$$h(x|R = R_0) = \frac{1}{R_0}h\left(\frac{x}{R_0}\,\middle|\, R = 1\right).$$

Let Γ be the random variable defined by $\Gamma = r/R$. Using the conditional expectation, conditioning on $R = R_0$, and then using a substitution, we obtain

$$\mathbb{P}(\Gamma \le z) = \mathbb{E}\mathbb{P}(r \le zR_0 \mid R = R_0) = \mathbb{E}\left[\int_0^z R_0 h(sR_0|R = R_0)\,\mathrm{d}s\right]$$

for $z \ge 0$. Applying the scaling property and conditional expectation we see that a density of Γ is given by (3.10). When Q and R are independent, R and Γ are independent by construction.

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Using (3.3) with $R = 1$, the cumulative distribution function of the random variable Γ in Proposition 3.1 is immediately obtained as

$$F_\Gamma(z) = 1 - \frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}\bigg[B\bigg(1 - \bigg(1 - \frac{(1-z^2)}{Q^2}\bigg)^+; \frac{1}{2}, \frac{n-1}{2}\bigg)\bigg], \qquad 0 \le z < 1. \tag{3.11}$$

Remark 3.2. Since the distribution of Γ in Proposition 3.1 depends only on (R, Q) through Q, we may write $c_k(Q) = \mathbb{E}\Gamma^k$, $k = 0, 1, 2, \ldots$, for the kth moment of Γ. These moments are given by

$$c_k(Q) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\bigg[\frac{1}{Q^{n-2}}\int_{\sqrt{1-Q^2}}^1 \frac{z^{k+1}}{\sqrt{1-z^2}}(Q^2 - 1 + z^2)^{(n-3)/2}\,\mathrm{d}z\bigg] \tag{3.12}$$

for $k = 0, 1, 2, \ldots$. In particular, for $n = 3$, the substitution $x = z^2$ yields

$$c_k(Q) = \frac{1}{2}\mathbb{E}\bigg[\frac{1}{Q}\bigg(\frac{\sigma_{k+3}}{\sigma_{k+2}} - B\bigg(1 - Q^2, \frac{k+2}{2}, \frac{1}{2}\bigg)\bigg)\bigg]. \tag{3.13}$$

Denote the kth moment of R by M_k, and that of r by m_k. When R and Q are independent, Proposition 3.1 gives the moment relation

$$m_k = c_k(Q)M_k, \tag{3.14}$$

where $c_k(Q)$ is given by (3.12).

4. Uniqueness

The distribution F_q uniquely determines F_Q. This can be seen from (3.8), which can be rewritten as

$$f_q(y) = \frac{2\sigma_{n-1}y^{n-2}}{\sigma_n(1-y^2)^{n/2}}\mathbf{1}\{y > 0\}\int_y^1 \frac{(1-s^2)^{(n-1)/2}}{s^{n-2}\sqrt{s^2-y^2}}\,\mathrm{d}F_Q(s).$$

This is essentially an Abel transform of the positive measure $(1-s^2)^{(n-1)/2}/s^{n-2}\,\mathrm{d}F_Q(s)$. The Abel transform has a unique solution (see, e.g. [9, Section 1.2]) and, hence, F_q uniquely determines F_Q. An explicit solution is given in Proposition 5.1 for $n = 3$.

When the spatial parameters R and Q are independent, we can also show that F_r determines F_R uniquely.

Theorem 4.1. *Adopt the setup in Theorem 3.1. If R and Q are independent, $F_{(r,q)}$ uniquely determines $F_{(R,Q)}$.*

Proof. From Proposition 3.1 we have $r = \Gamma R$ and, thus, $\log r = \log\Gamma + \log R$. When R and Q are independent, Γ and R are independent and, hence, the characteristic functions obey

$$\varphi_{-\log r}(t) = \varphi_{-\log\Gamma}(t)\varphi_{-\log R}(t), \qquad t \in \mathbb{R}. \tag{4.1}$$

We want to show that the characteristic function $\varphi_{-\log\Gamma}$ is an analytic function. We note that $F_{-\log\Gamma}(z) = 1 - F_\Gamma(\mathrm{e}^{-z})$. Using the dominated convergence theorem, we obtain $\lim_{z\to 0} f_\Gamma(z) = 0$. Hence, applying l'Hôpital's rule, we obtain

$$\lim_{z\to\infty}\frac{1 - F_{-\log\Gamma}(z)}{\mathrm{e}^{-z}} = \lim_{z\to 0}\frac{F_\Gamma(z)}{z} = \lim_{z\to 0} f_\Gamma(z) = 0,$$

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that is, $1 - F_{-\log \Gamma}(z) = o(\mathrm{e}^{-z})$ as $z \to \infty$. As $F_{-\log \Gamma}(-z) = 0$, when $z \geq 0$, we have

$$1 - F_{-\log \Gamma}(z) + F_{-\log \Gamma}(-z) = o(\mathrm{e}^{-z}).$$

Therefore, $\varphi_{-\log \Gamma}$ is an analytic function [17, p. 137]. This implies that $\varphi_{-\log \Gamma}$ has only countably many roots. Therefore, in view of (4.1), $\varphi_{-\log r}$ uniquely determines the continuous function $\varphi_{-\log R}$. It then follows from the Fourier uniqueness theorem that F_r determines F_R uniquely. As F_q uniquely determines F_Q, this completes the proof.

