J. Inst. Math. Jussieu (2024), **23**(4), 1521–1566
doi:10.1017/S1474748023000282

OPTIMAL GEVREY STABILITY OF HYDROSTATIC APPROXIMATION FOR THE NAVIER-STOKES EQUATIONS IN A THIN DOMAIN

CHAO WANG[1] AND YUXI WANG[2]

[1] *School of Mathematical Sciences, Peking University, Beijing 100871, China*
(wangchao@math.pku.edu.cn)

[2] *School of Mathematics, Sichuan University, Chengdu 610064, China*
(wangyuxi@scu.edu.cn)

(Received 27 July 2022; revised 29 June 2023; accepted 12 July 2023;
first published online 6 September 2023)

Abstract In this paper, we study the hydrostatic approximation for the Navier-Stokes system in a thin domain. When we have convex initial data with Gevrey regularity of optimal index $\frac{3}{2}$ in the x variable and Sobolev regularity in the y variable, we justify the limit from the anisotropic Navier-Stokes system to the hydrostatic Navier-Stokes/Prandtl system. Due to our method in the paper being independent of ε, by the same argument, we also obtain the well-posedness of the hydrostatic Navier-Stokes/Prandtl system in the optimal Gevrey space. Our results improve upon the Gevrey index of $\frac{9}{8}$ found in [15, 35].

1. Introduction

1.1. Presentation of the problem and related results

In this article, we study 2-D incompressible Navier-Stokes equations in a thin domain where the aspect ratio and the Reynolds number have certain constraints:

$$\begin{cases} \partial_t U + U\cdot\nabla U - \varepsilon^2(\partial_x^2+\eta\partial_y^2)U + \nabla P = 0, \\ \mathrm{div}\ U = 0, \\ U|_{y=0} = U|_{y=\varepsilon} = 0, \end{cases} \tag{1.1}$$

where $t\geq 0, (x,y)\in\mathcal{S}^\varepsilon = \{(x,y)\in\mathbb{T}\times\mathbb{R} : 0<y<\varepsilon\}$. Here, $U(t,x,y), P(t,x,y)$ stand for the velocity and pressure function, respectively, and η is a positive constant independent of ε. The width of domain $\mathcal{S}^\varepsilon$ is ε, and the boundary condition in (1.1) corresponds to the non-slip condition at the walls $y=0,\varepsilon$. In addition, the system is prescribed with the initial data of the form

Keywords: Navier-Stokes equations; hydrostatic approximation; thin domain

2020 Mathematics subject classification: Primary 35Q30

$$U|_{t=0} = \left(u_0\left(x,\frac{y}{\varepsilon}\right),\varepsilon v_0\left(x,\frac{y}{\varepsilon}\right)\right) = U_0^\varepsilon \quad \text{in} \quad \mathcal{S}^\varepsilon. \tag{1.2}$$

This is a classical model with applications to oceanography, meteorology and geophysical flows, where the vertical dimension of the domain is very small compared with the horizontal dimension of the domain.

To study the process $\varepsilon \to 0$, we first fix the domain independent of ε. Here, we rescale the system (1.1) as follows:

$$U(t,x,y) = \left(u^\varepsilon\left(t,x,\frac{y}{\varepsilon}\right),\varepsilon v^\varepsilon\left(t,x,\frac{y}{\varepsilon}\right)\right) \quad \text{and} \quad P(t,x,y) = p^\varepsilon\left(t,x,\frac{y}{\varepsilon}\right).$$

We put above relations into (1.1), and then (1.1) is reduced to a scaled anisotropic Navier-Stokes system:

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon - \varepsilon^2\partial_x^2 u^\varepsilon - \eta\partial_y^2 u^\varepsilon + \partial_x p^\varepsilon = 0, \\ \varepsilon^2(\partial_t v^\varepsilon + u^\varepsilon\partial_x v^\varepsilon + v^\varepsilon\partial_y v^\varepsilon - \varepsilon^2\partial_x^2 v^\varepsilon - \eta\partial_y^2 v^\varepsilon) + \partial_y p^\varepsilon = 0, \\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0, \\ (u^\varepsilon,v^\varepsilon)|_{y=0,1} = 0, \\ (u^\varepsilon,v^\varepsilon)|_{t=0} = (u_0,v_0), \end{cases} \tag{1.3}$$

where $(x,y) \in \mathcal{S} = \{(x,y) \in \mathbb{T}\times(0,1)\}$.

To simplify the notations, we take $\eta = 1$ in this paper and denote $\Delta_\varepsilon = \varepsilon^2\partial_x^2 + \partial_y^2$.

Formally, taking $\varepsilon \to 0$ in (1.3), we derive the hydrostatic Navier-Stokes/Prandtl system (see [22, 31]):

$$\begin{cases} \partial_t u_p + u_p\partial_x u_p + v_p\partial_y u_p - \eta\partial_y^2 u_p + \partial_x p_p = 0 \quad \text{in } \mathcal{S}\times(0,\infty), \\ \partial_y p_p = 0 \quad \text{in } \mathcal{S}\times(0,\infty), \\ \partial_x u_p + \partial_y v_p = 0 \quad \text{in } \mathcal{S}\times(0,\infty), \\ (u_p,v_p)|_{y=0,1} = 0, \\ u_p|_{t=0} = u_0 \quad \text{in } \mathcal{S}. \end{cases} \tag{1.4}$$

This paper aims to justify the limit from the scaled anisotropic Navier-Stokes system (1.3) to the hydrostatic Navier-Stokes/Prandtl system (1.4) for a class of convex data in the optimal Gevrey class with index $\gamma = \frac{3}{2}$.

Before presenting the precise statement of the main result in this paper, we recall some results on system (1.4). If $\eta = 0$ in the system (1.4), we get the hydrostatic Euler system:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0 \quad \text{in } \mathcal{S}\times(0,\infty), \\ \partial_y p = 0 \quad \text{in } \mathcal{S}\times(0,\infty), \\ \partial_x u + \partial_y v = 0 \quad \text{in } \mathcal{S}\times(0,\infty), \\ v|_{y=0,1} = 0, \\ u|_{t=0} = u_0 \quad \text{in } \mathcal{S}. \end{cases} \tag{1.5}$$

There is a lot of research on the system (1.5), and readers can refer to [3, 4, 5, 16, 20, 19, 26, 31, 38]. Renardy [31] proved the linearization of (1.5) has a growth like $e^{|k|t}$ if

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

the initial data is not uniform convexity (or concavity) with respect to the variable y. Local well-posedness in the analytic setting was established in [20]. Under the convexity condition, Masmoudi and Wong [26] got the well-posedness of (1.5) in the Sobolev space.

Next, we recall some results on the well-posedness of the hydrostatic Navier-Stokes/Prandtl system (1.4). Similar to the classical Prandtl equation, (1.4) loses one derivative because of the term $v_p\partial_y u_p$. Paicu, Zhang and Zhang [30] obtained the global well-posedness of system (1.4) when the initial data is small in the analytical space. Meanwhile, Renardy [31] also proved that the linearization of the hydrostatic Navier-Stokes equations at certain parallel shear flows is ill-posed and may have a growth $e^{|k|t}$ which is the same as (1.5) when the initial data is not convex. Thus, to obtain well-posedness results that break through the analytic space, one may need the convexity condition on the velocity. For that, under the convexity condition, Gérard-Varet, Masmoudi and Vicol proved (1.4) is local well-posedness in the Gevrey class with index 9/8 in [15]. In [15], they first derive the vorticity equations $\omega = \partial_y u$:

$$\partial_t(\partial_x\omega) + \partial_x v\partial_y\omega + \cdots = 0,$$

where the worst term is $\partial_x v$ leading to one derivative loss. Then, they use the 'hydrostatic trick' which means that they take the inner product with $\partial_x\omega/\partial_y\omega$ ($\partial_y\omega \geq c_0 > 0$) instead of $\partial_x\omega$ to take advantage of the cancellation:

$$\int \partial_x v\partial_y\omega \cdot \frac{\partial_x\omega}{\partial_y\omega} = \int \partial_x v\partial_x\omega = -\int \partial_x\partial_y v\partial_x u = 0.$$

Such an idea was used previously in [26]. To close the energy estimates, the 'hydrostatic trick' is not enough due to the 'bad' boundary condition of ω

$$\partial_y\omega|_{y=0} = -\partial_x \int_0^1 u^2 dy + \cdots,$$

which loses one derivative too. To overcome that, [15] introduce the following decomposition:

$$\omega = \omega^{bl} + \omega^{in},$$

where ω^{bl} is the boundary corrector which satisfies that

$$\partial_t\omega^{bl} - \partial_y^2\omega^{bl} = 0, \quad \partial_y\omega^{bl}|_{y=0} = -\partial_x \int_0^1 u^2 dy.$$

Following the above decomposition, [15] obtain the well-posedness results of (1.4) in the Gevrey class with index $\gamma = \frac{9}{8}$.

To search the best functional space for the system (1.4), based on the Tollmien-Schlichting instabilities for Navier-Stokes [17], Gérard-Varet, Masmoudi and Vicol also give the following conjecture: *'Our conjecture - based on a formal parallel with Tollmien-Schlichting instabilities for Navier-Stokes [18] - is that the best exponent possible should be* $\gamma = \frac{3}{2}$*, but the such result is for the time being out of reach'.*

While studying the anisotropic Navier-Stokes system (1.3) and the hydrostatic Navier-Stokes/Prandtl system (1.4), another important problem is to justify the inviscid limit. Under the analytical setting, Paicu, Zhang and Zhang [30] justified the limit from (1.3) to (1.4). Based on the work [15], we [35] justified the limit in the Gevrey class with index $\gamma = \frac{9}{8}$.

In this paper, we aim to prove the conjecture of Gérard-Varet, Masmoudi and Vicol. To do that, we use some ideas from the classical inviscid limit theory. Next, we recall the recent development of the classical Prandtl equation and the inviscid limit theory.

There are a lot of papers studying the well-posedness of the Prandtl equation in some special functional space. For monotonic initial data, [29, 1, 27] used a different method to get the local existence and uniqueness of classical solutions to the Prandtl equation in Sobolev space. Without monotonic condition, [24, 32] proved that the Prandtl equation is well-posedness in the analytic class; [14, 23, 6] proved the well-posedness of the Prandtl equations in the Gevrey class for a class of concave initial data. Without any structure assumption, Dietert and Gérard-Varet [8] proved well-posedness in the Gevrey space with index $\gamma = 2$. According to [10], $\gamma = 2$ may be the optimal index for the well-posedness theory. For more results on the Prandtl equation, see [18, 37, 36, 39, 40].

On the inviscid limit problem, we refer to [33, 34, 21, 28, 25, 9] for the analytical class. Note that going from analytic to Gevrey data is a challenging problem. The first result in the Gevrey class is given by [12]. Gérard-Varet, Masmoudi and Maekawa [12] proved the stability of the Prandtl expansion for the perturbations in the Gevrey class when $U^{BL}(t,Y)$ is a monotone and concave function where the boundary layer is the shear type like

$$u_s^\nu = (U^e(t,y),0) + (U^{BL}(t,\frac{y}{\sqrt{\nu}}),0),$$

where ν is the viscosity coefficient. Later, Chen, Wu and Zhang [7] improved the results in [12] to get the $L^2 \cap L^\infty$ stability. Very recently, Gérard-Varet, Masmoudi and Maekawa [13] used a very clever decomposition to get the optimal Prandtl expansion around the concave boundary layer. Their results generalized the one obtained in [12, 7], which restricted to expansions of shear flow type. In their paper, they decompose the stream function ϕ as follows:

$$\phi = \phi_{slip} + \phi_{bc},$$

where ϕ_{slip} enjoys a "good" boundary condition and ϕ_{bc} is a corrector which recover the boundary condition. This kind of decomposition is also used in [7]. To estimate ϕ_{bc}, they also need the following decomposition

$$\phi_{bc} = \phi_{bc,S} + \phi_{bc,T} + \phi_{bc,R},$$

where $\phi_{bc,S}$ satisfies the Stokes equation, $\phi_{bc,T}$ is to correct the stretching term with 'good' boundary condition and $\phi_{bc,R}$ solves formally the same system as ϕ_{slip}. In this paper, we apply the decomposition in [13] to justify the limit from (1.3) to (1.4).

1.2. Statement of the main results.

Before starting the main results, we give some assumptions on the initial data. Assume that initial data belong to the following Gevrey class:

$$\|e^{\langle D_x\rangle^{\frac{2}{3}}}\partial_y u_0\|_{H^{14,0}} + \|e^{\langle D_x\rangle^{\frac{2}{3}}}\partial_y^3 u_0\|_{H^{10,0}} := M < +\infty, \tag{1.6}$$

where $H^{r,s}$ is the anisotropic Sobolev space defined by

$$\|f\|_{H^{r,s}} = \|\|f\|_{H^r_x(\mathbb{T})}\|_{H^s_y(0,1)}.$$

More precisely, we consider the initial data of the form

$$u^\varepsilon(0,x,y) = u_0(x,y), \quad v^\varepsilon(0,x,y) = v_0(x,y),$$

which satisfies the compatibility conditions

$$\partial_x u_0 + \partial_y v_0(t,x,y) = 0, \quad u_0(t,x,0) = u_0(t,x,1) = v_0(t,x,0) = v_0(t,x,1) = 0, \tag{1.7}$$

$$\int_0^1 \partial_x u_0 dy = 0, \quad \partial_y^2 u_0|_{y=0,1} = \int_0^1 (-\partial_x u_0^2 + \partial_y^2 u_0) - \int_{\mathcal{S}} \partial_y^2 u_0 dxdy. \tag{1.8}$$

Moreover, we assume the initial velocity satisfies the convex condition

$$\inf_{\mathcal{S}} \partial_y^2 u_0 \geq 2c_0 > 0. \tag{1.9}$$

Now, we are in the position to state the main results of our paper.

Theorem 1.1. *Let initial data u_0 satisfy (1.6)-(1.9). Then there exist $T > 0$ and $C > 0$ independent of ε such that there exists a unique solution of the scaled anisotropic Navier–Stokes equations (1.3) in $[0,T]$, which satisfies that for any $t \in [0,T]$, it holds that*

$$\|(u^\varepsilon - u_p, \varepsilon v^\varepsilon - \varepsilon v_p)\|_{L^2_{x,y}\cap L^\infty_{x,y}} \leq C\varepsilon^2,$$

where (u_p,v_p) is the solution to (1.4).

Remark 1.2. Although we do not give the proof that the system (1.4) is well-posedness in Gevrey class $\frac{3}{2}$, one can follow the proof of Theorem 1.1 to obtain the well-posedness. To avoid repeatability in the proof, we omit the details. Actually, the main difference between $\varepsilon = 0$ and $\varepsilon \neq 0$ is on the construction boundary corrector $\phi_{bc,S}$, and readers can find more details in Remark 8.1.

Remark 1.3. In the recent work [11], they established the well-posedness of the linearized Hydrostatic Navier-Stokes system around shear flow in Gevrey class $\frac{3}{2}$. In our present work, we consider the general nonlinear system and focus on the inviscid limit problem.

