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HURWITZ GROUPS AND $G_2(q)$

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ABSTRACT. Finite factor groups of $G_{2,3,7} := \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1^2 = \sigma_2^3 = \sigma_3^7 = \sigma_1\sigma_2\sigma_3 = \iota \rangle$ are called Hurwitz groups. Here we prove that apart from ${}^2G_2(3)$, $G_2(2)$, $G_2(3)$ and $G_2(4)$, all the groups ${}^2G_2(3^{2n+1})$ and $G_2(q)$, $q = p^n$, $p \in \mathbb{P}$, are Hurwitz groups. For the proof, (2, 3, 7) structure constants are calculated from the character tables [2], [7], and then with the lists of maximal subgroups [8], [5], the existence of generating triples is deduced.

A (finite) group G is called *Hurwitz group* if it is a factor group of

$$G_{2,3,7} := \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1^2 = \sigma_2^3 = \sigma_3^7 = \sigma_1\sigma_2\sigma_3 = \iota \rangle,$$

or, to put it differently, it is generated by two elements of order two and three such that their product has order seven. The interest in such groups stems from the fact that they can be represented as a group of $84(\gamma - 1)$ automorphisms of a Riemann surface of genus γ. This is the biggest possible value for the order of a group acting on a surface of genus $\gamma \geqq 2$. As this bound was found by Hurwitz, such groups are called Hurwitz groups (see [9], and also [3] and the references cited there, for more recent investigations). The first example of such a group seems to have been found by Klein, namely $G = L_2(7)$.

In [9], for example, the family of simple groups $L_2(q)$ is studied and those of them which are Hurwitz groups are determined. Here we consider the Dickson-Cayley groups $G_2(q)$, $q = p^n$, $p \in \mathbb{P}$. They should be good candidates for an investigation, since they all have orders divisible by 2, 3 and 7. In contrast to the situation in [9], where at most groups with $q = p$ or $q = p^3$ are Hurwitz groups, almost all of the $G_2(q)$ turn out to be Hurwitz groups.

THEOREM 1. *Let $q = p^n$, $p \in \mathbb{P}$ and $q \geqq 5$. Then $G_2(q)$ is a Hurwitz group. On the other hand, none of $G_2(2)$, $G_2(3)$ or $G_2(4)$ are Hurwitz groups.*

THEOREM 2. *Let $q = 3^{2m+1}$, $m \geqq 1$. Then ${}^2G_2(q)$ is a Hurwitz group.*

An immediate corollary of the two theorems is

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Corollary. *All the groups $G_2(q)$ and ${}^2G_2(q)$ are factor groups of the modular group $SL_2(\mathbb{Z})$.*

Proof. The modular group $PSL_2(\mathbb{Z})$ is the free product of two cyclic groups of order two and three. Hence any group generated by two elements of orders two and three is a factor group. From the theorems, this is clear for $G_2(q)$, $q \neq 2, 3, 4$, and ${}^2G_2(q)$, $q \neq 3$. (For the Ree groups, this already follows from Theorem 2.1 in [10].) With the help of the Atlas [4], it is not hard to check that $(2B, 3B, 12C)$ of $G_2(2)$, $(2A, 3E, 13A)$ of $G_2(3)$, $(2B, 3B, 13A)$ of $G_2(4)$, and $(2A, 3B, 9D)$ of ${}^2G_2(3)$ contain generating triples for the respective groups. □

1. **Description of the method.** The proof of the theorems is based on the knowledge of the character table and the maximal subgroups of $G = G_2(q)$, which were calculated by several authors. To explain the strategy, we reformulate the condition for Hurwitz groups. Let G be a finite group, C_1, C_2, C_3 three conjugacy classes, then call $\mathfrak{C} = (C_1, C_2\ C_3)$ a *class structure*, as in constructive Galois theory [11]. Also define

$$\bar{\Sigma}(\mathfrak{C}) = \{(\sigma_1, \sigma_2, \sigma_3) | \sigma_i \in C_1,\ \sigma_1\sigma_2\sigma_3 = \iota\}$$

and

$$\Sigma(\mathfrak{C}) = \{(\sigma_1, \sigma_2, \sigma_3) \in \bar{\Sigma}(\mathfrak{C}) | \langle \sigma_1, \sigma_2, \sigma_3 \rangle = G\}.$$