It is of interest to ask whether the same holds without the independence assumption: does the joint distribution $F_{(r,q)}$ uniquely determine $F_{(R,Q)}$ when the independence assumption is dropped? This is an open question, but the following example shows that the answer is negative if only the marginals F_q and F_r are given.

Example 4.1. Assume that $n = 3$, and let the joint density of (R, Q) be given by

$$f_{(R,Q)}(t, s) = 3s\mathbf{1}\{0 < t < s < 1\}. \tag{4.2}$$

Then the two marginals are

$$f_Q(s) = 3s^2\mathbf{1}\{0 < s < 1\}, \qquad f_R(t) = \tfrac{3}{2}(1 - t^2)\mathbf{1}\{0 < t < 1\}.$$

We now show that there is another pair (R', Q') of size and shape variables which are independent, but lead to the same section marginals F_q and F_r as the pair (R, Q) with density (4.2). As $F_{Q'}$ is uniquely determined by F_q, we necessarily have $F_{Q'} = F_Q$. If we assume that R' has a density $f_{R'}$, this and (3.7) imply that this density must satisfy

$$f_r(x) = \frac{3}{2}x^3 \int_x^\infty \frac{f_{R'}(s)}{s^3\sqrt{s^2 - x^2}}\,\mathrm{d}s.$$

Solving this Abel transform (see, e.g. [9, p. 35]), we obtain

$$f_{R'}(s) = -\frac{4s^4}{3\pi}\int_s^\infty \frac{\mathrm{d}(f_r(x)x^{-3})/\mathrm{d}x}{\sqrt{x^2 - s^2}}\,\mathrm{d}x. \tag{4.3}$$

Hence, we would define $f_{R'}$ by (4.3) if we can guarantee that this function is a density. By (4.2) and (3.7), f_r is explicitly known. With this f_r, the right-hand side of (4.3) indeed is nonnegative and integrates to 1; see [33, pp. 12–13] for details. Hence, $f_{R'}$ is a density. We thus have shown that populations of balls with size-shape parameters (R, Q) and (R', Q') lead to the same size-shape distributions of their profiles, although $F_{(R,Q)} \neq F_{(R',Q')}$.

5. The unfolding problem

We mentioned in the introduction that there exists a reproducing distribution for the radii in the classical Wicksell problem (the Rayleigh distribution). In the local Wicksell problem a reproducing radii distribution does not typically exist. When the balls are not a.s. centered at O, the radii of the section profiles are smaller than the radii of the respective balls with positive probability. This implies that there does not exist a reproducing distribution in the local Wicksell problem under the assumption $\mathbb{P}(Q > 0) > 0$.

In the previous section we saw that, under the assumption that R and Q are independent, $F_{(r,q)}$ uniquely determines $F_{(R,Q)}$. In this section we will present analytical unfolding formulae and moment relations. In order to avoid technicalities, we restrict attention to the three-dimensional case, which is most relevant for practical applications. The following proposition gives F_Q in terms of F_q.

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Proposition 5.1. *If the assumptions of Theorem 3.1 hold and $n = 3$, then*

$$1 - F_Q(y) = -\frac{2y^3}{\pi}\left(\int_y^1 \frac{(s^2-2)(1-F_q(s))}{s^3\sqrt{1-s^2}\sqrt{s^2-y^2}}\,\mathrm{d}s - \int_y^1 \frac{\sqrt{1-s^2}}{s^2\sqrt{s^2-y^2}}\,\mathrm{d}F_q(s)\right) \tag{5.1}$$

for $y \in [0, 1)$.

Proof. Let $0 \le y < 1$. From (3.6) we know that the distributions of Q and q are connected by

$$F_q(y) = 1 - \mathbb{E}\left[\frac{\sqrt{(Q^2-y^2)^+}}{Q\sqrt{1-y^2}}\right].$$

Using integration by parts and rearranging, we obtain

$$1 - F_q(y) = \frac{y^2}{\sqrt{1-y^2}}\int_y^1 \frac{1}{s^2\sqrt{s^2-y^2}}(1-F_Q(s))\,\mathrm{d}s. \tag{5.2}$$

Define

$$h(y) = \frac{\sqrt{1-y^2}}{y^2}(1-F_q(y)) \quad \text{and} \quad g(s) = \frac{1-F_Q(s)}{s^2}. \tag{5.3}$$

Equation (5.2) is an Abel transform of $g(s)$ with solution given by

$$g(y) = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\int_y^1 \frac{sh(s)}{\sqrt{s^2-y^2}}\,\mathrm{d}s = -\frac{2y}{\pi}\int_y^1 \frac{h'(s)}{\sqrt{s^2-y^2}}\,\mathrm{d}s; \tag{5.4}$$

see, for instance, [9, p. 35]. Substituting for g and the derivative of h, (5.1) is obtained.

