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ON THE GLOBAL DIMENSIONS OF D+M

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1. **Introduction and notation.** This note answers affirmatively a question of the author [4, p. 456], by producing an example of an integrally closed quasi-local nonvaluation domain of global dimension 3, each of whose overrings is a going-down ring. Although [4, Proposition 4.5] shows that such an example cannot be constructed by means of restrained power series, an approach via the more general $D+M$ construction succeeds. The main tool, Proposition 3.1, concerns weak (flat) global dimension. Together with a bound of Jensen, it leads via cardinal arithmetic to the desired result, Example 3.2.

Background material on the $D+M$ construction and weak dimension may be found in [8, Appendix 2] and [3, pp. 122–123], respectively. Weak dimension, projective dimension, weak global dimension, and global dimension are denoted by w.d., p.d., w.gl. dim, and gl. dim, respectively.

To fix notation, let V be a valuation ring of the form $K+M$, where K is a field and $M(\neq 0)$ is the maximal ideal of V. Let D be a proper subring of K; let k, viewed inside K, be the quotient field of D. Finally, set $R=D+M$.

2. **Shaping the example.** We begin by examining the global dimensions of R in case $k=K$.

PROPOSITION 2.1. *Let $k=K$. Then:*
(1) *If $n=$*gl. dim(V) *and $m=$*gl. dim(D), *then*

$$\text{gl. dim}\,(R) = \begin{cases} n & \text{if } n > m \\ m & \text{if } m \geq n \text{ and } \text{p.d.}_D(K) < m \\ m+1 & \text{if } m \geq n \text{ and } \text{p.d.}_D(K) = m. \end{cases}$$

(2) w.gl. dim$(R)=$w.gl. dim(D).

Proof. (1) In the terminology of Greenberg [9], R is an F-ring with F-ideal M if $k=K$. (The converse is also valid, by [8, Theorem $A(h)$, p. 561], since $R_M=k+M$ in general.) Thus [9, Theorem 4.3] specializes to the assertion in (1).

(2) (Sketch) The desired result follows by aping Greenberg's route to [9, Theorem 4.3]: replace "projective" by "flat" as needed; use [7, Lemme] in place of [13, Theorem 1.1], to obtain the "flat" analogue of [9, Proposition 2.6]; note that

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w.gl. $\dim(R_M)$=w.gl. $\dim(V)=1$, by [3, Proposition 2.9, p. 112], to eliminate the analogue of case (1) in the proof of [9, Theorem 4.3]; and, for the analogue of case (2) in the proof of [9, Theorem 4.3], use the reasoning in [11, p. 35, ll. 1–8] to reduce to the consideration of finitely generated ideals, for which case (b) is eliminated, as $\text{w.d.}_{R/M}(R_M/M)=0$.

To aid in our search for a context hospitable to the desired example, it will be convenient to refer to any integrally closed quasi-local nonvaluation domain of global dimension 3, all of whose overrings are going-down rings, as a *solution.*

COROLLARY 2.2. *If R is a solution and D is not a solution, then $k\neq K$.*

Proof. Deny. Note that D then inherits from R the properties of being quasi-local, integrally closed and nonvaluation, by [8, Theorem A(c), (d), (b), (h), pp. 560–561]. If T is any overring of D, then $T+M$, being an overring of R, is going-down, so that [5, Corollary] implies T is also going-down. As D is not a solution, the process of elimination yields gl. $\dim(D)\neq 3$ (=gl. $\dim(R)$). By Proposition 2.1(1), gl. $\dim(D)$ is either 1 or 2. This leads to D being valuation: in the first case, since it would be local Dedekind; in the second case, by coherence and treedness, as explained in [4, pp. 442–443]. This (desired) contradiction completes the proof.

The preceding result allows us to restrict attention to the case $k\neq K$. (Indeed, if D were a solution, why consider R?) The final result of this section permits the further restriction $D=k$ and re-explains the inadequacy of restrained power series for our purposes.

PROPOSITION 2.3. *If R is a solution and $k\neq K$, then $k+M$ is a solution and $M=M^2$.*

Proof. By [8, Theorem A(c), (d), p. 560], $k+M$ is quasi-local (whether or not R is a solution). As R *is* a solution, D is integrally closed in K [8, Theorem A(b), p. 560]; thus, k is integrally (algebraically) closed in K, so that $k+M$ is integrally closed. Moreover, each overring of $k+M$ is going-down, since it is also an overring of R. Of course, $k\neq K$ forces $k+M$ to be nonvaluation [8, Theorem $A(h)$, p. 561]. Finally, since $R_M=k+M$, we have gl. $\dim(k+M)\leq$gl. $\dim(R)=3$. The cases gl. $\dim(k+M)=1, 2$ are ruled out as at the close of the proof of Corollary 2.2, and so $k+M$ is a solution. As noted in [6, Remark 10], it follows from [6, Theorem 8] and the proof of [4, Proposition 4.5] that, if $M\neq M^2$, then $\text{w.d.}_R(M)=\infty$. In fact, $\text{w.d.}_R(M)\leq$w.gl. $\dim(R)\leq$gl. $\dim(R)=3$, and so $M=M^2$, completing the proof.