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

1.3. Sketch of the proof.

In this subsection, we sketch the main ingredients in our proof.

(1) **Introduce the error equations.** In Section 3, we deduce the error equations. We introduce the error

$$u^R = u^\varepsilon - u^p, \quad v^R = v^\varepsilon - v^p, \quad p^R = p^\varepsilon - p^p,$$

which satisfies

$$\begin{cases} \partial_t u^R - \Delta_\varepsilon u^R + v^R \partial_y u^p + \partial_x p^R = \cdots, \\ \varepsilon^2(\partial_t v^R - \Delta_\varepsilon v^R) + \partial_y p^R = \cdots. \end{cases} \tag{1.10}$$

Here, (u^p, v^p, p^p) is an approximate solution given in (3.1). The key point in this paper is to obtain the uniform estimate (in ε) of $(u^R, \varepsilon v^R)$ in the Gevrey class with index $\gamma = \frac{3}{2}$. In view of (1.10), since v^R is controlled via the relation $v^R = -\int_0^y \partial_x u^R dy'$, the main difficulty comes from the term $v^R \partial_y u^p$, which loses one tangential derivative. In [35], we justify the limit in Gevrey class $\frac{9}{8}$. For the data in the Gevrey class with optimal index $\gamma = \frac{3}{2}$, we need to introduce new ideas.

(2) **Introduce the vorticity formulation.** In order to eliminate p^R, we introduce vorticity $\omega^R = -\varepsilon^2 \partial_x v^R + \partial_y u^R$ and rewrite the equation of ω^R by stream function ϕ which satisfies

$$v^R = -\partial_x \phi, \quad u^R = \partial_y \phi + C(t), \quad C(t) = \frac{1}{2\pi} \int_{\mathcal{S}} u^R dx dy.$$

Thus, we get

$$\begin{cases} (\partial_t - \Delta_\varepsilon)\Delta_\varepsilon \phi - \partial_x \phi \partial_y \omega^p = \cdots, & (x,y) \in \mathcal{S}, \\ \phi|_{y=0,1} = 0, \quad \partial_y \phi|_{y=0,1} = C(t), & x \in \mathbb{T}. \end{cases} \tag{1.11}$$

We notice the term $\partial_x \phi \partial_y \omega^p$ also loses one tangential derivative. But under the convexity condition $\partial_y \omega^p \geq c_0 > 0$, one can use the 'hydrostatic trick' to deal with this term. Testing $\frac{\omega^R}{\partial_y \omega^p}$ to the (1.11) instead of ω^R, we have the following cancellation:

$$-\int_{\mathcal{S}} \partial_x \phi \partial_y \omega^p \cdot \frac{\omega^R}{\partial_y \omega^p} dx dy = -\int_{\mathcal{S}} \partial_x \phi \triangle_\varepsilon \phi dx dy = \int_{\mathcal{S}} \partial_x |\nabla_\varepsilon \phi|^2 dx dy = 0,$$

where we use $\phi|_{y=0,1} = 0$. However, the boundary condition of ϕ is $\partial_y \phi|_{y=0,1} = C(t)$, which brings an essential difficulty.

By the energy estimates, taking the inner product in X^r(the definition is given in section 2) with $-\partial_t \phi$, we get

$$\begin{aligned} &\sup_{s \in [0,t]} (\lambda \|\nabla_\varepsilon \phi(s)\|^2_{X^{\frac{7}{3}}} + \|\Delta_\varepsilon \phi(s)\|^2_{X^2}) \\ &\leq C \int_0^t (\varepsilon^{-2} \|\varphi \Delta_\varepsilon \phi\|^2_{X^2} + \varepsilon^{-2} \|\nabla_\varepsilon \phi\|^2_{X^2} + \cdots) ds, \end{aligned} \tag{1.12}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

where $\varphi(y)=y(1-y)$. All we need to do is to control $\varepsilon^{-1}\|\varphi\Delta_\varepsilon\phi\|_{X^2}$ and $\varepsilon^{-1}\|\nabla_\varepsilon\phi\|_{X^2}$ by the left-hand side of (1.12).

Motivated by [13], we expect to achieve that by a decomposition of stream function, $\phi=\phi_{slip}+\phi_{bc}$ in Gevrey $\frac{3}{2}$ regularity. Here, ϕ_{slip} enjoys a 'good' boundary condition and ϕ_{bc} is a corrector which recovers the boundary condition. In the following, we present the decomposition precisely.

(3) **Gevrey estimate under artificial boundary conditions.** ϕ_{slip} enjoying a good boundary condition is defined by

$$\begin{cases}(\partial_t-\Delta_\varepsilon)\omega_{slip}-\partial_x\phi_{slip}\partial_y\omega^p=\cdots, & (x,y)\in\mathcal{S}\\ \phi_{slip}|_{y=0,1}=0,\quad \omega_{slip}|_{y=0,1}=0, & x\in\mathbb{T},\end{cases}\tag{1.13}$$

where $\omega_{slip}=\Delta_\varepsilon\phi_{slip}$. By 'hydrostatic trick' and Navier-slip boundary conditions, we obtain

$$\lambda\int_0^t(\|\omega_{slip}\|^2_{X^{\frac{7}{3}}}+\|\nabla_\varepsilon\phi_{slip}\|^2_{X^{\frac{7}{3}}}+|\nabla_\varepsilon\phi_{slip}|_{y=0,1}|^2_{X^{\frac{7}{3}}})ds\le\frac{C}{\lambda}\int_0^t\|\varepsilon\Delta_\varepsilon\phi\|^2_{X^2}ds+\cdots.\tag{1.14}$$

The full study of the Orr-Sommerfeld formulation (1.13) with Navier-slip boundary conditions is given in Section 7.

(4) **Recovery the non-slip boundary condition.** In Step (3), we use the slip boundary condition, not the real boundary condition $\partial_y\phi|_{y=0,1}=C(t)$. To recover the boundary condition, we introduce the following system:

$$\begin{cases}(\partial_t-\Delta_\varepsilon)\Delta_\varepsilon\phi_{bc}-\partial_x\phi_{bc}\partial_y\omega^p=0, & (x,y)\in\mathcal{S}\\ \phi_{bc}|_{y=0,1}=0,\quad \partial_y\phi_{bc}|_{y=i}=h^i, & x\in\mathbb{T},\end{cases}\tag{1.15}$$

where $\omega_{bc}=\Delta_\varepsilon\phi_{bc}$ and $i=0,1$. And we need to choose a suitable h^i such that

$$\partial_y\phi_{bc}|_{y=0,1}=-\partial_y\phi_{slip}|_{y=0,1}+C(t).$$

Next, we give the main idea for proving the existence of h^i:

1. We define $\phi_{bc,S}=\phi^0_{bc,S}+\phi^1_{bc,S}$, where $\phi^i_{bc,S}$ solve

$$\begin{cases}(\partial_t-\Delta_\varepsilon)\Delta_\varepsilon\phi^i_{bc,S}=0,\\ \phi^i_{bc,S}|_{y=i}=0,\quad \partial_y\phi^i_{bc,S}|_{y=i}=h^i,\\ \phi^i_{bc,S}|_{t=0}=0,\end{cases}\tag{1.16}$$

where $x\in\mathbb{T}$, $y\in(0,+\infty)$ for $i=0$ and $y\in(-\infty,1)$ for $i=1$. Taking Fourier transformation on t and x, we can write the precise expression of the solution to obtain the Gevrey estimate for $\phi^i_{bc,S}$:

$$\int_0^t\|\nabla_\varepsilon\phi^i_{bc,S}\|^2_{X_i^{\frac{5}{2}}}+\|\varphi^i\Delta_\varepsilon\phi^i_{bc,S}\|^2_{X_i^{\frac{5}{2}}}+\|\partial_x\phi^i_{bc,S}\|^2_{X^{\frac{5}{3}}}ds\le\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t|h^i|^2_{X^{\frac{7}{3}}}ds,\tag{1.17}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

where $\varphi^0(y)=y,\ \varphi^1(y)=1-y$. Compared with the decomposition in [35], we get more regularity of $\partial_x\phi^i_{bc,S}$, which is a key point to get the optimal Gevrey regularity. The details for this step are given in Section 8.1.

2. We correct the nonlocal term constructed in the above step by considering the following equations:

$$\begin{cases}(\partial_t-\Delta_\varepsilon)\Delta_\varepsilon\phi^i_{bc,R}-\partial_x\phi^i_{bc,R}\partial_y\omega^p=\partial_x\phi^i_{bc,S}\partial_y\omega^p, & (x,y)\in\mathcal{S}\\ \phi^i_{bc,R}|_{t=0}=0, \quad (x,y)\in\mathcal{S}\end{cases} \tag{1.18}$$

with Navier-slip conditions. By the same process as Step (3) and combining with the sharp estimate (1.17) to get an estimate for $\phi^i_{bc,R}$,

$$\begin{aligned}\lambda\int_0^t\|\omega^i_{bc,R}\|^2_{X^{\frac{7}{3}}}ds+\int_0^t(\|\nabla_\varepsilon\phi^i_{bc,R}\|^2_{X^{\frac{7}{3}}}+|\partial_y\phi^i_{bc,R}|_{y=0,1}|^2_{X^{\frac{7}{3}}})ds \\ \le\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t|h^i|^2_{X^{\frac{7}{3}}}ds+\cdots,\qquad t\in[0,T].\end{aligned} \tag{1.19}$$

More details are given in Section 8.3.

3. We define $\phi_{bc}=\phi_{bc,S}+\phi_{bc,R}$, where $\phi_{bc,S}=\sum_{i=0,1}\phi^i_{bc,S}$ and $\phi_{bc,R}=\sum_{i=0,1}\phi^i_{bc,R}$, which solves system (1.15). To match the boundary condition on the derivative of $\partial_y\phi|_{y=0,1}=C(t)$, we need

$$\partial_y\phi_{bc,S}|_{y=0,1}+\partial_y\phi_{bc,R}|_{y=0,1}=\partial_y\phi_{bc}|_{y=0,1}=-\partial_y\phi_{slip}|_{y=0,1}+C(t).$$

On one hand, $\phi_{bc,S}$ and $\phi_{bc,R}$ are defined by h^i. We define a 0-order operator R_{bc} given in (8.71) such that

$$(1+R_{bc})h^i=-\partial_y\phi_{slip}|_{y=0,1}+C(t).$$

Moreover, by the estimate in Step 1 and Step 2, we can get

$$\int_0^t\left|R_{bc}[h^0,h^1]\right|^2_{X^{\frac{7}{3}}}ds\le\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t|(h^0,h^1)|^2_{X^{\frac{7}{3}}}ds,$$

which means that $(1+R_{bc})$ is an invertible operator when λ is large. That means that $\phi_{bc,S}$ and $\phi_{bc,R}$ are well-defined and (1.15) is well-posedness. Details are given in Section 8.4.
Due to the transport terms, we need to introduce a new auxiliary function $\phi_{bc,T}$ between Step 1 and Step 2. For more details, see Section 8.2.

(5) **Close the energy estimates (1.12).** Summing estimates (1.17) and (1.19) in Step (4), we get an estimate for ϕ_{bc} :

$$\begin{aligned}\int_0^t\|\nabla_\varepsilon\phi_{bc}\|^2_{X^{\frac{7}{3}}}+\|\varphi\Delta_\varepsilon\phi_{bc}\|^2_{X^{\frac{7}{3}}}ds\le\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t|(h^0,h^1)|^2_{X^{\frac{7}{3}}}ds \\ \le C\int_0^t|\nabla_\varepsilon\phi_{slip}|_{y=0,1}|^2_{X^{\frac{7}{3}}}ds+\cdots,\end{aligned}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

which along with (1.14), we have

$$\int_0^t (\|\varphi\omega\|^2_{X^{\frac{7}{3}}} + \|\nabla_\varepsilon \phi\|^2_{X^{\frac{7}{3}}})ds \le \frac{C}{\lambda}\int_0^t \|\varepsilon\Delta_\varepsilon\phi\|^2_{X^2} ds + \cdots,$$

and then we put the above estimate into (1.12) to close the estimate for system (1.11).

1.4. Notations

- $\mathcal{S}^\varepsilon = \{(x,y)\in\mathbb{T}\times\mathbb{R} : 0<y<\varepsilon\}$ and $\mathcal{S} = \{(x,y)\in\mathbb{T}\times\mathbb{R} : 0<y<1\}$.
- $\nabla_\varepsilon = (\varepsilon\partial_x,\partial_y)$ and $\Delta_\varepsilon = \varepsilon^2\partial_x^2 + \partial_y^2$.
- Vorticity of Prandtl part ω^p is defined by $\omega^p = \partial_y u^p$.
- Vorticity of reminder part $\omega^R = \Delta_\varepsilon\phi$ is defined by $\omega^R = \varepsilon^2\partial_x v^R - \partial_y u^R$. In this paper, we also define $\omega^i_{bc,j} = \Delta_\varepsilon\phi^i_{bc,j}$, where $i = 0,1$ and $j\in\{R,T\}$
- Cut-off functions $\varphi(y) = y(1-y)$ and $\varphi^i(y) = i + (-1)^i y$.
- $C(t) = \frac{1}{2\pi}\int_{\mathcal{S}} u^R dxdy$.
- The Fourier transform of f_Φ is defined by $e^{(1-\lambda t)\langle k\rangle^{\frac{2}{3}}}\widehat{f}(k)$.

2. Gevrey norms and preliminary lemmas

At the beginning of this section, we define the functional space X^r and the Gevrey class. First, we define

$$f_\Phi = \mathcal{F}^{-1}(e^{\Phi(t,k)}\widehat{f}(k)) = e^{\Phi(t,D_x)}f, \quad \Phi(t,k) \stackrel{\text{def}}{=} \tau(t)\langle k\rangle^{\frac{2}{3}}, \tag{2.1}$$

where $\tau(t)\ge 0$. Moreover, it is easy to get that $\Phi(t,k)$ satisfies the subadditive inequality

$$\Phi(t,k) \le \Phi(t,k-\ell) + \Phi(t,\ell). \tag{2.2}$$

Now, we are in the position to define X^r_τ, which is defined by

$$\|f\|_{X^r_\tau} = \|f_\Phi\|_{H^{r,0}}.$$

We say that a function f belongs to the Gevrey class $\frac{3}{2}$ if $\|f\|_{X^r_\tau} < +\infty$.

Moreover, we need to deal with some Gevrey class functions defined on the boundary. Thus, we introduce the following functional space:

$$|f|_{X^r_\tau} = \|f_\Phi\|_{H^r_x(\mathbb{T})},$$

where f depends on variable x.

By the definition of X^r_τ, it is easy to see that if $r'\ge r$, then $\|\cdot\|_{X^{r'}_\tau} \ge \|\cdot\|_{X^r_\tau}$. For simplicity, we drop subscript τ in the notations $\|f\|_{X^r_\tau}, |f|_{X^r_\tau}$ etc. In the sequel, we always take

$$\tau(t) = 1-\lambda t,$$

with $\lambda\ge 1$ determined later. Thus, if we take t small enough, we have $\tau > 0$.