Using similar arguments as in the proof of Proposition 5.1, we find that, when Q has a density f_Q, it is given by

$$f_Q(s) = \frac{8s^2}{\pi}\frac{\mathrm{d}^2}{\mathrm{d}(s^2)^2}\int_s^1 \frac{t\sqrt{1-t^2}}{\sqrt{t^2-s^2}}(1-F_q(t))\,\mathrm{d}t, \qquad 0 \le s < 1. \tag{5.5}$$

We can use (5.4) to write the moments of Γ in Proposition 3.1 as functions of q only.

Corollary 5.1. *Adopt the setup in Theorem 3.1. For $n = 3$, the moments of the random variable Γ in Proposition 3.1 can be written as*

$$\mathbb{E}\Gamma^k = 1 - \frac{k}{\pi}\mathbb{E}[\tilde{\Gamma}(q)], \qquad k = 0, 1, 2, \ldots, \tag{5.6}$$

where

$$\tilde{\Gamma}(y) = \int_0^y \frac{\sqrt{1-s^2}}{s}\int_0^{s^2} \frac{\sqrt{t}(1-t)^{(k-2)/2}}{\sqrt{s^2-t}}\,\mathrm{d}t\,\mathrm{d}s. \tag{5.7}$$

Proof. Using (3.11) with $n = 3$, we note that the distribution function of Γ can be written as

$$F_\Gamma(z) = \mathbb{E}\left[\left(1 - \frac{1}{Q}\sqrt{1-z^2}\right)^+\right] = \sqrt{1-z^2}\int_{\sqrt{1-z^2}}^1 g(s)\,\mathrm{d}s, \qquad 0 \le z < 1,$$

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where $g(s)$ is given by (5.3). Using the first equality in (5.4) and substituting $t = 1 - z^2$, the moments of Γ are given by

$$\mathbb{E}\Gamma^k = \int_0^1 kz^{k-1}(1 - F_\Gamma(z))\,\mathrm{d}z = 1 - \frac{k}{\pi}\int_0^1 (1-t)^{(k-2)/2}\sqrt{t}\int_{\sqrt{t}}^1 \frac{yh(y)}{\sqrt{y^2 - t}}\,\mathrm{d}y\,\mathrm{d}t.$$

Substituting for h and using Tonelli's theorem, we arrive at

$$\begin{aligned}\mathbb{E}\Gamma^k &= 1 - \frac{k}{\pi}\int_0^1 \frac{\sqrt{1-y^2}}{y}(1 - F_q(y))\int_0^{y^2} \frac{\sqrt{t}(1-t)^{(k-2)/2}}{\sqrt{y^2 - t}}\,\mathrm{d}t\,\mathrm{d}y \\ &= 1 - \frac{k}{\pi}\mathbb{E}[\tilde{\Gamma}(q)],\end{aligned}$$

where $\tilde{\Gamma}$ is given by (5.7).

If $k = 2m,\ m \geq 1$, then (5.6) simplifies to

$$\mathbb{E}\Gamma^{2m} = 1 - \frac{m}{\pi}\sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^j\frac{\sigma_{2j+4}}{\sigma_{2j+3}}\mathbb{E}\bigg[B\bigg(q^2; j+1, \frac{3}{2}\bigg)\bigg]$$

and, in particular,

$$\mathbb{E}\Gamma^2 = \tfrac{1}{3}(2 + \mathbb{E}[(1-q^2)^{3/2}]).$$

Thus, the average surface area of balls in $\mathbb{R}^3$ can be estimated by the ratio unbiased expression

$$\frac{(12\pi/N)\sum_{i=1}^N r_i^2}{2 + (1/N)\sum_{i=1}^N (1-q_i^2)^{3/2}},$$

where $(r_1, q_1), \ldots, (r_N, q_N)$ are N independent observations of profile parameters.

When Q has a density, similar arguments as applied in the proof of (5.5) can be used to show that the moments of Γ can be written as

$$\mathbb{E}\Gamma^k = \frac{1}{2\pi}\mathbb{E}[\tilde{\Gamma}_1(q^2)] + \frac{1}{\pi}\mathbb{E}[\tilde{\Gamma}_2(q^2)], \tag{5.8}$$

where

$$\tilde{\Gamma}_1(y) = \int_0^y \frac{(1-t)^{k/2}}{\sqrt{t}}B\bigg(\frac{y-t}{1-t}; \frac{1}{2}, \frac{1}{2}\bigg)\,\mathrm{d}t, \qquad \tilde{\Gamma}_2(y) = \sqrt{1-y}\int_0^y \frac{(1-t)^{k/2}}{\sqrt{t}\sqrt{y-t}}\,\mathrm{d}t.$$

By (3.14) and Corollary 5.1, the moments of R can be estimated from the section profiles when R and Q are independent. A number of stochastic simulation studies were carried out, varying F_Q, F_R, and the number of observations. The function `quad` in the language and interactive environment MATLAB® was used to determine $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ numerically. We used (5.8) instead of the more general expression given in Corollary 5.1 as it is more straightforward to implement. Then M_k was estimated by dividing the crude Monte Carlo (CMC) estimate of m_k by the CMC estimate obtained for $\mathbb{E}\Gamma^k$ using (5.8). The simulations suggest that the estimation procedure is quite stable for moments up to the 7th order. The difference between the coefficient of error of this estimator and of M_k estimated by CMC, using the simulated particle radii, is typically less than 10% for $k = 1, \ldots, 7$. This seems to be true irrespective of the choice of F_Q and even after adding moderate measurement errors to r and q.