3. **Weak dimension and the example.** The comments of the preceding section suggest the hypotheses of the next result.

PROPOSITION 3.1. *Let $D=k$ $(\neq K)$. If $M=M^2$, then* w.gl. $\dim(R)=2$ *and* gl. $\dim(R)\geq 3$.

Proof. Let I be a nonzero finitely generated ideal of R. By [8, Theorem $A(k)$, p. 562], $I=Wm+Mm$, for some nonzero finite-dimensional k-subspace W of K and some nonzero m in M. Let $\{b_i:1\le i\le n\}$ be any k-basis of W. If R^n is R-free, the R-module homomorphism $g:R^n\to I$ determined by $g(e_i)=b_im$ is surjective, since $M=b_iM$. One checks readily that ker(g) consists of those sums $\sum m_ie_i$ such that each m_i is in M and $\sum m_ib_i=0$. Again since $M=b_iM$, we have ker$(g)\simeq M^{n-1}$. However, M is R-flat, as $M=M^2$ [6, Theorem 8]. Thus, w.d.$_R(I)$ is 0 or 1 according as $n=1$ or $n>1$. (Indeed, if I were R-flat, it would be principal, generated by some m_1 in M. As we can write $m_1=vm$ with v in V, it follows that $W+M=(k+M)v$, so that W is cyclic over k, and $n=1$.) Now, we can arrange $n>1$ since $k\ne K$, so that sup{w.d.$_R(J):0\ne J$, finitely generated ideal of $R\}=1$. Then (cf. [11, p. 35, 11. 1–8]), w.gl. dim$(R)=2$. As before, $k\ne K$ implies R nonvaluation, so that rerevisiting the close of the proof of Corollary 2.2 (and bearing in mind that $k+M$ is going-down) rules out the cases gl. dim$(R)=1$, 2, and completes the proof.

Example 3.2. Solutions exist.

Proof. Let k be a countable field; choose a field $K\supset k$ such that k is algebraically closed in K and tr. deg$_k(K)=1$. (For example, let $K=k(x)$, where x is transcendental.) Observe that card$(K)=\aleph_0$. Let Γ be a nonzero countable subgroup of $\mathbb{R}$ (under addition) such that $\Gamma=2\Gamma$. (For example, let $\Gamma=\mathbb{Q}$.) Subjecting K and Γ to the construction in [2, Exemple 6, p. 107] leads to a valuation ring $V=K+M$ with value group Γ. We claim that $R=k+M$ is a solution.

Before verifying the claim, recall from [2, p. 107] that the quotient field L of V is the quotient field of an algebra which, as a k-space, is free on Γ^+. By standard cardinal arithmetic, one easily checks now that card$(V)=\aleph_0$.

If v is the valuation associated to V, then $M=\{b$ in $L:v(b)>0\}$ and $V\backslash M=\{b$ in $L:v(b)=0\}$. Since $\Gamma=2\Gamma$, it is clear that $M=M^2$, and Proposition 3.1 implies w.gl. dim$(R)=2$ and gl. dim$(R)\ge 3$. If each ideal of R is $\aleph_n$-generated, a key result of Jensen-Osofsky [10, Corollary 2.47] now implies that gl. dim$(R)\le 3+n$. However, one upshot of the preceding paragraph is card$(R)\le\aleph_0$, and so we may certainly take $n=0$, giving gl. dim$(R)=3$.

Finally, appeals to the now-familiar parts of [8, Theorem A, p. 560] show that R is quasi-local, integrally closed and nonvaluation. It remains only to show that each overring T of R is going-down. According to [1, Theorem 3.1], such T are either valuation (hence, going-down) or of the form $T=E+M$, where $k\subset E\subset K$. By [5, Corollary], we need only show that each ring between k and K is going-down, and this follows as in the proof of [4, Theorem 4.2 (iii)] since tr. deg$_k(K)=1$. The proof is complete.

Remark 3.3. In the spirit of [4, Corollary 4.4], we note that the construction employed in Example 3.2 actually yields a family of quasi-local noncoherent going-down rings of global dimension 3. (The noncoherence may be shown by either [4, Proposition 2.5] or [6, Corollary 5].) By removing the condition that k be algebraically closed in K, we produce such rings which are not integrally closed. Permitting tr. $\deg_k(K)>1$ results in examples with (some) overrings that are not going-down. Although the particular ring in Example 3.2 has (Krull) dimension 1 and valuative dimension 2, examples exist with arbitrary finite positive dimension and with arbitrary larger finite valuative dimension. For instance, an example with dimension 2 and valuative dimension 5 may be constructed by taking $K=k(x, y, z)$ and setting $\Gamma=\mathbb{Q}\times\mathbb{Q}$, lexicographically ordered.

These examples suggest that pullback descriptions, adequate for quasi-local rings of global dimension 2 (cf. [12], [9]), no longer suffice for global dimension 3. We close by raising the problem of developing enough information about non-coherent rings in order to classify the quasi-local going-down rings of global dimension 3.

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