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

In the following, we present some lemmas on product estimates in the Gevrey class and the readers can refer to Lemmas 2.1–2.3 in [34] for details. The first lemma is the commutator estimate in Sobolev space:

Lemma 2.1. *Let* $r \geq 0,\ s_1 > \frac{3}{2},\ s > \frac{1}{2}$ *and* $0 \leq \delta \leq 1$. *Then it holds that*

$$\big\|[\langle D\rangle^r, f]\partial_x g\big\|_{L^2_x} \leq C\|f\|_{H^{s_1}_x}\|g\|_{H^r_x} + C\|f\|_{H^{r+1-\delta}_x}\|g\|_{H^{s+\delta}_x}.$$

In the Gevrey class, we have the following:

Lemma 2.2. *Let* $r \geq 0$ *and* $s > \frac{1}{2}$. *Then it holds that*

$$|fg|_{X^r} \leq C|f|_{X^s}|g|_{X^r} + C|f|_{X^r}|g|_{X^s}.$$

For the commutator in the Gevrey class, we have the following:

Lemma 2.3. *Let* $r \geq 0,\ s_1 > \frac{3}{2},\ s > \frac{1}{2}$ *and* $0 \leq \delta \leq 1$. *Then it holds that*

$$\|(f\partial_x g)_\Phi - f\partial_x g_\Phi\|_{H^r_x} \leq C|f|_{X^{s_1}}|g|_{X^{r+\frac{2}{3}}} + C|f|_{X^{r+1-\delta}}|g|_{X^{s+\delta}}.$$

3. Approximate equations and Error equations

3.1. Approximate equations

By the Hilbert asymptotic method, we can obtain the approximate solutions. We define approximate solutions as the following:

$$\begin{cases} u^p(t,x,y) = u_p^0(t,x,y) + \varepsilon^2 u_p^2(t,x,y), \\ v^p(t,x,y) = v_p^0(t,x,y) + \varepsilon^2 v_p^2(t,x,y), \\ p^p(t,x,y) = p_p^0(t,x,y) + \varepsilon^2 p_p^2(t,x,y), \end{cases} \tag{3.1}$$

where (u_p^0, v_p^0, p_p^0) satisfies equation (1.4) and (u_p^2, v_p^2, p_p^2) satisfies equation

$$\begin{cases} \partial_t u_p^2 + u_p^0\partial_x u_p^2 + v_p^0\partial_y u_p^2 + u_p^2\partial_x u_p^0 + v_p^2\partial_y u_p^0 + \partial_x p_p^2 - \partial_y^2 u_p^2 = -\partial_x^2 u_p^0, \\ \partial_y p_p^2 = -(\partial_t v_p^0 + u_p^0\partial_x v_p^0 + v_p^0\partial_y v_p^0 - \partial_y^2 v_p^0), \\ \partial_x u_p^2 + \partial_y v_p^2 = 0, \\ (u_p^2, v_p^2)|_{y=0,1} = 0, \\ u_p^2|_{t=0} = 0. \end{cases} \tag{3.2}$$

We point here that $(u_p^1, v_p^1, p_p^1) = 0$ by matching the equation of order ε. Based on the equation of (u_p^0, v_p^0, p_p^0) and (u_p^2, v_p^2, p_p^2), we deduce the approximate solution (u^p, v^p, p^p)

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

which satisfies the following equation:

$$\begin{cases} \partial_t u^p + u^p \partial_x u^p + v^p \partial_y u^p + \partial_x p^p - \Delta_\varepsilon u^p = -R_1, \\ \varepsilon^2(\partial_t v^p + u^p \partial_x v^p + v^p \partial_y v^p - \Delta_\varepsilon v^p) + \partial_y p^p = -R_2, \\ \partial_x u^p + \partial_y v^p = 0, \\ (u^p, v^p)|_{y=0,1} = 0, \\ (u^p, v^p)|_{t=0} = (u_0, v_0), \end{cases} \tag{3.3}$$

where reminder (R_1, R_2) is given by

$$R_1 = \varepsilon^4(u_p^2 \partial_x u_p^2 + v_p^2 \partial_y v_p^2 - \partial_x^2 u_p^2), \tag{3.4}$$

$$R_2 = \varepsilon^4\Big(\partial_t v_p^2 + u_p^0 \partial_x v_p^2 + u_p^2 \partial_x v_p^0 + \varepsilon^2 u_p^2 \partial_x v_p^2 + v_p^0 \partial_y v_p^2 + v_p^2 \partial_y v_p^0 \tag{3.5}$$

$$+ \varepsilon^2 v_p^2 \partial_y v_p^2 - \partial_x^2(v_p^0 + \varepsilon^2 v_p^2) - \partial_y^2 v_p^2\Big).$$

By the definition of R_1 and R_2, it is easy to get that

$$(R_1, R_2) \sim O(\varepsilon^4).$$

3.2. Equations of error functions

We define error functions (u^R, v^R, p^R):

$$u^R = u^\varepsilon - u^p, \quad v^R = v^\varepsilon - v^p, \quad p^R = p^\varepsilon - p^p.$$

It is easy to deduce the system of error functions:

$$\begin{cases} \partial_t u^R - \Delta_\varepsilon u^R + \partial_x p^R + u^\varepsilon \partial_x u^R + u^R \partial_x u^p + v^\varepsilon \partial_y u^R + v^R \partial_y u^p = R_1, \\ \varepsilon^2(\partial_t v^R - \Delta_\varepsilon v^R + u^\varepsilon \partial_x v^R + u^R \partial_x v^p + v^\varepsilon \partial_y v^R + v^R \partial_y v^p) + \partial_y p^R = R_2, \\ \partial_x u^R + \partial_y v^R = 0, \\ (u^R, v^R)|_{y=0} = (u^R, v^R)|_{y=1} = 0, \\ (u^R, v^R)|_{t=0} = 0. \end{cases} \tag{3.6}$$

For convenience, we rewrite (3.6) as

$$\begin{cases} \partial_t u^R - \Delta_\varepsilon u^R + u^p \partial_x u^R + u^R \partial_x u^p + v^R \partial_y u^p + v^p \partial_y u^R + \partial_x p^R = \mathcal{N}_u + R_1, \\ \varepsilon^2(\partial_t v^R - \Delta_\varepsilon v^R + u^p \partial_x v^R + u^R \partial_x v^p + v^R \partial_y v^p + v^p \partial_y v^R) + \partial_y p^R = \varepsilon^2 \mathcal{N}_v + R_2, \\ \partial_x u^R + \partial_y v^R = 0, \\ (u^R, v^R)|_{y=0} = (u^R, v^R)|_{y=1} = 0, \\ (u^R, v^R)|_{t=0} = 0. \end{cases} \tag{3.7}$$

Here, $(\mathcal{N}_u, \mathcal{N}_v)$ is a nonlinear term given by

$$\mathcal{N}_u = -(u^R \partial_x u^R + v^R \partial_y u^R), \quad \mathcal{N}_v = -(u^R \partial_x v^R + v^R \partial_y v^R). \tag{3.8}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

Based on the above system, we get the equations of the vorticity $\omega^R = \partial_y u^R - \varepsilon^2 \partial_x v^R$:

$$\begin{aligned}&\partial_t \omega^R - \Delta_\varepsilon \omega^R + u^p \partial_x \omega^R + u^R \partial_x \omega^p + v^p \partial_y \omega^R + v^R \partial_y \omega^p \\ &\quad = \partial_y \mathcal{N}_u - \varepsilon^2 \partial_x \mathcal{N}_v + \varepsilon^2 f_1 + f_2,\end{aligned} \tag{3.9}$$

where f_1, f_2 are defined by

$$f_1 = -(u^R \partial_x^2 v^p + v^R \partial_x \partial_y v^p), \tag{3.10}$$

$$f_2 = \partial_y R_1 - \varepsilon^2 \partial_x R_2, \tag{3.11}$$

$$\omega^p = \partial_y u^p. \tag{3.12}$$

Moreover, following the calculations in [34], we can obtain the boundary conditions of ω^R:

$$(\partial_y + \varepsilon |D|)\omega^R|_{y=0} = \partial_y (\Delta_{\varepsilon,D})^{-1}(f - \mathcal{N})|_{y=0} + \frac{1}{2\pi}\int_{\mathcal{S}} \partial_t u^R dxdy, \tag{3.13}$$

$$(\partial_y - \varepsilon |D|)\omega^R|_{y=1} = \partial_y (\Delta_{\varepsilon,D})^{-1}(f - \mathcal{N})|_{y=1} + \frac{1}{2\pi}\int_{\mathcal{S}} \partial_t u^R dxdy, \tag{3.14}$$

where

$$\mathcal{N} = \partial_y \mathcal{N}_u - \varepsilon^2 \partial_x \mathcal{N}_v = -u^R \partial_x \omega^R - v^R \partial_y \omega^R, \tag{3.15}$$

$$f = f_3 - \varepsilon^2 f_1 - f_2, \tag{3.16}$$

$$f_3 = u^p \partial_x \omega^R + u^R \partial_x \omega^p + v^p \partial_y \omega^R + v^R \partial_y \omega^p, \quad \omega^p = \partial_y u^p. \tag{3.17}$$

3.3. Equations of stream function

Thanks to $\partial_x u^R + \partial_y v^R = 0$ and $v^R|_{y=0,1} = 0$, there exists a stream function ϕ satisfying the following system:

$$-\partial_x \phi = v^R, \quad \partial_y \phi = u^R - \frac{1}{2\pi}\int_{\mathcal{S}} u^R dxdy. \tag{3.18}$$

Since $\int_{\mathbb{T}} v^R dx = 0$, the function ϕ is periodic in x. Thanks to $\partial_x \phi|_{y=0,1} = 0$ and $\phi(1,x) - \phi(0,x) = 0$, we may assume that $\phi|_{y=0,1} = 0$. Thus, there holds that

$$\Delta_\varepsilon \phi = \omega^R \quad \text{in } \mathcal{S}, \quad \phi|_{y=0,1} = 0. \tag{3.19}$$

Taking (3.18) and (3.19) into (3.9) and using the boundary condition $(u^R, v^R)|_{y=0,1} = 0$, we obtain

$$\begin{cases} (\partial_t - \Delta_\varepsilon)\Delta_\varepsilon \phi + u^p \partial_x \Delta_\varepsilon \phi + v^p \partial_y \Delta_\varepsilon \phi + \partial_y \phi \partial_x \omega^p - \partial_x \phi \partial_y \omega^p \\ \qquad\qquad = \partial_y \mathcal{N}_u - \varepsilon^2 \partial_x \mathcal{N}_v + \varepsilon^2 f_1 + f_2 - C(t)\partial_x \omega^p, \\ \phi|_{y=0,1} = 0, \quad \partial_y \phi|_{y=0,1} = C(t), \end{cases} \tag{3.20}$$

where $C(t) = \frac{1}{2\pi}\int_{\mathcal{S}} u^R dxdy$ and $(\mathcal{N}_u, \mathcal{N}_v)$, f_1, f_2 are given in (3.8), (3.10) and (3.11).

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

At the end of the subsection, we state some elliptic estimates which can be obtained by classical theory. First, by elliptic estimate and Hardy inequality, we have

$$\|\nabla_\varepsilon \phi\|_{L^2} \le C\|\varphi\omega^R\|_{L^2}, \tag{3.21}$$

where $\varphi(y) = y(1-y)$ and $\nabla_\varepsilon = (\partial_y, \varepsilon\partial_x)$. Indeed, taking the L^2 inner product with $-\phi$ in (3.19) and integrating by parts, we use boundary condition $\phi|_{y=0,1} = 0$ and the Hardy inequality to have

$$\|\nabla_\varepsilon \phi\|_{L^2}^2 = -\langle \omega^R, \phi\rangle_{L^2} = -\langle \varphi\omega^R, \frac{\phi}{\varphi}\rangle_{L^2} \le C\|\varphi\omega^R\|_{L^2}\|\partial_y\phi\|_{L^2}.$$

Since (u^R, v^R) satisfies the elliptic equations

$$\begin{cases} \Delta_\varepsilon u^R = \partial_y \omega^R, \\ u^R|_{y=0,1} = 0, \end{cases} \qquad \begin{cases} \Delta_\varepsilon v^R = -\partial_x \omega^R, \\ v^R|_{y=0,1} = 0, \end{cases}$$

we arrive at

$$\big\|(u^R, \varepsilon v^R, \partial_y u^R, \varepsilon\partial_x u^R, \varepsilon\partial_y v^R, \varepsilon^2\partial_x v^R)\big\|_{X^r} \le C\|\omega^R\|_{X^r}, \tag{3.22}$$

for any $r \ge 0$.

4. Estimate of $\nabla_\varepsilon\phi$ and $\Delta_\varepsilon\phi$ in the Gevrey space

Before giving the estimate of $\nabla_\varepsilon\phi$ and $\Delta_\varepsilon\phi$, we need the estimates of the reminder terms R_1 and R_2 which are defined by the approximate solution u^p and v^p. For (u^p, v^p), we have the following bound:

Lemma 4.1. *Let initial data u_0 of (1.4) satisfy (1.6)-(1.9). There exists a time T_p such that (u_p^i, v_p^i), $i = 0, 2$ defined in (1.4) and (3.2) have the following estimates:*

$$\begin{aligned} &\|v_p^0\|_{X^{11}} + \|(u_p^0, \varepsilon v_p^0)\|_{X^{12}} + \|\partial_y u_p^0\|_{X^{12}} + \|\partial_y^3 u_p^0\|_{X^8} \le C, \\ &\|v_p^2\|_{X^9} + \|(u_p^2, \varepsilon v_p^2)\|_{X^{10}} + \|\partial_y u_p^2\|_{X^{10}} + \|\partial_y^3 u_p^2\|_{X^6} \le C, \end{aligned}$$

for $t \in [0, T_p]$.

Moreover, according to (3.1), it holds that

$$\|v^p\|_{X^9} + \|(u^p, \varepsilon v^p)\|_{X^{10}} + \|\partial_y u^p\|_{X^{10}} + \|\partial_y^3 u^p\|_{X^6} \le C, \qquad t \in [0, T_p]$$

and

$$\partial_y \omega^p \ge c_0, \qquad t \in [0, T_p].$$

Proof. Here, the key of this lemma is to prove that (1.4) is well-posedness in the Gevrey class $\frac{3}{2}$ which is the conjecture in [15]. If we set $\varepsilon = 0$ and follow the step-by-step process in this paper, we can get the conjecture proved. Here, to avoid repeatability, we leave the proof to the readers. □

Then, by the definition of (R_1,R_2) in (3.4)-(3.5), using Lemma 2.2, we get the following:

Lemma 4.2. *It holds that*

$$\|(R_1,R_2)\|_{X^3} \le C\varepsilon^4, \quad \|\nabla(R_1,R_2)\|_{X^2} \le C\varepsilon^4, \quad t\in[0,T_p].$$

Now, we state our main result in this section:

Proposition 4.3. *There exist* $0<T<\min\{T_p,\frac{1}{2\lambda}\}$ *and* $\lambda_0\ge 1$, *such that for any* $t\in[0,T]$ *and* $\lambda\ge\lambda_0$, *it holds that*

$$\begin{aligned}&\sup_{s\in[0,t]}\Big(\lambda\|\nabla_\varepsilon\phi(s)\|^2_{X^{\frac{7}{3}}}+\|\Delta_\varepsilon\phi(s)\|^2_{X^2}\Big)+\int_0^t\Big(\|\partial_t\nabla_\varepsilon\phi_\Phi\|^2_{H^{2,0}}+\|\nabla_\varepsilon\omega^R\|^2_{L^2}\Big)ds\\ &\le C\int_0^t\Big(\varepsilon^{-2}\|\varphi\triangle_\varepsilon\phi\|^2_{X^2}+\varepsilon^{-2}\|\nabla_\varepsilon\phi\|^2_{X^2}+\|\Delta_\varepsilon\phi\|^2_{X^2}+\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|^2_{X^2}+\|\mathcal{N}\|^2_{L^2}+\varepsilon^8\Big)ds,\end{aligned}$$

where $\Delta_\varepsilon\phi=\omega^R$, $\varphi(y)=y(1-y)$ *and* C *is a constant independent of* ε.