6. Reconstruction

In this section we will discuss the estimation of F_R and F_Q from data. We assume throughout this section that R and Q are independent, and that $n = 3$. To assure this independence, a priori knowledge on the particles is required. There are, however, cases where this independence holds trivially. We refer the reader to the concluding Section 9 for a discussion. As mentioned in the introduction, there exist various methods for numerically solving Wicksell's classical problem but none of these seems to be superior to all the others. In [6] six distribution-free methods are compared using several error criteria as well as studying their numerical stability and sensitivity to underlying distributions. Considering all criteria, the Scheil–Schwartz–Saltykov method (S^3M) from [24] is favored, in particular when the underlying distribution is smooth. The method can be classified as a finite difference method, a class of methods which are relatively easy to implement. We therefore chose to implement a variation of S^3M. The method and its advantages and disadvantages are discussed in [6]. Although this method is rather crude, we obtain satisfactory results. We have also studied the product integration method explained in [3]. The method has been claimed to yield accurate results for Wicksell's classical problem, but our reconstruction of F_Q based on Proposition 5.1 is not satisfactory. The reconstruction $\hat{F}_Q$ is not monotonic and fluctuates far too much around F_Q to be of any use. We have also implemented kernel density estimators for obtaining $\hat{F}_Q$ but without obtaining stable results.

In the following we describe the implementation of S^3M in our setup. We start using S^3M to obtain a discrete approximation of F_Q based on the Abel-type relation (5.2). By discretizing F_Q, the integral becomes a finite sum. This produces a system of linear equations that can be solved. After having obtained an estimate $\hat{F}_Q$ we can apply S^3M again, this time discretizing F_R to solve (3.5) numerically (where $\hat{F}_Q$ is substituted for the unknown F_Q). We used MATLAB for all simulations and for generating Figure 2.

Assume that we observe N pairs $(r_1, q_1), \dots, (r_N, q_N)$ of size and shape variables in independent sections of (independent) particles. Assume further that all observed rs are less than or equal to a constant c. Divide the intervals $(0, 1]$ and $(0, c]$ into classes of constant width. Let k_1 denote the number of classes for q, k_2 the number of classes for Q, k_3 for r, and k_4 for R. Different to [6] we allow k_1, k_3 to be greater than k_2, k_4, respectively. Define $\Delta_i = 1/k_i$, $i = 1, 2$, and $\Delta_i = c/k_i$, $i = 3, 4$. Let $n_i = \mathbb{P}(q \in ((i-1)\Delta_1, i\Delta_1])$ be the probability that q is in class $i \in \{1, \dots, k_1\}$, $N_j = \mathbb{P}(Q \in ((j-1)\Delta_2, j\Delta_2])$ be the probability that Q is in class $j \in \{1, \dots, k_2\}$, $m_a = \mathbb{P}(r \in ((a-1)\Delta_3, a\Delta_3])$ be the probability that r is in class $a \in \{1, \dots, k_3\}$, and $M_b = \mathbb{P}(R \in ((b-1)\Delta_4, b\Delta_4])$ be the probability that R is in class $b \in \{1, \dots, k_4\}$.

The S^3M method approximates $F_Q(u)$ by the step function

$$u \mapsto \sum_{\{j:\ \Delta_2 j \le u\}} N_j.$$

Note that we use the standard definition of a cumulative distribution function and, hence, the notation is slightly different from [6] where left-continuous distribution functions are considered. Inserting the approximation for F_Q in (5.2) and simplifying, we obtain

$$1 - F_q(i\Delta_1) \approx \frac{1}{\sqrt{1 - i^2\Delta_1^2}\,\Delta_2} \sum_{j=1}^{k_2} N_j \mathbf{1}\{i\Delta_1 < j\Delta_2\} \frac{\sqrt{j^2\Delta_2^2 - i^2\Delta_1^2}}{j},$$

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where the approximation is exact if and only if Q only has mass in the points $\Delta_2 j$, $j = 1, \dots, k_2$. Hence, n_i can be approximated by

$$n_i = (1 - F_q((i-1)\Delta_1)) - (1 - F_q(i\Delta_1)) \approx \sum_{j=1}^{k_2} b_{ij} N_j \tag{6.1}$$

for $i = 1, \dots, k_1$, where the constants b_{ij} depend only on Δ_1 and Δ_2; see [33, p. 17] for an explicit formula. If F_q in (6.1) is replaced by its empirical cumulative distribution function, we obtain $\hat{n}_i = (1/N) \sum_{j=1}^{N} \mathbf{1}\{q_j \in ((i-1)\Delta_1, i\Delta_1]\}$ as approximations of the left-hand side of (6.1). Using these and solving the corresponding linear system yields approximations $\hat{N}_j$ of the unknown probabilities N_j, $j = 1, \dots, k_2$. The function

$$\hat{F}_Q(u) = \sum_{\{j\colon \Delta_2 j \le u\}} \hat{N}_j \tag{6.2}$$

is then the estimator for F_Q.