Then G is a Hurwitz group if there exists a class structure $\mathfrak{C} = (C_1, C_2, C_3)$ such that C_1 contains involutions, C_2 and C_3 elements of orders three, seven respectively, and $|\Sigma(\mathfrak{C})| > 0$. We will consider triples in $\Sigma(\mathfrak{C})$ modulo conjugation in G. Then it is known that if $Z(G) = 1$ the *normalized structure constant*

$$n(\mathfrak{C}) := \frac{|G|}{|C_G(\sigma_1)| \cdot |C_G(\sigma_2)| \cdot |C_G(\sigma_3)|} \sum_{i=1}^{h} \frac{\chi_i(\sigma_1)\chi_i(\sigma_2)\chi_i(\sigma_3)}{\chi_i(\iota)}$$

(where we sum over all irreducible characters χ_i of G) counts the number of classes of triples $(\sigma_1, \sigma_2, \sigma_3) \in \bar{\Sigma}(\mathfrak{C})$ modulo conjugation, each contributing $(|Z(K)| \cdot (\mathcal{N}_G(K) : K))^{-1}$ to $n(\mathfrak{C})$ if $K = \langle \sigma_1, \sigma_2 \rangle$ [11]. Hence if $K_1, \ldots, K_s$ is a full system of representatives of the conjugacy classes of maximal subgroups of G, we get that G is a Hurwitz group if

$$n_G(\mathfrak{C}) - \sum_{i=1}^{s} \sum_{j=1}^{t_i} n_{K_i}(\mathfrak{C}_{ij}) > 0,$$

where the second summation ranges over all class structures $\mathfrak{C}_{ij}$ of K_i fusing into $\mathfrak{C}$. It is this condition which we will verify for $G = G_2(q)$.

We now collect some information about suitable classes in the groups $G_2(q)$. It can be deduced from the classification of the conjugacy classes in [1] and [6]. The abbreviation $\Phi_i = \Phi_i(q) :=$ *i-th cyclotomic polynomial evaluated at q* will be used throughout.

PROPOSITION 1. *(a) Let $p \neq 7$. Then $G_2(q)$ contains a single class of regular elements of order seven. (b) Let $p \neq 2$. Then $G_2(q)$ contains a single class of involutions. (c) Let $p \neq 3$. Then $G_2(q)$ contains just two classes of elements of order three. All these classes are rational.*

PROOF. We have $|G_2(q)| = q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$. If $q \equiv 2, 3, 4, 5 \pmod 7$ then $\Phi_3(q)$ or $\Phi_6(q)$ are divisible by seven, hence elements of order seven are contained in a maximal torus T of order $\Phi_3(q)$ or $\Phi_6(q)$. Apart from possibly elements of order three, all elements in those tori are regular. The Weyl group $W(T)$ of those tori has order six, hence the elements of order seven are all conjugate. If $q \equiv 1 \pmod 7$, then $7|\Phi_1(q)$ and the maximal torus $T \cong Z_{q-1} \times Z_{q-1}$ contains a Sylow 7-subgroup. From the action of the Weyl group $W(G_2) \cong D_{12}$ on T one sees that there exists exactly one class of regular elements of order seven. The same arguments apply for $q \equiv -1$ (mod 7) with $T \cong Z_{q+1} \times Z_{q+1}$. Again from the action of the Weyl groups of the relevant tori, the assertion about the rationality of that class is deduced.

Parts (b) and (c) follow from the lists of conjugacy classes in [1] and [6]. □

2. **The general case.** We now choose for $p \neq 7$ the class $7A$ to contain the regular elements of order seven, for $p \neq 2$ the class $2A$ to be the class of involutions, and for $p \neq 3$ the class $3B$ to be the class of elements of order three with centralizer order $q\Phi_1\Phi_2(q-\epsilon)$, $q \equiv \epsilon \pmod 3$. If $G_2(q_0) < G_2(q)$ with $\mathbb{F}_{q_0} < \mathbb{F}_q$, then it is clear from the description of the classes that exactly the three classes $(2A, 3B, 7A)$ of $G_2(q_0)$ fuse into $(2A, 3B, 7A)$ of $G_2(q)$.