Proof. Acting $e^{\Phi(t,D_x)}$ on the both sides of the first equation of (3.20), we get

$$\begin{aligned}&(\partial_t+\lambda\langle D_x\rangle^{\frac{2}{3}}-\Delta_\varepsilon)\Delta_\varepsilon\phi_\Phi+(u^p\partial_x\Delta_\varepsilon\phi+v^p\partial_y\Delta_\varepsilon\phi)_\Phi+(\partial_y\phi\partial_x\omega^p-\partial_x\phi\partial_y\omega^p)_\Phi\\ &\qquad=\partial_y(\mathcal{N}_u)_\Phi-\varepsilon^2\partial_x(\mathcal{N}_v)_\Phi+(\varepsilon^2f_1+f_2)_\Phi-C(t)\partial_x\omega^p_\Phi.\end{aligned}$$

Taking $H^{2,0}$ inner product with $-\partial_t\phi_\Phi$ and using boundary conditions

$$\phi_\Phi|_{y=0,1}=0, \quad \partial_y\phi_\Phi|_{y=0,1}=C(t),$$

we integrate by parts to arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\lambda\|\nabla_\varepsilon\phi\|^2_{X^{\frac{7}{3}}}+\|\Delta_\varepsilon\phi\|^2_{X^2})+\|\partial_t\nabla_\varepsilon\phi_\Phi\|^2_{H^{2,0}}-\Big\langle\Delta_\varepsilon\phi_\Phi,\partial_t\partial_y\phi_\Phi\Big\rangle_{H^2_x}\Big|_{y=0}^{y=1} \qquad (4.1)\\ &=\Big\langle(u^p\partial_x\Delta_\varepsilon\phi+v^p\partial_y\Delta_\varepsilon\phi)_\Phi,\partial_t\phi_\Phi\Big\rangle_{H^{2,0}}+\Big\langle(\partial_y\phi\partial_x\omega^p-\partial_x\phi\partial_y\omega^p)_\Phi,\partial_t\phi_\Phi\Big\rangle_{H^{2,0}}\\ &\quad+\Big\langle\partial_y(\mathcal{N}_u)_\Phi-\varepsilon^2\partial_x(\mathcal{N}_v)_\Phi,-\partial_t\phi_\Phi\Big\rangle_{H^{2,0}}+\Big\langle(\varepsilon^2f_1+f_2)_\Phi,-\partial_t\phi_\Phi\Big\rangle_{H^{2,0}}\\ &\quad-\Big\langle C(t)\partial_x\omega^p_\Phi,-\partial_t\phi_\Phi\Big\rangle_{H^{2,0}}\\ &=I_1+\cdots+I_5.\end{aligned}$$

First, let's estimate I_i, $i=1,\cdots,5$ term by term.
Estimate of I_1. Since divergence free condition $\partial_xu^p+\partial_yv^p=0$, we get

$$(u^p\partial_x\Delta_\varepsilon\phi+v^p\partial_y\Delta_\varepsilon\phi)_\Phi=\partial_x(u^p\Delta_\varepsilon\phi)_\Phi+\partial_y(v^p\Delta_\varepsilon\phi)_\Phi.$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

According to $(u^p,v^p)|_{y=0,1}=0$, we use integration by parts and Lemma 2.2 to have

$$\begin{aligned}
I_1 =& -\Big\langle (u^p\Delta_\varepsilon\phi)_\Phi,\partial_t\partial_x\phi_\Phi\Big\rangle_{H^{2,0}} - \Big\langle (v^p\Delta_\varepsilon\phi)_\Phi,\partial_t\partial_y\phi_\Phi\Big\rangle_{H^{2,0}}\\
\le& C\Big\||\frac{u^p}{\varphi}|_{X^2}|\varphi\Delta_\varepsilon\phi|_{X^2}\Big\|_{L^2_y}\|\partial_t\partial_x\phi_\Phi\|_{H^{2,0}} + C\Big\||\frac{v^p}{\varphi}|_{X^2}|\varphi\Delta_\varepsilon\phi|_{X^2}\Big\|_{L^2_y}\|\partial_t\partial_y\phi_\Phi\|_{H^{2,0}}\\
\le& C\||\frac{u^p}{\varphi}|_{X^2}\|_{L^\infty_y}\|\varphi\Delta_\varepsilon\phi\|_{X^2}\|\partial_t\partial_x\phi_\Phi\|_{H^{2,0}} + C\Big\||\frac{v^p}{\varphi}|_{X^2}\|_{L^\infty_y}\|\varphi\Delta_\varepsilon\phi\|_{X^2}\|\partial_t\partial_y\phi_\Phi\|_{H^{2,0}}\\
\le& C\varepsilon^{-1}\|\varphi\Delta_\varepsilon\phi\|_{X^2}\|\partial_t\nabla_\varepsilon\phi_\Phi\|_{H^{2,0}}.
\end{aligned}$$

<u>Estimate of I_2.</u> Similarly, we write

$$(\partial_y\phi\partial_x\omega^p - \partial_x\phi\partial_y\omega^p)_\Phi = \Big(\partial_x(\partial_y\phi\omega^p) - \partial_y(\partial_x\phi\omega^p)\Big)_\Phi.$$

Then, along with $\phi|_{y=0,1}=0$, we use integration by parts and Lemma 2.2 to deduce

$$\begin{aligned}
I_2 =& \Big\langle \partial_x(\partial_y\phi\omega^p)_\Phi - \partial_y(\partial_x\phi_\Phi\omega^p)_\Phi,\partial_t\phi_\Phi\Big\rangle_{H^{2,0}}\\
=& -\Big\langle (\partial_y\phi\omega^p)_\Phi,\partial_t\partial_x\phi_\Phi\Big\rangle_{H^{2,0}} + \Big\langle (\partial_x\phi\omega^p)_\Phi,\partial_t\partial_y\phi_\Phi\Big\rangle_{H^{2,0}}\\
\le& C\||\omega^p|_{X^2}\|_{L^\infty_y}\|\partial_y\phi\|_{X^2}\|\partial_t\partial_x\phi_\Phi\|_{H^{2,0}} + C\||\omega^p|_{X^2}\|_{L^\infty_y}\|\partial_x\phi\|_{X^2}\|\partial_t\partial_y\phi_\Phi\|_{H^{2,0}}\\
\le& C\varepsilon^{-1}\|\nabla_\varepsilon\phi\|_{X^2}\|\partial_t\nabla_\varepsilon\phi_\Phi\|_{H^{2,0}}.
\end{aligned}$$

<u>Estimate of I_3.</u> Due to $\phi|_{y=0,1}=0$, taking integration by parts, it yields that

$$I_3 \le C\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{X^2}\|\partial_t\nabla_\varepsilon\phi_\Phi\|_{H^{2,0}}.$$

<u>Estimate of I_4.</u> Recall f_1 and f_2 in (3.10)-(3.11). According to (3.22) and Lemma 4.2, we have

$$\begin{aligned}
I_4 \le& C(\|\varepsilon^2 f_1\|_{X^2} + \|f_2\|_{X^2})\|\partial_t\phi_\Phi\|_{H^{2,0}}\\
\le& C(\varepsilon^2\|u^R\|_{X^2} + \varepsilon\|\varepsilon v^R\|_{X^2} + \varepsilon^4)\|\partial_t\partial_y\phi_\Phi\|_{H^{2,0}}\\
\le& C(\varepsilon\|\Delta_\varepsilon\phi\|_{X^2} + \varepsilon^4)\|\partial_t\partial_y\phi_\Phi\|_{H^{2,0}}.
\end{aligned}$$

<u>Estimate of I_5.</u> Poincaré inequality implies

$$\|\partial_t\phi_\Phi\|_{H^{2,0}} \le C\|\partial_t\partial_y\phi_\Phi\|_{H^{2,0}},$$

for $\phi|_{y=0,1}=0$. Since

$$|C(t)| = |\frac{1}{2\pi}\int_{\mathcal{S}} u^R dxdy| \le C\|u^R\|_{L^2} \le C\|\omega^R\|_{L^2} \le C\|\Delta_\varepsilon\phi\|_{L^2},$$

we get

$$I_5 \le C\|\Delta_\varepsilon\phi\|_{L^2}\|\partial_t\partial_y\phi_\Phi\|_{H^{2,0}}.$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

Collecting $I_1 - I_5$ together, it holds that

$$\begin{aligned} I_1+\cdots+I_5 \le\ & C\varepsilon^{-1}\|\varphi\Delta_\varepsilon\phi\|_{X^2}\|\partial_t\nabla_\varepsilon\phi_\Phi\|_{H^{2,0}}+C\varepsilon^{-1}\|\nabla_\varepsilon\phi\|_{X^2}\|\partial_t\nabla_\varepsilon\phi_\Phi\|_{H^{2,0}} \\ & +C\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{X^2}\|\partial_t\nabla_\varepsilon\phi_\Phi\|_{H^{2,0}}+C\|\Delta_\varepsilon\phi\|_{L^2}\|\partial_t\partial_y\phi_\Phi\|_{H^{2,0}} \\ & +C(\varepsilon\|\Delta_\varepsilon\phi\|_{X^2}+\varepsilon^4)\|\partial_t\partial_y\phi_\Phi\|_{H^{2,0}} \\ \le\ & \frac{1}{10}\|\partial_t\nabla_\varepsilon\phi_\Phi\|_{H^{2,0}}^2+C\Big(\varepsilon^{-2}\|\varphi\Delta_\varepsilon\phi\|_{X^2}^2+\varepsilon^{-2}\|\nabla_\varepsilon\phi\|_{X^2}^2+\|\Delta_\varepsilon\phi\|_{X^2}^2\Big) \\ & +C\Big(\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{X^2}+\varepsilon^8\Big). \end{aligned} \tag{4.2}$$

Next, we focus on the boundary term $\Big\langle\Delta_\varepsilon\phi_\Phi,\partial_t\partial_y\phi_\Phi\Big\rangle_{H_x^2}\Big|_{y=0}^{y=1}$. First, we give the estimate of $C'(t)$:

$$\int_{\mathcal{S}}\partial_t u^R dxdy=\int_{\mathcal{S}}\partial_y^2 u^R dxdy=\int_{\mathcal{S}}\partial_y\omega^R dxdy,$$

which gives

$$\Big|\int_{\mathcal{S}}\partial_t u^R dxdy\Big|\le\|\partial_y\omega^R\|_{L^1}. \tag{4.3}$$

Owing to

$$\partial_t\partial_y\widehat{\phi_\Phi}|_{y=0,1}(k)=C'(t)\delta(k),$$

where $\delta(k)$ is a Dirac function and $k\in\mathbb{Z}$, we have

$$\begin{aligned} \Big\langle\Delta_\varepsilon\phi_\Phi,\partial_t\partial_y\phi_\Phi\Big\rangle_{H_x^2}\Big|_{y=0}^{y=1} &= \Big\langle\Delta_\varepsilon\phi\Big|_{y=0}^{y=1},C'(t)\Big\rangle_{L_x^2}=\Big\langle\int_0^1\partial_y\Delta_\varepsilon\phi dy,C'(t)\Big\rangle_{L_x^2} \\ &\le CC'(t)\|\partial_y\omega^R\|_{L^2}\le C\|\partial_y\omega^R\|_{L^2}^2, \end{aligned}$$

where we used (4.3) in the last step.

Putting the above estimate and (4.2) into (4.1), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\lambda\|\nabla_\varepsilon\phi\|_{X^{\frac{7}{3}}}^2+\|\Delta_\varepsilon\phi\|_{X^2}^2) \\ &\quad\le C\Big(\varepsilon^{-2}\|\varphi\Delta_\varepsilon\phi\|_{X^2}^2+\varepsilon^{-2}\|\nabla_\varepsilon\phi\|_{X^2}^2+\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{X^2}^2+\|\Delta_\varepsilon\phi\|_{X^2}^2+\|\partial_y\omega^R\|_{L^2}^2+\varepsilon^8\Big). \end{aligned} \tag{4.4}$$

Next, we give the estimates of $\|\partial_y\omega^R\|_{L^2}^2$. First, we recall the equation of ω^R:

$$\begin{aligned} &\partial_t\omega^R-\Delta_\varepsilon\omega^R+u^p\partial_x\omega^R+u^R\partial_x\omega^p+v^p\partial_y\omega^R+v^R\partial_y\omega^p \\ &\quad=\partial_y\mathcal{N}_u-\varepsilon^2\partial_x\mathcal{N}_v+\varepsilon^2 f_1+f_2, \end{aligned} \tag{4.5}$$

with boundary conditions

$$(\partial_y+\varepsilon|D|)\omega^R|_{y=0}=\partial_y(\Delta_{\varepsilon,D})^{-1}(f-\mathcal{N})|_{y=0}+\frac{1}{2\pi}\int_{\mathcal{S}}\partial_t u^R dxdy, \tag{4.6}$$

$$(\partial_y-\varepsilon|D|)\omega^R|_{y=1}=\partial_y(\Delta_{\varepsilon,D})^{-1}(f-\mathcal{N})|_{y=1}+\frac{1}{2\pi}\int_{\mathcal{S}}\partial_t u^R dxdy, \tag{4.7}$$

where $f_1,\ f_2, f$ and $\mathcal{N}$ are given in (3.10)-(3.16).