In a second step (3.5) is inverted using S^{3}M and the estimator $\hat{F}_Q$. This is in analogy to the treatment of F_Q and the derivation is deferred to Appendix A.

In our simulations, we used N independent realizations of (r_i, q_i) to estimate $\hat{n}_i$, $i = 1, \dots, k_1$ and $\hat{m}_a$, $a = 1, \dots, k_3$. Then we solved (6.1) using constrained minimum least squares, `lsqlin`, in MATLAB. To ensure that the estimated distribution functions are non-decreasing, we required that $\hat{N}_j \ge 0$, $j = 1, \dots, k_2$, and $\hat{M}_b \ge 0$, $b = 1, \dots, k_4$. Non-negativity constraints have also been suggested in [32] for the classical S^{3}M. The distribution function F_Q is then estimated by (6.2). Using the estimate $\hat{F}_Q$, the linear system of equations involving the relative frequencies $\hat{m}_a$ in terms of $\hat{M}_b$ (see (A.2)) can be solved for $\hat{M}_b$, $b = 1, \dots, k_4$, in exactly the same way and F_R is then estimated by $\hat{F}_R(u) = \sum_{\{b\colon \Delta_4 b \le u\}} \hat{M}_b$.

The simulations in [6] show that classes with overestimation of the distribution function are usually close to classes with underestimation and the authors refer to this phenomenon quite intuitively as *waves*. In order to decrease the occurrence of waves, we allowed k_1, k_3 to be greater than k_2, k_4, respectively. However, this does not seem to be of great importance in our setting. We ran two types of simulation, one with an equal number of classes and another with an unequal number of classes. Then the Kullback–Leibler divergence between the true probability distribution and each of the estimated distributions was calculated using `KLDiv` in MATLAB. The difference was negligible (even after adding independent measurement errors to r_j and q_j). Therefore we chose an equal number of classes in Figure 2.

In Figure 2 true and reconstructed distribution functions of Q (left-hand diagram) and R (right-hand diagram) are compared, where Q follows the beta distribution with parameters 5 and 2 while R is exponentially distributed with mean 1. The number of classes were chosen to be 20 for all four variables and 100 observations were used.

These results indicate that the S^{3}M method works well for reconstructing F_R and F_Q from simulated data. This is true even when the profile variables are measured with random multiplicative errors. The errors were chosen to be independent, lognormally (truncated lognormal in the case of q) distributed variables with mean 1 and moderate variance. Even with variance $\frac{1}{2}$ the mean Kullback–Leibler divergence between the true probability distribution and that obtained using the perturbed profile variables did not differ from the corresponding mean Kullback–Leibler divergence using the unperturbed profile variables.

In practice, for instance in a confocal microscopy image, the boundary of an object can be quite blurry and it can therefore be difficult to evaluate profile variables when the reference

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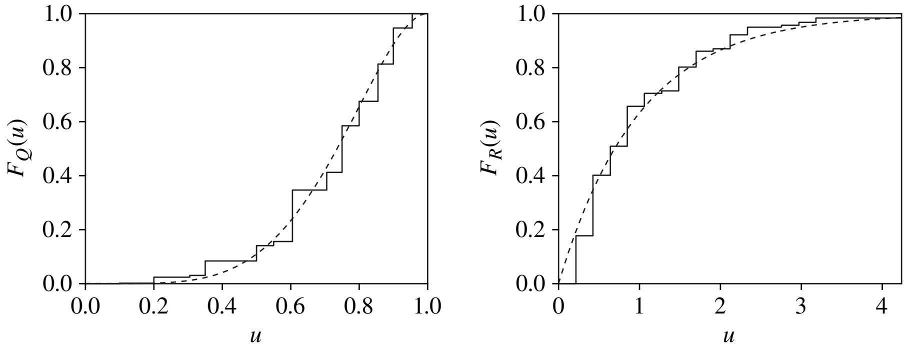

FIGURE 2: The dashed curves are the true distribution functions $F_Q(u)$ (*left*) and $F_R(u)$ (*right*) when $Q \sim \text{Beta}(5, 2)$ and $R \sim \text{e}(1)$, whereas the step functions are the estimated distributions. Parameter choice: $k_1 = k_2 = k_3 = k_4 = 20$, $N = 100$.

point is close to the boundary and the IR section is such that it produces a small profile. We analysed how this influenced the reconstructions by omitting small profiles obtained when $Q > 1 - 0.1/R$ and $0 < r < 0.5$. Otherwise, the setup was the same as for Figure 2, with 100 valid observations. This did not influence the reconstruction of F_Q, but to obtain a satisfactory reconstruction of F_R a similar recommendation as in [6] applies: the sample size should be large (more than 400 profile sections) but the number of classes small (around 7).