PROPOSITION 2. *Let $(\sigma) = (\sigma_1, \sigma_2, \sigma_3)$ be a $(2, 3, 7)$-system of $G = G_2(q)$ with $\sigma_1\sigma_2\sigma_3 = \iota$. If $2 \nmid q$, or $2|q$ and $\sigma_3 \in 7A$, then $\langle\sigma\rangle$ is not contained in a maximal parabolic subgroup of G.*

PROOF. Maximal parabolic subgroups P of $G_2(q)$ have the structure $P = [q^5] : GL_2(q)$. If P has a $(2, 3, 7)$-system, then so has $GL_2(q)$, the factor group of P by the unipotent radical. Inspection of the determinants shows that this triple even has to lie inside $SL_2(q)$. But for $2 \nmid q$ this group does not have $(2, 3, 7)$-system, since its only involution lies in the center. If $2|q$, a different argument is needed. Here we use that no regular semisimple element (like σ_3) can lie inside the $SL_2(q)$ of the factor group of P by the unipotent radical. This follows from the action of the Weyl group on the maximally split torus. □

PROPOSITION 3. *Theorem 1 holds for $p \neq 2, 3, 7$ with $\mathfrak{C} = (2A, 3B, 7A)$.*

PROOF. Following the reformulated criterion, we first calculate the structure constant $n_G(\mathfrak{C})$. This is possible using the character table of $G = G_2(q)$ in [2]. Different cases have to be distinguished, according to the congruences of q modulo 3 and 7. In Table 1, the respective values of the normalized structure constant $n(2A, 3B, 7A)$ are displayed (the values were checked by Gerhard Hiss, using the MAPLE computer system):

TABLE 1: The (2,3,7) normalized structure constants in $G_2(p^n)$, $p \neq 2,3,7$.

$q \equiv$	1 (mod 7)	−1 (mod 7)	2,4 (mod 7)	3,5 (mod 7)
1 (mod 3)	q^2+3q+1	q^2-q+1	q^2+1	q^2+2q+1
−1 (mod 3)	q^2+q+1	q^2-3q+1	q^2-2q+1	q^2+1

Taking into account the congruences for q this shows that for any $q = p^n$, $p \neq 2, 3, 7$, we have $\frac{1}{2}q^2 \leqq n(\mathfrak{C}) \leqq 2q^2$.

We must now decide which contribution to the structure constant comes from proper subgroups of G. By the considerations in the first part it is sufficient to do this for a representative system of classes of maximal subgroups. In Table 2 we list all classes of maximal subgroups K of $G_2(q)$ for $p \geqq 5$ [8], except for the parabolic subgroups (they were excluded in Proposition 2).

TABLE 2: Some maximal subgroups of $G_2(q)$, $q = p^n$, $p \geq 5$.

K	occurs for	$n_K(2,3,7)$
$(SL_2(q) \circ SL_2(q)) \cdot 2$	always	≤ 18
$2^3 \cdot L_3(2)$	$q = p$	4
$SL_3(q){:}2$	always	≤ 3
$SU_3(q){:}2$	always	≤ 3
$PGL_2(q)$	$p \geq 7, q \geq 11$	≤ 3
$L_2(8)$	$\mathbb{F}q = \mathbb{F}p[\zeta_7 + \overline{\zeta_7}]$	3
$L_2(13)$	$\mathbb{F}q = \mathbb{F}p[\sqrt{13}]$	6
$G_2(2)$	$q = p$	1
J_1	$q = 11$	7
$G_2(q_0)$	$q = q_0^t, t \in \mathbb{P}$	$\leq 2q_0^2$

We have added a column showing the number of (2, 3, 7)-triples in K which can possibly fuse into our G-class structure $\mathfrak{C}$. For the involution centralizer we use that $L_2(q)$ possesses at most six such classes, hence the central product can have no more than 36 (remember that the class 7A is regular). The 2 on top fuses them into at most 18. For $SL_3(q)$ and $SU_3(q)$ we could either invoke the lemma of Belyi [11], p. 106, telling us that any class structure of $SL_3(q)$ with an involution contains at most one generating class of triples, and then use the list of maximal subgroups of $SL_3(q)$ to determine the contribution from proper subgroups of this maximal subgroup. But it is easier to explicitely calculate $n_{SL_3}(\mathfrak{C})$ from the character tables in [12].