Taking the L^2 inner product with ω^R on (4.5) and integration by parts, it follows from $(\mathcal{N}_u,\varepsilon\mathcal{N}_v)|_{y=0,1}=0$ and $(u^p,v^p)|_{y=0,1}=0$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega^R\|_{L^2}^2+\|\nabla_\varepsilon\omega^R\|_{L^2}^2-\int_{\mathbb{T}}\partial_y\omega^R\omega^R dx\Big|_{y=0}^{y=1}\\ \le C\|(u^R,v^R)\|_{L^2}\|\omega^R\|_{L^2}+C\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}\|\nabla_\varepsilon\omega^R\|_{L^2}\\ +C(\|\varepsilon^2u^R\|_{L^2}+\|\varepsilon^2v^R\|_{L^2}+\varepsilon^4)\|\omega^R\|_{L^2}\\ \le\frac{1}{10}\|\nabla_\varepsilon\omega^R\|_{L^2}^2+C(\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2+\|\omega^R\|_{H^{1,0}}^2+\varepsilon^8).\end{aligned}\tag{4.8}$$

For the boundary term, we use (4.6)-(4.7) to write

$$\begin{aligned}\int_{\mathbb{T}}\partial_y\omega^R\omega^R dx\Big|_{y=0}^{y=1}&=\int_{\mathbb{T}}\Big(\varepsilon|D|\omega^R|_{y=1}+\partial_y(\Delta_{\varepsilon,D})^{-1}(f-\mathcal{N})|_{y=1}+C(t)\Big)\omega^R|_{y=1}dx\\ &\quad-\int_{\mathbb{T}}\Big(-\varepsilon|D|\omega^R|_{y=0}+\partial_y(\Delta_{\varepsilon,D})^{-1}(f-\mathcal{N})|_{y=0}+C(t)\Big)\omega^R|_{y=0}dx\\ &=\int_{\mathbb{T}}(\varepsilon|D|\omega^R\ \omega^R)|_{y=0,1}dx+C(t)\int_{\mathbb{T}}\omega^R|_{y=0}^{y=1}dx\\ &\quad+\int_{\mathcal{S}}\partial_y\Big(\partial_y(\Delta_{\varepsilon,D})^{-1}(f-\mathcal{N})\omega^R\Big)dxdy=B_1+B_2+B_3.\end{aligned}$$

Let $y_0\in[0,1]$ so that

$$\|\varepsilon|D|\omega^R(y_0)\|_{L_x^2}\le\|\varepsilon|D|\omega^R\|_{L^2}.$$

Then, along with Gagliardo-Nirenberg inequality

$$\|g\|_{L_y^\infty}\le C\|g\|_{L_y^2}^{\frac{1}{2}}\big(\|g\|_{L_y^2}^{\frac{1}{2}}+\|\partial_y g\|_{L_y^2}^{\frac{1}{2}}\big),\tag{4.9}$$

it infers that

$$\begin{aligned}B_1&=\int_{y_0}^1\partial_y(\varepsilon|D|\omega^R\omega^R)dxdy+\int_{y_0}^0\partial_y(\varepsilon|D|\omega^R\omega^R)dxdy+2\int_{\mathbb{T}}(\varepsilon|D|\omega^R\ \omega^R)|_{y=y_0}dx\\ &\le C\|\varepsilon|D|\omega^R\|_{L^2}\|\partial_y\omega^R\|_{L^2}+C\|\varepsilon|D|\omega^R\|_{L^2}\|\omega^R\|_{L_y^\infty(L_x^2)}\\ &\le C\varepsilon\|\omega^R\|_{H^{1,0}}^2+C\varepsilon\|\partial_y\omega^R\|_{L^2}^2.\end{aligned}$$

Similarly, we use (4.9) and $|C(t)|\le C\|u^R\|_{L^2}\le C\|\omega^R\|_{L^2}$ to have

$$\begin{aligned}B_2&\le C|C(t)|\|\omega^R\|_{L_y^\infty(L_x^2)}\le C\|\omega^R\|_{L^2}^{\frac{3}{2}}(\|\omega^R\|_{L^2}^{\frac{1}{2}}+\|\partial_y\omega^R\|_{L^2}^{\frac{1}{2}})\\ &\le\frac{1}{10}\|\partial_y\omega^R\|_{L^2}^2+C\|\omega^R\|_{L^2}^2.\end{aligned}$$

All we have left is to do B_3. With the fact that operator $\partial_y(\triangle_{\varepsilon,D})^{-1}$, $\partial_y(\triangle_{\varepsilon,D})^{-1}(\partial_y,\varepsilon\partial_x)$ and $\partial_y^2(\triangle_{\varepsilon,D})^{-1}$ are bounded from $L^2\to L^2$, we have

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

$$
\begin{aligned}
B_3 = & \int_{\mathcal{S}} \partial_y^2 (\triangle_{\varepsilon,D})^{-1}(f-\mathcal{N})\omega^R dxdy + \int_{\mathcal{S}} \partial_y (\triangle_{\varepsilon,D})^{-1}\partial_y(v^p\omega^R)\partial_y\omega^R dxdy \\
& + \int_{\mathcal{S}} \partial_y (\triangle_{\varepsilon,D})^{-1}(f-\partial_y(v^p\omega^R) - \partial_y\mathcal{N}_u - \varepsilon^2\partial_x\mathcal{N}_v)\partial_y\omega^R dxdy \\
\leq & \, C\|f-\mathcal{N}\|_{L^2}\|\omega^R\|_{L^2} + C\|v^p\omega^R\|_{L^2}\|\partial_y\omega^R\|_{L^2} \\
& + C(\|f-\partial_y(v^p\omega^R)\|_{L^2} + \|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2})\|\partial_y\omega^R\|_{L^2}.
\end{aligned}
$$

According to the definition of (3.16) and (3.15), we have

$$
\begin{aligned}
\|f\|_{L^2} & \leq C(\|\partial_x\omega^R\|_{L^2} + \|\partial_y\omega^R\|_{L^2} + \|(u^R,v^R)\|_{L^2} + \varepsilon^4) \\
& \leq C(\|\omega^R\|_{H^{1,0}} + \|\partial_y\omega^R\|_{L^2} + \varepsilon^4),
\end{aligned}
$$

and

$$
\begin{aligned}
\|f-\partial_y(v^p\omega^R)\|_{L^2} & \leq C(\|\partial_x\omega^R\|_{L^2} + \|(u^R,v^R)\|_{L^2} + \varepsilon^4) \\
& \leq C(\|\omega^R\|_{H^{1,0}} + \varepsilon^4),
\end{aligned}
$$

which give that

$$
\begin{aligned}
B_3 \leq & \, C(\|\omega^R\|_{H^{1,0}} + \|\partial_y\omega^R\|_{L^2} + \|\mathcal{N}\|_{L^2} + \varepsilon^4)\|\omega^R\|_{L^2} \\
& + C(\|\omega^R\|_{H^{1,0}} + \|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2} + \varepsilon^4)\|\partial_y\omega^R\|_{L^2} \\
\leq & \, \frac{1}{10}\|\partial_y\omega^R\|_{L^2}^2 + C(\|\omega^R\|_{H^{1,0}}^2 + \|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2 + \|\mathcal{N}\|_{L^2}^2 + \varepsilon^8).
\end{aligned}
$$

Summarizing $B_1 - B_3$ together, we obtain

$$
\Big|\int_{\mathbb{T}} \partial_y\omega^R\omega^R dx\Big|_{y=0}^{y=1}\Big| \leq (\frac{1}{5} + C\varepsilon)\|\partial_y\omega^R\|_{L^2}^2 + C(\|\omega^R\|_{H^{1,0}}^2 + \|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2 + \|\mathcal{N}\|_{L^2}^2 + \varepsilon^8). \tag{4.10}
$$

Substituting (4.10) into (4.8), we take ε small enough to arrive at

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\omega^R\|_{L^2}^2 + \frac{1}{2}\|\nabla_\varepsilon\omega^R\|_{L^2}^2 \\
& \qquad \leq C(\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2 + \|\omega^R\|_{H^{1,0}}^2 + \varepsilon^8) \\
& \qquad \quad + C(\|\omega^R\|_{H^{1,0}}^2 + \|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2 + \|\mathcal{N}\|_{L^2}^2 + \varepsilon^8) \\
& \qquad \leq C(\|\Delta_\varepsilon\phi\|_{H^{1,0}}^2 + \|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2 + \|\mathcal{N}\|_{L^2}^2 + \varepsilon^8).
\end{aligned}
$$

Bring the above estimate into (4.4) and integrate time from 0 to t to get the desired results. □

5. Sketch the proof to Theorem 1.1

In this section, we shall sketch the proof of Theorem 1.1. In the paper, we use the continued argument. Here, we define

$$
T^* \stackrel{\text{def}}{=} \sup\{t>0 | \sup_{s\in[0,t]} \|\omega^R\|_{X^2} \leq \mathfrak{C}\varepsilon^3\}. \tag{5.1}
$$

5.1. The key *a priori* estimates.

In this subsection, we shall present the key *a priori* estimates used in the proof of Theorem 1.1.

By Proposition 4.3, we need the estimates of $\int_0^t (\|\nabla_\varepsilon \phi\|_{X^2}^2 + \|\varphi \Delta_\varepsilon \phi\|_{X^2}^2) ds$ to close the energy.

Proposition 5.1. *Let ϕ be the solution of (3.20). Then there exists $\lambda_0 \geq 1$ and $0 < T < \min\{T_p, \frac{1}{2\lambda}\}$ such that for $\lambda \geq \lambda_0$ and $t \in [0,T]$, it holds that*

$$\int_0^t (\|\nabla_\varepsilon \phi\|_{X^2}^2 + \|\varphi \Delta_\varepsilon \phi\|_{X^2}^2) ds \leq C \int_0^t (\|(\mathcal{N}_u, \varepsilon \mathcal{N}_v)\|_{X^2}^2 + \|\varepsilon \Delta_\varepsilon \phi\|_{X^{\frac{5}{3}}}^2 + \varepsilon^8) ds, \tag{5.2}$$

with $t \in [0,T]$.

The proof of the above proposition is the main part of this paper, and we prove it in Section 6.

5.2. Proof of Theorem 1.1

Before we prove Theorem 1.1, we first give the estimates for the nonlinear terms:

Proposition 5.2. *Under the assumption (5.1), there holds that*

$$\int_0^t \|(\mathcal{N}_u, \varepsilon \mathcal{N}_v)\|_{X^2}^2 ds \leq C\varepsilon^4 \int_0^t \|\omega^R\|_{X^2}^2 ds, \tag{5.3}$$

$$\int_0^t \|\mathcal{N}\|_{L^2}^2 ds \leq C\varepsilon^4 \int_0^t \|\nabla_\varepsilon \omega^R\|_{L^2}^2 ds, \tag{5.4}$$

where $t \in [0,T^*]$.

Proof. By the definition of $\mathcal{N}_u$, we have

$$\int_0^t \|\mathcal{N}_u\|_{X^2}^2 ds \leq \int_0^t \|u^R \partial_x u^R\|_{X^2}^2 ds + \int_0^t \|v^R \partial_y u^R\|_{X^2}^2 ds = I_1 + I_2.$$

It follows from Lemma 2.2 and (3.22) that

$$\begin{aligned} I_1 &\leq C \int_0^t \|\frac{u_\Phi^R}{\varepsilon}\|_{L_y^\infty(H_x^2)}^2 \|\varepsilon \partial_x u^R\|_{X^2}^2 ds \\ &\leq C\varepsilon^{-2} \int_0^t \|u^R\|_{X^2} (\|u^R\|_{X^2} + \|\partial_y u^R\|_{X^2}) \|\varepsilon \partial_x u^R\|_{X^2}^2 ds \\ &\leq C\varepsilon^{-2} \int_0^t \|\omega^R\|_{X^2}^4 ds, \end{aligned}$$

where we use the Gagliardo-Nirenberg inequality (4.9) in the second step.

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

Similarly, we use Lemma 2.2 and (3.22) to deduce

$$
\begin{aligned}
I_2 &\le C\int_0^t \|v_\Phi^R\|^2_{L^\infty_y(H^2_x)}\|\partial_y u^R\|^2_{X^2}ds \le C\varepsilon^{-2}\int_0^t \|\varepsilon\partial_x u^R\|^2_{X^2}\|\partial_y u^R\|^2_{X^2}ds\\
&\le C\varepsilon^{-2}\int_0^t \|\omega^R\|^4_{X^2}ds,
\end{aligned}
$$

where we use $v^R = -\int_0^y \partial_x u^R dy'$ in the second step.

Collecting I_1 and I_2 together and using (5.1), it holds that

$$
\int_0^t \|\mathcal{N}_u\|^2_{X^2}ds \le C\varepsilon^{-2}\int_0^t \|\omega^R\|^4_{X^2}ds \le C\varepsilon^4\int_0^t \|\omega^R\|^2_{X^2}ds.
$$

The estimate for $\varepsilon\mathcal{N}_v$ is obtained by changing u^R into εv^R in the above argument and we omit details. Thus, we obtain (5.3).

For (5.4), we use the definition of $\mathcal{N}$ to have

$$
\begin{aligned}
\int_0^t \|\mathcal{N}\|^2_{L^2}ds \le{}& \int_0^t \|u^R\partial_x\omega^R\|^2_{L^2}ds + \int_0^t \|v^R\partial_y\omega^R\|^2_{L^2}ds\\
&\le \varepsilon^{-2}\int_0^t \|u^R\|^2_{L^\infty_y(H^1_x)}\|\varepsilon\partial_x\omega^R\|^2_{L^2}ds + \int_0^t \|v^R\|^2_{L^\infty_y(H^1_x)}\|\partial_y\omega^R\|^2_{L^2}ds\\
&\le C\varepsilon^{-2}\int_0^t \|\omega^R\|^2_{H^{1,0}}\|\varepsilon\partial_x\omega^R\|^2_{L^2}ds + C\int_0^t \|\omega^R\|^2_{H^{2,0}}\|\partial_y\omega^R\|^2_{L^2}ds\\
&\le C\varepsilon^{-2}\sup_{s\in[0,t]}\|\omega^R\|^2_{X^2}\int_0^t \|\nabla_\varepsilon\omega^R\|^2_{L^2}ds\\
&\le C\varepsilon^4\int_0^t \|\nabla_\varepsilon\omega^R\|^2_{L^2}ds
\end{aligned}
$$

by (5.1), and we obtain (5.4). □

With Proposition 5.1 and Proposition 5.2 in hand, we are in the position to prove Theorem 1.1. By Proposition 4.3, Proposition 5.1 and Proposition 5.2, we get

$$
\sup_{s\in[0,t]}(\lambda\|\nabla_\varepsilon\phi(s)\|^2_{X^{\frac{7}{3}}} + \|\Delta_\varepsilon\phi(s)\|^2_{X^2}) + \int_0^t \|\partial_t\nabla_\varepsilon\phi_\Phi\|^2_{H^{2,0}} \le Ct\varepsilon^6 + C\int_0^t \|\Delta_\varepsilon\phi(s)\|^2_{X^2}ds,
$$

for $t\in[0,T]$. By the Gronwall inequality and choosing a small $T<\min\{T_p,\frac{1}{2\lambda}\}$, we get that

$$
\sup_{s\in[0,t]}(\lambda\|\nabla_\varepsilon\phi(s)\|^2_{X^{\frac{7}{3}}} + \|\omega^R\|^2_{X^2}) + \int_0^t \|\partial_t\nabla_\varepsilon\phi_\Phi\|^2_{H^{2,0}} \le \frac{\mathfrak{C}}{2}\varepsilon^6.
$$

By the Sobolev embedding theorem and Lemma 4.1, we get Theorem 1.1 proved.