7. Examples

We now discuss some special cases and variants of the above general theory. We restrict our attention to the three-dimensional case. Recall that we considered Q to be a variable describing the 'shape' of the random ball under consideration. We first show that the formulae simplify if Q is (a.s.) the same for all particles, meaning that all reference points have the same relative distance from their respective ball centers.

Example 7.1. Assume that $Q = Q_0$ a.s., $Q_0 > 0$. Then the marginal distribution function of q given by (5.2) becomes

$$F_q(y) = 1 - \mathbf{1}\{y < Q_0\} \frac{\sqrt{Q_0^2 - y^2}}{Q_0\sqrt{1 - y^2}}.$$

Furthermore, the moments of Γ in (3.13) simplify to

$$c_k(Q) = \frac{1}{2Q_0}\left(\frac{\sigma_{k+3}}{\sigma_{k+2}} - B\left(1 - Q_0^2; \frac{k+2}{2}, \frac{1}{2}\right)\right).$$

For $k = 1, 2$, we obtain

$$c_1(Q) = \frac{\pi}{4Q_0} - \frac{\sin^{-1}(\sqrt{1 - Q_0^2})}{2Q_0} + \frac{\sqrt{1 - Q_0^2}}{2}, \qquad c_2(Q) = 1 - \frac{Q_0^2}{3}.$$

The formulae simplify, in particular, if the reference point lies on the boundary of the object, that is, when $Q_0 = 1$.

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Example 7.2. Let the reference point be located on the boundary of the ball $RB_3 + O'$. Then $Q = 1$ a.s. and, from Example 7.1, we immediately obtain $q = 1$ a.s. and $c_k(Q) = \sigma_{k+3}/(2\sigma_{k+2})$. Note that this constant is the same as c_{k+1} given by (1.4), which can be explained by the fact that the section plane is an IUR plane hitting the ball, as shown in Example 7.4. The moments of Γ can in fact be calculated explicitly in n-dimensional space and (3.14) becomes

$$m_k = \frac{\sigma_{n-1}\sigma_{k+n}}{\sigma_n \sigma_{k+n-1}} M_k.$$

Furthermore, (3.9) becomes an Abel transform of the positive measure $\mathbb{P}_R(\mathrm{d}t)/t$. Hence, when R has a density function f_R, it is given by

$$f_R(t) = -\frac{2t^2}{\pi}\int_t^\infty \frac{1}{\sqrt{x^2 - t^2}} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f_r(x)}{x}\right)\mathrm{d}x = -\frac{2t}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{f_r(x)}{\sqrt{x^2 - t^2}}\,\mathrm{d}x;$$

see, for instance, [9, p. 35].

In practice a particle may have an easily identifiable kernel that cannot be treated as a mathematical point. Thus, one has to work with a reference set of positive volume. In general, our methods do not apply in this case, but whenever the reference set is (approximately) ball shaped, concentric with the whole particle, and has a radius proportional to the particle size, our method can be used. We suggest in the following examples two possible schemes for this situation.

Example 7.3. Assume that the particle $RB_3 + O'$ contains a reference set $Q_0 RB_3 + O'$, $Q_0 \in (0, 1]$. The first sampling design we suggest is the choice of an isotropic plane through a uniformly chosen boundary point of the reference set. Let $L^3_{2[O]}$ be an IR plane and z a uniformly distributed point on the boundary of the reference set, chosen independently of $L^3_{2[O]}$. Define $L = L^3_{2[O]} + z$ and adopt L as the section plane, that is, (r, q) refers to the parameters of the disk $(RB_3 + O') \cap L$. By construction, (r, q) can be interpreted as section variables from a local IR plane through the reference point z, which has relative distance Q_0 from the ball's center. Hence, we can use the local Wicksell theory directly with $Q = Q_0$. The simplifications in Example 7.1 apply.

Example 7.4. As in Example 7.3, assume that the particle $RB_3 + O'$ contains a reference set $Q_0 RB_3 + O'$, $Q_0 \in (0, 1]$. In contrast to Example 7.3 we now use an IUR section plane L^3_2 hitting the reference set. By definition, the distribution of L^3_2 is

$$\mathbb{P}_{L^3_2}(A) = \frac{1}{2\sigma_3 Q_0}\int_{S^2}\int_{-Q_0}^{Q_0} \mathbf{1}_A(ru + u^\perp)\,\mathrm{d}r\,\mathrm{d}u^2, \qquad A \in \mathcal{B}(\mathcal{L}^3_2).$$

Using cylindrical coordinates, we have, equivalently,

$$\mathbb{P}_{L^3_2}(A) = \frac{1}{\sigma_3^2}\int_{S^2}\int_{S^2} \mathbf{1}_A(Q_0 v + u^\perp)\,\mathrm{d}u^2\,\mathrm{d}v^2,$$

so L^3_2 is an isotropic plane through the (independent) point $z = Q_0 v$, which is uniform on the boundary of the reference set. Concluding, we see that L^3_2 has the same distribution as L in Example 7.3. Hence, the two designs lead to the same sample distribution, and, again, Wicksell's local theory with $Q = Q_0$ applies.