The case of the involution centralizer contributes to $n(\mathfrak{C})$ only if seven divides its order, i.e. $q \equiv \pm 1 \pmod 7$. The smallest such q in our analysis is $q = 13$. From the table and the value for $n(\mathfrak{C})$ above, the proposition immediately follows for $q = 5$. So we have $q \geqq 11$. Then the contribution from the maximal subgroups different from $G_2(q_0)$ is at most 48. Hence for $q = p$, we are done. If $q = p^2$, the contribution from $G_2(p)$ is at most $2p^2$, and $\frac{1}{2}p^4 \geqq 2p^2 + 48$.

So $q \geqq p^3 \geqq 125$. We can estimate $48 \leqq 2p^2$. Let $q = p^n$ and $n = \prod_{i=1}^{r} t_i^{\nu_i}$ the

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prime decomposition of n. Then we have to subtract at most

$$\sum_{i=1}^{r} 2p^{2n/t_i} \leqq \sum_{i=1}^{r} 2p^n = 2rp^n$$

as an upper bound for the part of $n_G(\mathfrak{C})$ lying inside all $G_2(q_0)$. So the number $l(\mathfrak{C})$ of generating triples in the class structure (i.e. the order of $\Sigma(\mathfrak{C})$ modulo $\mathrm{Inn}(G)$) is at least

$$l(\mathfrak{C}) \geqq \tfrac{1}{2}p^{2n} - 2rp^n - 2p^2 \geqq \tfrac{1}{2}p^n(p^n - 4r - \epsilon) \geqq 1, \text{ with an } \epsilon < 1,$$

for all prime powers under consideration. This proves the proposition. □

3. **The case** $p = 2$. Let us now turn to the exceptional cases, namely $p \in \{2, 3, 7\}$. If $p = 2$, the involutions are unipotent. We take $2U$ to be the class of involutions of $G_2(2^n)$ with centralizer order $q^4\Phi_1\Phi_2$ [6].

Proposition 4. *Theorem 1 holds for* $q = 2^n \geqq 8$ *with* $\mathfrak{C} = (2U, 3B, 7A)$.

Proof. The character values in $G_2(2^n)$ may be calculated by the Deligne-Lusztig theory from the values of the Green functions. From them we obtain

Table 3: The (2,3,7) normalized structure constants in $G_2(2^n)$.

$q \equiv$	1 (mod 7)	2,4 (mod 7)
1 (mod 3)	$q^2 + 3q$	q^2
−1 (mod 3)	$q^2 + q$	$q^2 - 2q$

Hence we get $\frac{1}{2}q^2 \leqq n(\mathfrak{C}) \leqq 2q^2$ for all $q = 2^n$. As in the argument for $p \neq 2, 3, 7$, we now need a list of maximal subgroups of $G_2(2^n)$ [5] to see which part of the structure constant comes from proper subgroups.

Table 4: Some maximal subgroups of $G_2(2^n)$, $n \geq 3$.

K	occurs for	$n_K(2,3,7)$
$L_2(q) \times L_2(q)$	always	≤ 10
$SL_3(q)$:2	always	≤ 3
$SU_3(q)$:2	always	≤ 3
$G_2(q_0)$	$q = q_0^t,\ t \in \mathbb{P}$	$\leq 2q_0^2$

The maximal parabolics can be disregarded by Proposition 2. So, with the prime decomposition $n = \prod_{i=1}^{r} t_i^{\nu_i}$ we have

$$l(\mathfrak{C}) \geqq \tfrac{1}{2}q^2 - 16 - \sum_{i=1}^{r} 2 \cdot 2^{\frac{2n}{t_i}} \geqq 2^{2n-1} - 16 - \sum_{i=1}^{r} 2^{n+1} = 2^{n+1}(2^{n-2} - r - 2^{3-n}) \geqq 1,$$

proving the assertion. (The case $q = 8$ needs a bit more careful estimations.) □

4. **The case** $p = 3$. Next let $p = 3$. We will need the (2, 3, 7) structure constants of ${}^2G_2(3^n)$ to handle $G_2(3^n)$. From them, it is immediate to verify

PROPOSITION 5. *Theorem 2 holds for* ${}^2G_2(3^{2m+1})$, $m \geqq 1$, *with* $\mathfrak{C} = (2A,\ 3B,\ 7A)$.