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

6. The proof of Proposition 5.1

All we have left is to prove Proposition 5.1. To prove that, we first give the following decomposition of ϕ:

$$\phi = \phi_{slip} + \phi_{bc}, \tag{6.1}$$

where ϕ_{slip} satisfies that

$$\begin{cases} (\partial_t - \Delta_\varepsilon)\Delta_\varepsilon \phi_{slip} + u^p \partial_x \Delta_\varepsilon \phi_{slip} + v^p \partial_y \Delta_\varepsilon \phi_{slip} + \partial_y \phi_{slip} \partial_x \omega^p - \partial_x \phi_{slip} \partial_y \omega^p \\ \qquad\qquad\qquad = \partial_y \mathcal{N}_u - \varepsilon^2 \partial_x \mathcal{N}_v + \varepsilon^2 f_1 + f_2 - C(t)\partial_x \omega^p, \\ \phi_{slip}|_{y=0,1} = 0, \quad \Delta_\varepsilon \phi_{slip}|_{y=0,1} = 0, \\ \phi_{slip}|_{t=0} = 0, \end{cases} \tag{6.2}$$

and ϕ_{bc} satisfies that

$$\begin{cases} (\partial_t - \Delta_\varepsilon)\Delta_\varepsilon \phi_{bc} + u^p \partial_x \Delta_\varepsilon \phi_{bc} + v^p \partial_y \Delta_\varepsilon \phi_{bc} + \partial_y \phi_{bc} \partial_x \omega^p - \partial_x \phi_{bc} \partial_y \omega^p = 0, \\ \phi_{bc}|_{y=0,1} = 0, \quad \partial_y \phi_{bc}|_{y=0,1} = -\partial_y \phi_{slip}|_{y=0,1} + C(t) \\ \phi_{bc}|_{t=0} = 0. \end{cases} \tag{6.3}$$

To prove Proposition 5.1, we need the estimates of ϕ_{slip} and ϕ_{bc}. First, we notice that ϕ_{slip} has a good boundary condition. We use the 'hydrostatic trick' method to get its estimates. The proof of the following proposition is given in Section 7.

Proposition 6.1. *There exists $\lambda_0 > 1$ and $0 < T < \min\{T_p, \frac{1}{2\lambda}\}$ such that for all $t \in [0,T]$, $\lambda \geq \lambda_0$, there holds that*

$$\begin{aligned} \|\Delta_\varepsilon \phi_{slip}\|_{X^2}^2 + \lambda \int_0^t (\|\Delta_\varepsilon \phi_{slip}\|_{X^{\frac{7}{3}}}^2 + \|\nabla_\varepsilon \phi_{slip}\|_{X^{\frac{7}{3}}}^2 + |\nabla_\varepsilon \phi_{slip}|_{y=0,1}|_{X^{\frac{7}{3}}}^2) ds + \int_0^t \|\nabla_\varepsilon \Delta_\varepsilon \phi_{slip}\|_{X^2}^2 ds \\ \leq C \int_0^t \|(\mathcal{N}_u, \varepsilon \mathcal{N}_v)\|_{X^2}^2 ds + \frac{C}{\lambda} \int_0^t \|(C(t), \varepsilon^2 f_1, f_2)\|_{X^{\frac{5}{3}}}^2 ds. \end{aligned}$$

The estimates of ϕ_{bc} are much more difficult. Here, we state the main results on it:

Proposition 6.2. *There exists $\lambda_0 > 1$ and $0 < T < \min\{T_p, \frac{1}{2\lambda}\}$ such that for all $t \in [0,T]$, $\lambda \geq \lambda_0$, there holds that*

$$\int_0^t \|\nabla_\varepsilon \phi_{bc}\|_{X^{\frac{7}{3}}}^2 + \|\varphi \Delta_\varepsilon \phi_{bc}\|_{X^2}^2 ds \leq \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t \Big(|\nabla_\varepsilon \phi_{slip}|_{y=0,1}|_{X^{\frac{7}{3}}}^2 + |C(s)|^2 \Big) ds, \tag{6.4}$$

where C is a universal constant.

The proof of Proposition 6.2 is given in Section 8.

Based on the above two propositions, we are in the position to prove Proposition 5.1. First, we give the estimates of $\|u^R\|_{L^2}$ which are used to control the $C(t)$.

Lemma 6.3. *There exist $0 < T < \min\{T_p, \frac{1}{2\lambda}\}$ and $\lambda_0 \geq 1$ such that for $t \in [0,T]$ and $\lambda \geq \lambda_0$, it holds that*

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

$$\begin{aligned}&\|e^{(1-\lambda t)}(u^R,\varepsilon v^R)\|_{L^2}^2+\lambda\int_0^t\|e^{(1-\lambda s)}(u^R,\varepsilon v^R)\|_{L^2}^2ds+\int_0^t\|e^{(1-\lambda s)}\nabla_\varepsilon(u^R,\varepsilon v^R)\|_{L^2}^2ds \\ &\quad\le C\int_0^t(\|e^{(1-\lambda s)}(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2+\varepsilon^8)ds+\frac{C}{\lambda}\int_0^t\|\nabla_\varepsilon\phi\|_{X^2}^2ds.\end{aligned}\tag{6.5}$$

Remark 6.4. We use weighted quantity $\|e^{(1-\lambda t)}(u^R,\varepsilon v^R)\|_{L^2}$ instead of $\|(u^R,\varepsilon v^R)\|_{L^2}$ to obtain a small constant factor in front of $\int_0^t\|\nabla_\varepsilon\phi\|_{X^2}^2ds$ in (6.5).

Proof. Taking the L^2 inner product with $e^{2(1-\lambda t)}u^R$ in the first equation of (3.7) and with $e^{2(1-\lambda t)}v^R$ in the second equation of (3.7), we use the fact

$$\partial_t(e^{2(1-\lambda t)}f)=e^{2(1-\lambda t)}\partial_t f+2\lambda e^{2(1-\lambda t)}f$$

and integrate by parts by boundary condition $(u^R,v^R)|_{y=0,1}=0$ to yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|e^{(1-\lambda t)}(u^R,\varepsilon v^R)\|_{L^2}^2+\lambda\|e^{(1-\lambda t)}(u^R,\varepsilon v^R)\|_{L^2}^2+\|e^{(1-\lambda t)}\nabla_\varepsilon(u^R,\varepsilon v^R)\|_{L^2}^2\\ &\quad\le C(\|e^{(1-\lambda t)}(u^R,v^R)\|_{L^2}+\|e^{(1-\lambda t)}(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}+\|(R_1,R_2)\|_{L^2})\|e^{(1-\lambda t)}(u^R,\varepsilon v^R)\|_{L^2}\\ &\quad\le\frac{\lambda}{2}\|e^{(1-\lambda t)}(u^R,\varepsilon v^R)\|_{L^2}^2+C(\|e^{(1-\lambda t)}(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2+\varepsilon^8)+\frac{C}{\lambda}\|e^{(1-\lambda t)}\partial_x u^R\|_{L^2}^2,\end{aligned}$$

where we write $v^R=-\int_0^y\partial_x u^R dy'$ and use the fact $\partial_x u^p+\partial_y v^p=0$ to eliminate the transport term and $\partial_x u^R+\partial_y v^R=0$ to eliminate the pressure term, respectively.

Afterwards, integrating time from 0 to t and using $\partial_x u^R=-\partial_x\partial_y\phi$, we obtain

$$\begin{aligned}&\|e^{(1-\lambda t)}(u^R,\varepsilon v^R)(t)\|_{L^2}^2+\lambda\int_0^t\|e^{(1-\lambda s)}(u^R,\varepsilon v^R)\|_{L^2}^2ds+\int_0^t\|e^{(1-\lambda s)}\nabla_\varepsilon(u^R,\varepsilon v^R)\|_{L^2}^2ds\\ &\quad\le C\int_0^t(\|e^{(1-\lambda s)}(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2+\varepsilon^8)ds+\frac{C}{\lambda}\int_0^t\|e^{(1-\lambda s)}\partial_x\partial_y\phi\|_{L^2}^2ds.\end{aligned}$$

Finally, we use $\|e^{(1-\lambda s)}\partial_x\partial_y\phi\|_{L^2}\le C\|\nabla_\varepsilon\phi\|_{X^2}$ to complete the proof. □

Proof of Proposition 5.1. Now, we give the proof Proposition 5.1. We divide this proof into two parts. □

Estimates of $\int_0^t\|\nabla_\varepsilon\phi\|_{X^2}^2$. Since

$$|C(t)|=|\frac{1}{2\pi}\int_{\mathcal{S}}u^R dxdy|\le C\|u^R\|_{L^2}\le C\|e^{(1-\lambda t)}u^R\|_{L^2},$$

by Lemma 6.3, we ensure

$$|C(t)|^2\le C\int_0^t(\|e^{(1-\lambda s)}(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{L^2}^2+\varepsilon^8)ds+\frac{C}{\lambda}\int_0^t\|\nabla_\varepsilon\phi\|_{X^2}^2ds.\tag{6.6}$$

By the definition of f_1 and f_2, we obtain that

$$\int_0^t\|f_1\|_{X^{\frac{5}{3}}}^2+\|f_2\|_{X^{\frac{5}{3}}}^2ds\le C\int_0^t(\|\varepsilon\triangle_\varepsilon\phi\|_{X^{\frac{5}{3}}}^2+\varepsilon^8)ds,$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

and we get

$$\lambda\int_0^t(\|\Delta_\varepsilon\phi_{slip}\|^2_{X^{\frac{7}{3}}}+\|\nabla_\varepsilon\phi_{slip}\|^2_{X^{\frac{7}{3}}}+|\nabla_\varepsilon\phi_{slip}|_{y=0,1}|^2_{X^{\frac{7}{3}}})ds \tag{6.7}$$
$$\leq C\int_0^t\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|^2_{X^2}ds+\frac{C}{\lambda}\Big(|C(t)|^2+\int_0^t(\|\varepsilon\Delta_\varepsilon\phi\|^2_{X^{\frac{5}{3}}}+\varepsilon^8)ds\Big).$$

Then, it follows $\phi=\phi_{slip}+\phi_{bc}$ and (6.4) to deduce

$$\begin{aligned}\int_0^t\|\nabla_\varepsilon\phi\|^2_{X^2}ds&\leq\int_0^t\|\nabla_\varepsilon\phi_{slip}\|^2_{X^2}ds+\int_0^t\|\nabla_\varepsilon\phi_{bc}\|^2_{X^2}ds\\&\leq\frac{C}{\lambda}\int_0^t\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|^2_{X^2}ds+\frac{C}{\lambda^2}\Big(\int_0^t\|\varepsilon\Delta_\varepsilon\phi\|^2_{X^{\frac{5}{3}}}+\varepsilon^8ds\Big)\\&\quad+\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t|\nabla_\varepsilon\phi_{slip}|_{y=0,1}|^2_{X^{\frac{7}{3}}}ds+\frac{C}{\lambda^{\frac{1}{2}}}|C(t)|^2,\\&\leq\frac{C}{\lambda^{\frac{1}{2}}}|C(t)|^2+\frac{C}{\lambda}\int_0^t\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|^2_{X^2}ds+\frac{C}{\lambda^2}\int_0^t(\|\varepsilon\Delta_\varepsilon\phi\|^2_{X^{\frac{5}{3}}}+\varepsilon^8)ds.\end{aligned} \tag{6.8}$$

Plusing (6.6) and above estimates together and taking λ large enough, we get

$$|C(t)|^2+\int_0^t\|\nabla_\varepsilon\phi\|^2_{X^2}ds\leq C\int_0^t\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|^2_{X^2}ds+C\int_0^t(\|\varepsilon\Delta_\varepsilon\phi\|^2_{X^{\frac{5}{3}}}+\varepsilon^8)ds. \tag{6.9}$$

Estimates of $\int_0^t\|\varphi\triangle_\varepsilon\phi\|^2_{X^2}$.
It follows from (6.7) and (6.9) that

$$\begin{aligned}\int_0^t\|\Delta_\varepsilon\phi_{slip}\|^2_{X^2}ds&\leq\frac{C}{\lambda}\int_0^t\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|^2_{X^2}ds+\frac{C}{\lambda}\Big(|C(t)|^2+\int_0^t(\|\varepsilon\Delta_\varepsilon\phi\|^2_{X^{\frac{5}{3}}}+\varepsilon^8)ds\Big)\\&\leq\frac{C}{\lambda}\int_0^t(\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|^2_{X^2}+\|\varepsilon\Delta_\varepsilon\phi\|^2_{X^{\frac{5}{3}}}+\varepsilon^8)ds.\end{aligned}$$

Applying Proposition 6.2 again, we get

$$\int_0^t\|\varphi\Delta_\varepsilon\phi_{bc}\|^2_{X^2}ds\leq C\int_0^t(\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|^2_{X^2}+\|\varepsilon\Delta_\varepsilon\phi\|^2_{X^{\frac{5}{3}}}+\varepsilon^8)ds.$$

Combining the above two estimates, we get

$$\int_0^t\|\varphi\Delta_\varepsilon\phi\|^2_{X^2}ds\leq C\int_0^t(\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|^2_{X^2}+\|\varepsilon\Delta_\varepsilon\phi\|^2_{X^{\frac{5}{3}}}+\varepsilon^8)ds.$$

By now, we get the desired results.

7. Vorticity estimates under artificial boundary condition: Proof of Proposition 6.1

In this section, we give the proof of Proposition 6.1. To simplify the notation, we drop the subscript in the system (6.2):

$$\begin{cases} (\partial_t-\Delta_\varepsilon)\Delta_\varepsilon\phi+u^p\partial_x\Delta_\varepsilon\phi+v^p\partial_y\Delta_\varepsilon\phi+\partial_y\phi\partial_x\omega^p-\partial_x\phi\partial_y\omega^p \\ \qquad\qquad =\partial_y\mathcal{N}_u-\varepsilon^2\partial_x\mathcal{N}_v+\varepsilon^2 f_1+f_2-C(t)\partial_x\omega^p, \\ \phi|_{y=0,1}=0, \quad \Delta_\varepsilon\phi|_{y=0,1}=0, \\ \phi|_{t=0}=0. \end{cases} \tag{7.1}$$

The goal in this section is to establish a uniform (in ε) estimate of vorticity $\omega=\triangle_\varepsilon\phi$.