The last two (coinciding) sampling schemes can in particular be applied when the reference set is taken to be the whole ball. This is equivalent to choosing $Q_0 = 1$ and the formulae in Example 7.2 can be applied in this case.

8. Stereology of extremes

In some practical applications, for instance when studying damage of materials [21], the distribution of the maximal size parameter is of more interest than the whole distribution. When extremal parameters are studied based on lower-dimensional sections we speak of stereology of extremes. Here we will discuss stereology of extremes in the context of the local Wicksell problem.

The reader is referred to [10] for results in extreme value theory. Following the notation in [10] we write $F \in \mathcal{D}(L)$ if F is in the domain of attraction of the distribution function L. Up to affine transformations, L must be one of the extreme value distributions: Fréchet $L_{1,\gamma}$, Weibull $L_{2,\gamma}$, and Gumbel L_3, where $\gamma > 0$ is a parameter.

We assume that R and Q are independent and that $n = 3$. Given independent observations $(r_1, q_1), \ldots, (r_N, q_N)$ we are interested in the distribution of the extremal particle radius. Therefore, relations between the domains of attraction of the distributions of the size parameters are of interest. These are given in the following theorem.

Proposition 8.1. *Let $n = 3$, $\gamma > 0$, and assume that R and Q are independent and have probability densities. The following statements hold:*

- *if $F_R \in \mathcal{D}(L_{1,\gamma})$ then $F_r \in \mathcal{D}(L_{1,\gamma})$,*
- *if $F_R \in \mathcal{D}(L_{2,\gamma})$ then $F_r \in \mathcal{D}(L_{2,\gamma+1/2})$,*
- *if $F_R \in \mathcal{D}(L_3)$ then $F_r \in \mathcal{D}(L_3)$.*

A proof of this proposition can be found in [33, pp. 22–24]. Using similar arguments, an analogous result for the shape parameters can be shown: $F_Q \in \mathcal{D}(L_{2,\gamma})$ implies that $F_q \in \mathcal{D}(L_{2,\gamma+1/2})$ and $F_Q \in \mathcal{D}(L_3)$ implies that $F_q \in \mathcal{D}(L_3)$.

In order to use these results in practical applications, the normalizing constants for both F_r and F_R are required. They can be estimated by a semiparametric approach as in the classical Wicksell problem: first a parametric model for F_R is chosen. We know from Proposition 8.1 that F_R and F_r belong to the same domain of attraction. Hence, normalizing constants based on $(r_1, q_1), \ldots, (r_N, q_N)$ can be found, for example, using maximum likelihood estimators based on the k largest observations; cf. [34]. One then has to derive normalizing constants for F_R from the estimated normalizing constants for F_r. Methods regarding this are discussed in, e.g. [12] and [27]. When normalizing constants for F_R have been obtained, they can be used to approximate the distribution of the extremal particle radius.

9. Conclusion

The present paper gives a detailed survey of Wicksell's famous corpuscle problem in a local setting. We have derived several results analogous to those existing for the classical problem and described differences between the two. We have, in particular, given the distributions and marginal densities of the profile parameters in terms of the particle parameters, derived moment relations, uniqueness results, and domain of attraction relations as well as having given some examples and reconstructions of the particle distributions from experimental data. Many of these results require that the particle parameters are independent. Given experimental data, we might hope that the independence assumption of the particle parameters could be assured by checking if the profile parameters can be assumed to be independent. However, this is not possible since the profile parameters are always dependent apart from in mathematically trivial cases, for instance, when $Q = 1$ a.s. Nevertheless, we believe that in many practical applications it is not unrealistic to assume a priori that the relative position of the reference point does not depend on the size of the object.

It should be mentioned that the case $Q = 1$ a.s. (certainly implying the independence of R and Q) is also relevant for applications. For instance, white fat cells are spherical in shape and have their nucleus at the cell's periphery; see [18, p. 163]. Using this nucleus as the reference point allows us to use the formulae in Example 7.2 and the S^3M algorithm to estimate moments or even the distribution of R. As we have seen in Example 7.4, the local formulae can even be applied when there is no natural reference point, but the section planes are just chosen to be IUR hitting the particle under consideration.

Appendix A. S^3M for F_R

We have described shortly in the main text how S^3M can be used to invert (5.2). The following inversion of (3.5) is slightly more involved, since it is based on the approximation $\hat{F}_Q$. We therefore give some additional details here. We note that (3.5) can be rewritten as

$$F_r(x) = (1 - F_R(x)) + \mathbb{E}\bigg[\mathbf{1}\{x < R\}\frac{\sqrt{R^2 - x^2}}{R}\int_{(1/R)\sqrt{R^2-x^2}}^{1}\frac{1}{s^2}(1 - F_Q(s))\,\mathrm{d}s\bigg]$$

for $x \in [0, \infty)$. Approximating F_Q, by $\hat{F}_Q$ given by (6.2), we find that