PROOF. The groups $G = {}^2G_2(q)$, $q = 3^{2m+1}$, possess one class $2A$ of involutions, one class $7A$ of (rational) elements of order seven and three classes of elements of order three [13]. We will choose $3B$ to be one of the two nonrational such classes. Then the character table of G in [13] yields $\frac{1}{2}(q - \sqrt{3q}) \leqq n(\mathfrak{C}) \leqq \frac{1}{2}(q + \sqrt{3q})$. The maximal subgroups of G are found in [8], and by similar but easier arguments to the G_2-cases, the proposition is deduced. Clearly, since $({}^2G_2(3))' = L_2(8) < {}^2G_2(3)$, this group can not be a Hurwitz group. □

For $G = G_2(3^n)$ it proves suitable to work with two classes of 3-elements. Namely, let $3U$ be the union of the two unipotent classes with representatives A_{41} and A_{42} in [7]. By abuse of notation, still write $\mathfrak{C} = (2A,\ 3U,\ 7A)$, so that $n(\mathfrak{C}) = n(2A,\ [A_{41}],\ 7A) + n(2A,\ [A_{42}],\ 7A)$. This is necessary because an element A_{42} of a subgroup $G_2(3^m) < G_2(3^n)$ may fuse either into A_{41} or A_{42}, as is seen from the Steinberg representatives of the two classes in Proposition 6.4 in [6]. (There, the relevant elements are called x_4 and x_5.)

PROPOSITION 6. *Theorem* 1 *holds for* $q = 3^n \geqq 9$ *with* $\mathfrak{C} = (2A,\ 3U,\ 7A)$.

PROOF. From the character table of $G_2(3^n)$ in [7] the following table of normalized structure constants is calculated:

TABLE 5: The (2,3,7) normalized structure constants in $G_2(3^n)$.

$q \equiv$	1 (mod 7)	−1 (mod 7)	2,4 (mod 7)	3,5 (mod 7)
	$q^2 + 2q$	$q^2 - 2q$	$q^2 - q$	$q^2 + q$

Hence once more we have $\frac{1}{2}q^2 \leqq n(\mathfrak{C}) \leqq 2q^2$ for $q = 3^n$. Next, the list of maximal subgroups, taken from [8], is as follows

TABLE 6: Some maximal subgroups of $G_2(3^n)$, $n \geq 2$.

K	occurs for	$n_K(2,3,7)$
$(SL_2(q) \circ SL_2(q)) \cdot 2$	always	≤ 18
$SL_3(q){:}2$	always	≤ 1
$SL_3(q){:}2$	always	≤ 1
$SU_3(q){:}2$	always	≤ 1
$SU_3(q){:}2$	always	≤ 1
$G_2(q_0)$	$q = q_0^t,\ t \in \mathbb{P}$	$\leq 2q_0^2$
${}^2G_2(q)$	$q = 3^n$, n odd	$\leq 2q$

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The contribution from ${}^2G_2(q)$ is the sum over the three possible (2, 3, 7) structures of that group, see the proof of Proposition 5 and [13]. Estimations as in the preceeding sections now show that $l(\mathfrak{C}) > 0$. □

5. **The case** $p = 7$. The regular unipotent elements of $G_2(7^n)$ have order seven, as can be calculated from the Steinberg generators in [1], (2.1). (This contrasts to $G_2(5^n)$, where those elements have order 25.) Call the class of regular unipotent elements $7U$. Then by [1] or general theorems on regular unipotent elements in good characteristic, this is a rational class. Hence $\mathfrak{C} = (2A, 3B, 7U)$ is a rational class structure.