Proposition 7.1. *There exists $\lambda_0>0$ and $0<T<\min\{T_p,\frac{1}{2\lambda}\}$ such that for all $t\in[0,T]$, $\lambda\geq\lambda_0$, the following holds:*

$$\|\omega(t)\|_{X^2}^2+\lambda\int_0^t(\|\omega\|_{X^{\frac{7}{3}}}^2+\|\nabla_\varepsilon\phi\|_{X^{\frac{7}{3}}}^2+|\nabla_\varepsilon\phi|_{y=0,1}|_{X^{\frac{7}{3}}}^2)ds+\int_0^t\|\nabla_\varepsilon\omega\|_{X^2}^2ds$$
$$\leq C\int_0^t\|(\mathcal{N}_u,\varepsilon\mathcal{N}_v)\|_{X^2}^2ds+\frac{C}{\lambda}\int_0^t\|\varepsilon^2 f_1,f_2,C(t)\|_{X^{\frac{5}{3}}}^2ds.$$

Proof. By Lemma 4.1, we have

$$\partial_y\omega^p\geq c_0>0.$$

Hence, we use the 'hydrostatic trick' to get the desired results. First, acting operator $e^{\Phi(t,D_x)}$ on the first equation of (6.2), we get

$$\begin{aligned}&(\partial_t+\lambda\langle D_x\rangle^{\frac{2}{3}}-\Delta_\varepsilon)\omega_\Phi+u^p\partial_x\omega_\Phi+v^p\partial_y\omega_\Phi-\partial_x\phi_\Phi\partial_y\omega^p \\ &\quad=-(\partial_y\phi\partial_x\omega^p)_\Phi-[e^{\Phi(t,D_x)},u^p\partial_x]\omega-[e^{\Phi(t,D_x)},v^p\partial_y]\omega \\ &\qquad+[e^{\Phi(t,D_x)},\partial_y\omega^p]\partial_x\phi+\partial_y(\mathcal{N}_u)_\Phi-\varepsilon^2\partial_x(\mathcal{N}_v)_\Phi+(\varepsilon^2 f_1+f_2-C(t)\partial_x\omega^p)_\Phi.\end{aligned} \tag{7.2}$$

In view of (7.2), the terrible term comes from $\partial_x\phi_\Phi\partial_y\omega^p$, which loses one tangential derivative. In order to overcome the derivative loss, we take $\langle D_x\rangle^2$ on the (7.2) and then take the L^2 inner product with $\frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}$ to obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\|\frac{\langle D_x\rangle^2\omega_\Phi}{\sqrt{\partial_y\omega^p}}\Big\|_{L^2}^2+\lambda\Big\|\frac{\langle D_x\rangle^{\frac{7}{3}}\omega_\Phi}{\sqrt{\partial_y\omega^p}}\Big\|_{L^2}^2+\Big\|\frac{\nabla_\varepsilon\langle D_x\rangle^2\omega_\Phi}{\sqrt{\partial_y\omega^p}}\Big\|_{L^2}^2 \\ &=-\int_{\mathcal{S}}\langle D_x\rangle^2\omega_\Phi\cdot(\varepsilon\partial_x,\partial_y)\frac{1}{\partial_y\omega^p}\cdot(\varepsilon\partial_x,\partial_y)\langle D_x\rangle^2\omega_\Phi dxdy \\ &\quad+\int_{\mathcal{S}}|\langle D_x\rangle^2\omega_\Phi|^2\Big(\partial_x(\frac{u^p}{\partial_y\omega^p})+\partial_y(\frac{v^p}{\partial_y\omega^p})\Big)dxdy-\int_{\mathcal{S}}[\langle D_x\rangle^2,u^p\partial_x+v^p\partial_y]\omega_\Phi\ \frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}dxdy \\ &\quad-\int_{\mathcal{S}}\langle D_x\rangle^2(\partial_y\phi\partial_x\omega^p)_\Phi\ \frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}dxdy+\int_{\mathcal{S}}[\langle D_x\rangle^2,\partial_y\omega^p]\partial_x\phi_\Phi\ \frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}dxdy \\ &\quad+\int_{\mathcal{S}}\langle D_x\rangle^2\partial_x\phi_\Phi\langle D_x\rangle^2\omega_\Phi dxdy-\int_{\mathcal{S}}\langle D_x\rangle^2\Big([e^{\Phi(t,D_x)},u^p\partial_x]\omega\Big)\ \frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}dxdy\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathcal{S}}\langle D_x\rangle^2\Big([e^{\Phi(t,D_x)},v^p\partial_y]\omega\Big)\ \frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}dxdy\\
&+\int_{\mathcal{S}}\langle D_x\rangle^2\Big([e^{\Phi(t,D_x)},\partial_y\omega^p]\partial_x\phi\Big)\ \frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}dxdy\\
&+\int_{\mathcal{S}}\langle D_x\rangle^2\Big(\partial_y(\mathcal{N}_u)_\Phi-\varepsilon^2\partial_x(\mathcal{N}_v)_\Phi\Big)\ \frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}dxdy\\
&+\int_{\mathcal{S}}\langle D_x\rangle^2(\varepsilon^2f_1+f_2-C(t)\partial_x\omega^p)_\Phi\ \frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}dxdy\\
=\ &T^0+\cdots T^{10}.
\end{aligned}
$$

The boundary term is zero due to artificial boundary condition $\omega|_{y=0,1}=\Delta_\varepsilon\phi|_{y=0,1}=0$. Integrating on $[0,t)$ with $t\le T$ and using $\partial_y\omega^p\ge c_0$, we obtain

$$
\|\omega(t)\|_{X^2}^2+2\lambda\int_0^t\|\omega\|_{X^{\frac{7}{3}}}^2ds+2\int_0^t\|\nabla_\varepsilon\omega\|_{X^2}^2ds\le C\int_0^t|T^0|+\cdots+|T^{10}|ds.
$$

Now, we estimate $T^i,i=0,\cdots,10$ one by one.

Estimate of T^0 and T^1. Since $\partial_y\omega^p\ge c_0>0$ and Lemma 4.1 imply

$$
|(\varepsilon\partial_x,\partial_y)\frac{1}{\partial_y\omega^p}|\le C,\quad |\partial_x(\frac{u^p}{\partial_y\omega^p})|+|\partial_y(\frac{v^p}{\partial_y\omega^p})|\le C,
$$

it is easy to see

$$
|T^0|+|T^1|\le C\|\omega\|_{X^2}(\|\nabla_\varepsilon\omega\|_{X^2}+\|\omega\|_{X^2}).
$$

Estimate of T^2 and T^4. By using Lemma 2.1, we get

$$
\begin{aligned}
&\|[\langle D_x\rangle^2,u^p\partial_x+v^p\partial_y]\omega_\Phi\|_{L^2}\le C(\|\omega\|_{X^2}+\|\partial_y\omega\|_{X^2}),\\
&\|[\langle D_x\rangle^2,\partial_y\omega^p]\partial_x\phi_\Phi\|_{L^2}\le C\|\phi\|_{X^2}\le C\|\partial_y\phi\|_{X^2},
\end{aligned}
$$

where we used the Poincaré inequality and $\phi|_{y=0,1}=0$ to ensure

$$
\|\phi\|_{X^r}\le C\|\partial_y\phi\|_{X^r},\quad r\ge 0 \tag{7.3}
$$

in the last step.

According to

$$
\Delta_\varepsilon\phi=\omega,\quad \phi|_{y=0,1}=0, \tag{7.4}
$$

classical elliptic estimate and (7.3) imply

$$
\|\nabla_\varepsilon\phi\|_{L^2}^2\le\|\omega\|_{L^2}\|\phi\|_{L^2}\le C\|\omega\|_{L^2}\|\partial_y\phi\|_{L^2},
$$

which gives

$$
\|\nabla_\varepsilon\phi\|_{X^r}\le\|\omega\|_{X^r},\quad r\ge 0. \tag{7.5}
$$

Therefore, it follows from $\partial_y\omega^p\ge c_0>0$ to get

$$
|T^2|+|T^4|\le C(\|\partial_y\omega\|_{X^2}+\|\omega\|_{X^2})\|\omega\|_{X^2}.
$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

Estimate of T_3. Using Lemma 2.2 and (7.5), it shows

$$|T^3| \le C\|\partial_y\phi\|_{X^2}\|\omega\|_{X^2} \le C\|\omega\|_{X^2}^2.$$

Estimate of $T^{\cdot}$ This term is a troubling term because it loses one tangential derivative. However, the hydrostatic trick implies

$$\begin{aligned} T^5 &= \int_{\mathcal{S}} \langle D_x\rangle^2 \partial_x\phi_\Phi \langle D_x\rangle^2 \Delta_\varepsilon \phi_\Phi dxdy = -\int_{\mathcal{S}} \langle D_x\rangle^2 \partial_x \nabla_\varepsilon \phi_\Phi \langle D_x\rangle^2 \nabla_\varepsilon \phi_\Phi dxdy \\ &= -\frac{1}{2}\int_{\mathcal{S}} \partial_x |\langle D_x\rangle^2 \nabla_\varepsilon \phi_\Phi|^2 dxdy = 0, \end{aligned}$$

by using $\phi|_{y=0,1} = 0$.

Estimate of T^6,T^7 and T^8. Let's estimate commutators by Lemma 2.3. Since $\partial_y\omega^p \ge c_0 > 0$, we use Lemma 2.3 to ensure that

$$\begin{aligned} &|T^6| \le C\|(u^p\partial_x\omega)_\Phi - u^p\partial_x\omega_\Phi\|_{H^{2-\frac{1}{3},0}}\|\omega\|_{X^{\frac{7}{3}}} \le C\|\omega\|_{X^{2+1-\frac{1}{3}-\frac{1}{3}}}\|\omega\|_{X^{\frac{7}{3}}} = C\|\omega\|_{X^{\frac{7}{3}}}^2, \\ &|T^7| \le C\|\partial_y\omega\|_{X^2}\|\omega\|_{X^2}, \\ &|T^8| \le C\|(\partial_x\phi\partial_y\omega^p)_\Phi - \partial_x\phi_\Phi\partial_y\omega^p\|_{H^{2-\frac{1}{3}}}\|\omega\|_{X^{\frac{7}{3}}} \le C\|\partial_x\phi\|_{X^{2-\frac{1}{3}-\frac{1}{3}}}\|\omega\|_{X^{\frac{7}{3}}} \le C\|\phi\|_{X^{\frac{7}{3}}}\|\omega\|_{X^{\frac{7}{3}}} \\ &\qquad \le C\|\omega\|_{X^{\frac{7}{3}}}^2. \end{aligned}$$

Here, we use (7.3) and (7.5) in the last estimate.

Estimate of T^9 and T^{10}. Integration by parts and boundary condition $\omega|_{y=0,1} = 0$ give that

$$\begin{aligned} |T^9| &= \int_{\mathcal{S}} \langle D_x\rangle^2 (\mathcal{N}_u, \varepsilon\mathcal{N}_v)_\Phi \cdot \nabla_\varepsilon \Big(\frac{\langle D_x\rangle^2\omega_\Phi}{\partial_y\omega^p}\Big) dxdy \\ &\le C\|\mathcal{N}_u, \varepsilon\mathcal{N}_v\|_{X^2}(\|\omega\|_{X^2} + \|\nabla_\varepsilon\omega\|_{X^2}). \end{aligned}$$

However, using the Hölder inequality, we get

$$|T^{10}| \le C\|\varepsilon^2 f_1 + f_2 - C(t)\partial_x\omega^p\|_{X^{\frac{5}{3}}}\|\omega\|_{X^{\frac{7}{3}}}.$$

Collecting $T^0 - T^{10}$ together, we finally obtain

$$\begin{aligned} \int_0^t |T^0| + \cdots + |T^{10}| ds \le{}& C\int_0^t \|\mathcal{N}_u, \varepsilon\mathcal{N}_v\|_{X^2}(\|\nabla_\varepsilon\omega\|_{X^2} + \|\omega\|_{X^2}) \\ &+ \|\omega\|_{X^{\frac{7}{3}}}(\|\omega\|_{X^{\frac{7}{3}}} + \|\nabla_\varepsilon\omega\|_{X^2} + \|\varepsilon^2 f_1 + f_2 - C(t)\partial_x\omega^p\|_{X^{\frac{5}{3}}}) ds \\ \le{}& \frac{1}{10}\int_0^t \|\nabla_\varepsilon\omega\|_{X^2}^2 ds + C\int_0^t \|\mathcal{N}_u, \varepsilon\mathcal{N}_v\|_{X^2}^2 ds \\ &+ (C + \frac{\lambda}{4})\int_0^t \|\omega\|_{X^{\frac{7}{3}}}^2 ds + \frac{C}{\lambda}\int_0^t \|\varepsilon^2 f_1 + f_2 - C(t)\partial_x\omega^p\|_{X^{\frac{5}{3}}}^2 ds. \end{aligned}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

Taking λ large enough, we deduce

$$\|\omega(t)\|_{X^2}^2 + \lambda \int_0^t \|\omega\|_{X^{\frac{7}{3}}}^2 ds + \int_0^t \|\nabla_\varepsilon \omega\|_{X^2}^2 ds \tag{7.6}$$
$$\leq C \int_0^t (\|\mathcal{N}_u, \varepsilon \mathcal{N}_v\|_{X^2}^2 ds + \frac{C}{\lambda} \int_0^t \|\varepsilon^2 f_1 + f_2 - C(t)\partial_x \omega^p\|_{X^{\frac{5}{3}}}^2 ds.$$

However, (7.5) gives

$$\|\nabla_\varepsilon \phi\|_{X^{\frac{7}{3}}} \leq C\|\omega\|_{X^{\frac{7}{3}}}.$$

Calderon-Zygmund inequality and Gagliardo-Nirenberg inequality (4.9) imply

$$|\nabla_\varepsilon \phi|_{y=0,1}|_{X^{\frac{7}{3}}} \leq C\|\nabla_\varepsilon \phi\|_{X^{\frac{7}{3}}}^{\frac{1}{2}} (\|\nabla_\varepsilon \phi\|_{X^{\frac{7}{3}}}^{\frac{1}{2}} + \|\nabla_\varepsilon \partial_y \phi\|_{X^{\frac{7}{3}}}^{\frac{1}{2}}) \leq C\|\omega\|_{X^{\frac{7}{3}}}.$$

Along with (7.5) and (7.6), we get the desired result. □

8. Construction of the boundary corrector: Proof of Proposition 6.2

In the previous section, we construct a solution to the Orr-Sommerfeld equation with artificial boundary conditions: we replace condition $\partial_y \phi|_{y=0,1} = 0$ by $\Delta_\varepsilon \phi|_{y=0,1} = 0$. To go back to the original system, we need to correct Neumann's condition. Thus, we define ϕ_{bc} which satisfies the following system:

$$\begin{cases} (\partial_t - \Delta_\varepsilon)\Delta_\varepsilon \phi_{bc} + u^p \partial_x \Delta_\varepsilon \phi_{bc} + v^p \partial_y \Delta_\varepsilon \phi_{bc} + \partial_y \phi_{bc} \partial_x \omega^p - \partial_x \phi_{bc} \partial_y \omega^p = 0, \\ \phi_{bc}|_{y=0,1} = 0, \quad \partial_y \phi_{bc}|_{y=0,1} = -\partial_y \phi_{slip}|_{y=0,1} + C(t), \\ \phi|_{t=0} = 0, \end{cases} \tag{8.1}$$

To estimate ϕ_{bc}, we use the following decomposition:

$$\phi_{bc} = \phi_{bc,S} + \phi_{bc,T} + \phi_{bc,R}.$$

The definitions and estimates of $\phi_{bc,S}, \phi_{bc,T}$ and $\phi_{bc,R}$ are given in the following subsections.

8.1. The estimates of $\phi_{bc,S}$: Stokes equation

In this subsection, we deal with $\phi_{bc,S}$.