$$\begin{aligned}\mathbb{E}\bigg[&\mathbf{1}\{x < R\}\frac{\sqrt{R^2 - x^2}}{R}\int_{(1/R)\sqrt{R^2-x^2}}^{1}\frac{1}{s^2}(1 - \hat{F}_Q(s))\,\mathrm{d}s\bigg]\\ &= 1 - \sum_{j=1}^{k_2-1}\hat{N}_j\frac{1}{\Delta_2 j}\int_x^{x/\sqrt{1-\Delta_2^2 j^2}}\frac{x^2}{t^2\sqrt{t^2 - x^2}}(1 - F_R(t))\,\mathrm{d}t\\ &\quad - F_R(x) - \hat{N}_{k_2}\frac{1}{\Delta_2 k_2}\int_x^{\infty}\frac{x^2}{t^2\sqrt{t^2 - x^2}}(1 - F_R(t))\,\mathrm{d}t.\end{aligned}$$

Hence,

$$\begin{aligned}F_r(x) \approx \frac{1}{3}\bigg(2 &- \sum_{j=1}^{k_2-1}\hat{N}_j\frac{1}{\Delta_2 j}\int_x^{x/\sqrt{1-\Delta_2^2 j^2}}\frac{x^2}{t^2\sqrt{t^2 - x^2}}(1 - F_R(t))\,\mathrm{d}t\\ &- \hat{N}_{k_2}\frac{1}{\Delta_2 k_2}\int_x^{\infty}\frac{x^2}{t^2\sqrt{t^2 - x^2}}(1 - F_R(t))\,\mathrm{d}t\bigg).\end{aligned} \tag{A.1}$$

Recall that m_a is the probability that r is in class a, $a = 1, \ldots, k_3$, and M_b is the probability that R is in class b, $b = 1, \ldots, k_4$. The S^3M method approximates $F_R(u)$ by

$$u \mapsto \sum_{\{b:\ \Delta_4 b \le u\}} M_b.$$

Using this in (A.1), we obtain, again using elementary calculations,

$$\begin{aligned}F_r(a\Delta_3) \approx -\frac{1}{3}\bigg(&\sum_{j=1}^{k_2-1}\hat{N}_j\frac{1}{\Delta_2 j}\sum_{b=1}^{k_4}M_b\mathbf{1}\{\Delta_4 b \ge a\Delta_3\}\sqrt{1 - \bigg(\min\bigg\{\frac{\Delta_4^2 b^2}{a^2\Delta_3^2}, \frac{1}{1 - \Delta_2^2 j^2}\bigg\}\bigg)^{-1}}\\ &+ \hat{N}_{k_2}\frac{1}{\Delta_2 k_2}\sum_{b=1}^{k_4}M_b\mathbf{1}\{\Delta_4 b \ge a\Delta_3\}\sqrt{1 - \frac{a^2\Delta_3^2}{\Delta_4^2 b^2}} - 2\bigg).\end{aligned}$$

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We are thus led to the linear system

$$\hat{m}_a = \hat{F}_r(a\Delta_3) - \hat{F}_r((a-1)\Delta_3) = \sum_{b=1}^{k_4} c_{ab}\hat{M}_b \tag{A.2}$$

for $a = 1, \dots, k_3$, involving the relative frequencies

$$\hat{m}_a = \frac{1}{N}\sum_{j=1}^{N} \mathbf{1}\{r_j \in ((a-1)\Delta_3, a\Delta_3]\}.$$

Here

$$\begin{aligned} c_{ab} = \frac{1}{3}\sum_{j=1}^{k_2-1} \hat{N}_j \frac{1}{\Delta_2 j}\Bigg(&\mathbf{1}\{\Delta_4 b \geq (a-1)\Delta_3\}\sqrt{1-\left(\min\left\{\frac{\Delta_4^2 b^2}{(a-1)^2\Delta_3^2}, \frac{1}{1-\Delta_2^2 j^2}\right\}\right)^{-1}} \\ &- \mathbf{1}\{\Delta_4 b \geq a\Delta_3\}\sqrt{1-\left(\min\left\{\frac{\Delta_4^2 b^2}{a^2\Delta_3^2}, \frac{1}{1-\Delta_2^2 j^2}\right\}\right)^{-1}}\Bigg) \\ + \frac{\hat{N}_{k_2}}{3\Delta_2 k_2}\Bigg(&\mathbf{1}\{\Delta_4 b \geq (a-1)\Delta_3\}\sqrt{1-\frac{(a-1)^2\Delta_3^2}{\Delta_4^2 b^2}} - \mathbf{1}\{\Delta_4 b \geq a\Delta_3\}\sqrt{1-\frac{a^2\Delta_3^2}{\Delta_4^2 b^2}}\Bigg) \end{aligned}$$

for $a = 1, \dots, k_3$ and $b = 1, \dots, k_4$ depends only on Δ_3, Δ_4, $(\hat{N}_1, \dots, \hat{N}_{k_2})$, and Δ_2. The linear system of equations can be solved for the unknown $\hat{M}_b$ approximating M_b for $b = 1, \dots, k_4$.

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