PROPOSITION 7. *Theorem 1 holds for $q = 7^n$ with $\mathfrak{C} = (2A, 3B, 7U)$.*

PROOF. From the character table of G in [2] it is possible to calculate the following structure constants:

TABLE 7: The (2,3,7) normalized structure constants in $G_2(7^n)$.

$q \equiv$	
1 (mod 3)	$q^2 + q$
-1 (mod 3)	$q^2 - q$

The contribution from maximal subgroups of G was already collected in Table 2. (By Proposition 2, the maximal parabolic subgroups do not possess relevant (2, 3, 7)-systems.) Estimations as in the second section now show that there always remain generating triples for the whole group, and the proposition follows. □

6. **Exclusion of the remaining groups.**

PROPOSITION 8. *The groups $G_2(2)$, $G_2(3)$ and $G_2(4)$ are no Hurwitz groups.*

PROOF. For $G_2(2) = U_3(3) : 2$ this is trivial, since any group generated by a (2, 3, 7)-triple has to satisfy $G = G'$. It is less obvious for $G_2(3)$ and $G_2(4)$. Here we make heavy use of the Atlas tables for these groups and for various subgroups [4]. Moreover we use the fact that a class of (2, 3, 7)-systems generating a proper selfnormalizing subgroup H of G with trivial center contributes precisely 1 to the structures constant in G.

The group $G_2(3)$ has five classes of elements of order three, $3A, \ldots, 3E$. The other relevant conjugacy classes were already mentioned in Proposition 1 (there exist no nonregular elements of order seven in $G_2(3)$). The nonzero structure constants are

$$n(2A,\ 3C,\ 7A) = 1, \qquad n(2A,\ 3D,\ 7A) = n(2A,\ 3E,\ 7A) = 6.$$

For the maximal subgroup $L_2(8) : 3$, which contains elements from $2A$, $3C$ and $7A$, we find a (2, 3, 7)-structure constant of 1. This accounts for $n(2A,\ 3C,\ 7A)$. Next $L_2(13)$ has six classes of (2, 3, 7)-systems, and by [4] nontrivially intersects $3D$, killing $n(2A,\ 3D,\ 7A)$. For $2^3 \cdot L_3(2)$, we know $n(2, 3, 7) = 4$, and as the extension is nonsplit (see also [8]), each triple has to generate all of $2^3 \cdot L_3(2)$. Finally there are two classes of $U_3(3):2$ in $G_2(3)$, each having $n(2, 3, 7) = 1$. The group generated by these is at least an $L_2(7)$. By comparing the fusion of elements of order four, they are seen to be distinct. So $n(2A, 3E, 7A)$ also belongs to proper subgroups, and $G_2(3)$ can not be a Hurwitz group.

The interesting conjugacy classes in $G_2(4)$ are (in Atlas notation) two classes of involutions $2A$, $2B$, two classes of elements of order three $3A$, $3B$, and the class $7A$. We get the structure constants

$$n(2A, 3A, 7A) = n(2B, 3A, 7A) = 0,$$

$$n(2A, 3B, 7A) = 1, \qquad n(2B, 3B, 7A) = 16.$$

The maximal subgroup J_2 also has four interesting class structures with constants (the classes as in the Atlas)

$$n(2A, 3A, 7A) = n(2B, 3A, 7A) = 0,$$

$$n(2A, 3B, 7A) = 1, \qquad n(2B, 3B, 7A) = 10.$$

By inspection of the list of maximal subgroups of J_2 and their fusion into J_2 it is clear that all $(2B, 3B, 7A)$-triples generate J_2, giving a contribution of 10 to the structure constant of $G_2(4)$. On the other hand, the $(2A, 3B, 7A)$-triple generates an $L_2(7)$, which corresponds to the $(2A, 3B, 7A)$-triple in $G_2(4)$. We next study the maximal subgroup $L_2(13)$ of G. From Table 2 we see that this group has six $(2, 3, 7)$-systems, and they all generate the whole group. Hence another six of the $(2B, 3B, 7A)$-systems of G are taken away, and no one remains to generate all of $G_2(4)$. This proves the assertion. □

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