Because of two boundaries $y = 0$ and $y = 1$, we define

$$\phi_{bc,S} = \phi_{bc,S}^0 + \phi_{bc,S}^1,$$

where $\phi_{bc,S}^0$ satisfies the following Stokes equation:

$$\begin{cases} (\partial_t - \Delta_\varepsilon)\Delta_\varepsilon \phi_{bc,S}^0 = 0, \quad (x,y) \in \mathbb{T} \times (0, +\infty) \\ \phi_{bc,S}^0|_{y=0} = 0, \quad \partial_y \phi_{bc,S}^0|_{y=0} = h^0, \\ \phi_{bc,S}^0|_{t=0} = 0, \end{cases} \tag{8.2}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

and $\phi^1_{bc,S}$ satisfies the following Stokes equation:

$$\begin{cases}(\partial_t-\Delta_\varepsilon)\Delta_\varepsilon\phi^1_{bc,S}=0, & (x,y)\in\mathbb{T}\times(-\infty,1)\\ \phi^1_{bc,S}|_{y=1}=0, & \partial_y\phi^1_{bc,S}|_{y=1}=h^1,\\ \phi^1_{bc,S}|_{t=0}=0,\end{cases} \tag{8.3}$$

where $t\in[0,T]$. Here, (h^0,h^1) is a given boundary data satisfying $(h^0(t),h^1(t))=0$ for $t=0$ and $t\geq T$. Here, we point out that h^i is defined by

$$h^i=\mathcal{A}(-\partial_y\phi_{slip}|_{y=0,1}+C(t)),$$

where the operator $\mathcal{A}$ is a zero-order operator which is defined later.

In the following, we only give the process for $\phi^0_{bc,S}$. The case of $\phi^1_{bc,S}$ is almost the same and we leave details to readers.

At first, we give zero extension of $\phi^0_{bc,S}$ and h^0 with $t\leq 0$ such that we can take a Fourier transform in t. Let $\widehat{\phi^0_{bc,S}}=\widehat{\phi^0_{bc,S}}(\zeta,k,y)$ be the Fourier transform of $\phi^0_{bc,S}$ on x and t. Then $(\widehat{\phi^0_{bc,S}})_\Phi$ satisfies the ODE:

$$\begin{cases}-(\partial_y^2-\varepsilon^2|k|^2)^2(\widehat{\phi^0_{bc,S}})_\Phi+(i\zeta+\lambda\langle k\rangle^{\frac{2}{3}})(\partial_y^2-\varepsilon^2|k|^2)(\widehat{\phi^0_{bc,S}})_\Phi=0, & y>0,\\ (\widehat{\phi^0_{bc,S}})_\Phi|_{y=0}=0, \quad \partial_y(\widehat{\phi^0_{bc,S}})_\Phi|_{y=0}=\widehat{h^0_\Phi},\end{cases} \tag{8.4}$$

where $\zeta\in\mathbb{R}$ and $k\in\mathbb{Z}$. Assuming the decay of $(|k|\phi^0_{bc,S},\partial_y\phi^0_{bc,S})$ and the boundedness of $\partial_y\phi^0_{bc,S}$, we obtain the formula

$$(\widehat{\phi^0_{bc,S}})_\Phi(\zeta,k,y)=-\frac{e^{-\gamma y}-e^{-\varepsilon|k|y}}{\gamma-\varepsilon|k|}\widehat{h^0_\Phi}(\zeta,k), \quad y>0 \tag{8.5}$$

$$\gamma=\gamma(\zeta,k,\varepsilon,\lambda)=\sqrt{\varepsilon^2|k|^2+\lambda\langle k\rangle^{\frac{2}{3}}+i\zeta}, \tag{8.6}$$

where the square root is taken so that the real part is positive, and it follows that

$$\varepsilon|k|,\ \lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}\leq\sqrt{\varepsilon^2|k|^2+\lambda\langle k\rangle^{\frac{2}{3}}}\leq\mathbf{Re}(\gamma)\leq|\gamma|\leq 2\mathbf{Re}(\gamma). \tag{8.7}$$

This inequality will be used frequently. It is easy to calculate that

$$\partial_y(\widehat{\phi^0_{bc,S}})_\Phi=-e^{-\gamma y}\widehat{h^0_\Phi}-\varepsilon|k|(\widehat{\phi^0_{bc,S}})_\Phi, \tag{8.8}$$

$$(\partial_y^2-\varepsilon^2|k|^2)(\widehat{\phi^0_{bc,S}})_\Phi=(\gamma+\varepsilon|k|)e^{-\gamma y}\widehat{h^0_\Phi}. \tag{8.9}$$

The formula (8.8) will be used in estimating velocity and (8.9) will be used in estimating vorticity. With the same process above, we get the formula for $(\widehat{\phi^1_{bc,S}})_\Phi$:

$$(\widehat{\phi^1_{bc,S}})_\Phi(\zeta,k,y)=\frac{e^{-\gamma(1-y)}-e^{-\varepsilon|k|(1-y)}}{\gamma-\varepsilon|k|}\widehat{h^1_\Phi}(\zeta,k), \quad y<1, \tag{8.10}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

with γ given in (8.6). It is easy to see

$$\partial_y(\widehat{\phi^1_{bc,S}})_\Phi = e^{-\gamma(1-y)}\widehat{h^1_\Phi} + \varepsilon|k|(\widehat{\phi^1_{bc,S}})_\Phi, \tag{8.11}$$

$$(\partial_y^2 - \varepsilon^2|k|^2)(\widehat{\phi^1_{bc,S}})_\Phi = -(\gamma + \varepsilon|k|)e^{-\gamma(1-y)}\widehat{h^1_\Phi}. \tag{8.12}$$

Remark 8.1. For $\varepsilon = 0$ in (8.2), $\triangle_0 = \partial_y^2$.

$$(\widehat{\phi^0_{bc,S}})_\Phi(\zeta,k,y) = -\frac{\widehat{h^0_\Phi}}{\gamma_0}(e^{-\gamma_0 y} - 1), \quad \gamma_0 = \sqrt{\lambda\langle k\rangle^{\frac{2}{3}} + i\zeta} \tag{8.13}$$

solves (8.4) with $\varepsilon = 0$ and $(\widehat{\phi^0_{bc,S}})_\Phi$ holds $\lim_{y\to+\infty} = \frac{\widehat{h^0_\Phi}}{\gamma_0}$. Though $(\widehat{\phi^0_{bc,S}})_\Phi$ does not tend to zero as y tends to infinity, the solution $(\widehat{\phi^0_{bc,S}})_\Phi$ is only used to correct the boundary condition near $y = 0$, and we do not care about its value at infinity. It is easy to deduce

$$\partial_y(\widehat{\phi^0_{bc,S}})_\Phi = \widehat{h^0_\Phi}e^{-\gamma_0 y}, \quad \partial_y^2(\widehat{\phi^0_{bc,S}})_\Phi = -\gamma_0\widehat{h^0_\Phi}e^{-\gamma_0 y}, \tag{8.14}$$

and we find these two terms decay to zero as y tends to infinity. By the same method, we can get another solution near $y = 1$:

$$(\widehat{\phi^1_{bc,S}})_\Phi(\zeta,k,y) = \frac{\widehat{h^0_\Phi}}{\gamma_0}(e^{-\gamma_0(1-y)} - 1). \tag{8.15}$$

These constructions are the main difference between $\varepsilon = 0$ and $\varepsilon \neq 0$, but they enjoy the same properties stated below.

Lemma 8.2. *Let $\phi^i_{bc,S}$ be the solution of (8.2). It holds that*

$$\sum_{k\in\mathbb{Z}}\|(\varepsilon|k|(\widehat{\phi^i_{bc,S}})_\Phi, \partial_y(\widehat{\phi^i_{bc,S}})_\Phi)\|_{L^2_{\zeta,y}} \le \frac{C}{\lambda^{\frac{1}{4}}}\sum_{k\in\mathbb{Z}}\|\langle k\rangle^{-\frac{1}{6}}\widehat{h^i_\Phi}\|_{L^2_\zeta}, \tag{8.16}$$

where $i = 0,1$ and $L^2_{\zeta,y} = l^2_\zeta(L^2_y(0,+\infty))$ for $i = 0$ and $L^2_{\zeta,y} = l^2_\zeta(L^2_y(-\infty,1))$ for $i = 1$.
It is also held that

$$\sum_{k\in\mathbb{Z}}\|k(\widehat{\phi^i_{bc,S}})_\Phi\|_{L^2_{\zeta,y}} \le \frac{C}{\lambda^{\frac{1}{2}}}\|k\langle k\rangle^{-\frac{1}{3}}\widehat{h^i_\Phi}\|_{L^2_\zeta}, \tag{8.17}$$

where $i = 0,1$ and $L^2_{\zeta,y} = l^2_\zeta(L^2_y(0,1))$.

Proof. We only give the proof for $i = 0$. The case $i = 1$ is almost the same, and we omit details to readers.

(8.16) follows from (8.5), (8.8) and the Plancherel theorem by observing the estimate for multipliers

$$\|e^{-\mathbf{Re}(\gamma)y}\|_{L^2_y(0,\infty)} \le \frac{C}{\lambda^{\frac{1}{4}}\langle k\rangle^{\frac{1}{6}}}, \tag{8.18}$$

$$\|\varepsilon|k|\cdot e^{-\varepsilon|k|y}\cdot|\frac{1-e^{-(\gamma-\varepsilon|k|)y}}{\gamma-\varepsilon|k|}|\|_{L^2_y(0,+\infty)} \le \frac{C}{\lambda^{\frac{1}{4}}\langle k\rangle^{\frac{1}{6}}}. \tag{8.19}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

The estimate (8.18) is a direct consequence of

$$\mathbf{Re}(\gamma) \geq \frac{1}{\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}}. \tag{8.20}$$

For (8.19), we divide it into two cases: 1. $\varepsilon|k| \leq \frac{1}{2}\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}$, and 2. $\varepsilon|k| \geq \frac{1}{2}\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}$. In case 1,

$$\Big|\gamma - \varepsilon|k|\Big| \geq \frac{\varepsilon|k| + \lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}}{C},$$

which implies

$$\begin{aligned}\|\varepsilon|k|\cdot e^{-\varepsilon|k|y}\cdot|\frac{1-e^{-(\gamma-\varepsilon|k|)y}}{\gamma-\varepsilon|k|}|\|_{L^2_y(0,+\infty)} \leq& \frac{C}{\varepsilon|k|+\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}}\|\varepsilon|k|e^{-\varepsilon|k|y}\|_{L^2_y(0,+\infty)}\\ \leq& C\frac{(\varepsilon|k|)^{\frac{1}{2}}}{\varepsilon|k|+\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}} \leq \frac{C}{\lambda^{\frac{1}{4}}\langle k\rangle^{\frac{1}{6}}}.\end{aligned}$$

In case 2, we use the bound

$$|\frac{1-e^{-z}}{z}| \leq C,$$

for $\mathbf{Re}(z) > 0$, which implies that

$$\|\varepsilon|k|\cdot e^{-\varepsilon|k|y}\cdot|\frac{1-e^{-(\gamma-\varepsilon|k|)y}}{\gamma-\varepsilon|k|}|\|_{L^2_y(0,+\infty)} \leq \|y\varepsilon|k|e^{-\varepsilon|k|y}\|_{L^2_y(0,+\infty)} \leq \frac{C}{(\varepsilon|k|)^{\frac{1}{2}}} \leq \frac{C}{\lambda^{\frac{1}{4}}\langle k\rangle^{\frac{1}{6}}}.$$

Combining cases 1–2, we complete (8.19), which yields (8.16). The estimate (8.17) is proved by using (8.5), Placherel theorem and

$$\|e^{-\varepsilon|k|y}\cdot|\frac{1-e^{-(\gamma-\varepsilon|k|)y}}{\gamma-\varepsilon|k|}|\|_{L^2_y(0,1)} \leq \frac{C}{\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}}. \tag{8.21}$$

Indeed, note that the integral interval is $y \in (0,1)$ and we also divide it into $\varepsilon|k| \leq \frac{1}{2}\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}$ and $\varepsilon|k| \geq \frac{1}{2}\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}$. When $\varepsilon|k| \geq \frac{1}{2}\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}$, the similar argument above gives that

$$\|e^{-\varepsilon|k|y}\cdot|\frac{1-e^{-(\gamma-\varepsilon|k|)y}}{\gamma-\varepsilon|k|}|\|_{L^2_y(0,1)} \leq \frac{C}{\lambda^{\frac{3}{4}}\langle k\rangle^{\frac{1}{2}}}. \tag{8.22}$$

When $\varepsilon|k| \leq \frac{1}{2}\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}$(with $\varepsilon|k| \ll 1$), we compute as

$$\|e^{-\varepsilon|k|y}\cdot|\frac{1-e^{-(\gamma-\varepsilon|k|)y}}{\gamma-\varepsilon|k|}|\|_{L^2_y(0,1)} \leq C\|\frac{1}{\varepsilon|k|+\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}}\|_{L^2_y(0,1)} \leq \frac{C}{\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}}.$$

The finite interval $(0,1)$ is essential here. Thus, we complete this lemma. □

To express this clearly, we introduce norms related to $y > 0$ and $y < 1$, respectively. For any function f, we define

$$\|f\|_{X^r_i} = \|f_\Phi\|_{L^2_y(I_i;H^r_x)}, \tag{8.23}$$

where $I_0=(0,+\infty)$ and $I_1=(-\infty,1)$. It is obvious to see $\|\cdot\|_{X^r}\le\|\cdot\|_{X_i^r}$ for any $i=0,1$. Using Lemma 8.2 above, we can deduce the estimate for $\nabla_\varepsilon\phi^i$, where $i=0,1$.

Proposition 8.3. *Let $\phi^i_{bc,S}$ be the solution of (8.2). It holds that*

$$\int_0^t \|\nabla_\varepsilon\phi^i_{bc,S}\|^2_{X_i^{\frac{7}{3}+\frac{1}{6}}}\,ds \le \frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |h^i|^2_{X^{\frac{7}{3}}}\,ds, \tag{8.24}$$

$$\int_0^t \|\partial_x\phi^i_{bc,S}\|^2_{X^{\frac{5}{3}}}\,ds \le \frac{C}{\lambda}\int_0^t |\partial_x h^i|^2_{X^{\frac{4}{3}}}\,ds. \tag{8.25}$$

Proof. The proof is done by using (8.16) and (8.17). □

Next, we give the estimate to the boundary term $\phi^0_{bc,S}|_{y=1}$ and $\phi^1_{bc,S}|_{y=0}$.

Lemma 8.4. *For any $M\ge 0$ and $i=0,1$, it holds that*

$$\int_0^t |(\varepsilon\partial_x)^M\phi^i_{bc,S}|_{y=1-i}|^2_{X^{r+\frac{1}{3}}}\,ds \le \frac{C}{\lambda}\int_0^t |h^i|^2_{X^r}\,ds, \tag{8.26}$$

and

$$\int_0^t |(\varepsilon\partial_x)^M\partial_y\phi^i_{bc,S}|_{y=1-i}|^2_{X^{r+\frac{1}{3}}}\,ds \le \frac{C}{\lambda}\int_0^t |h^i|^2_{X^r}\,ds, \tag{8.27}$$

for any $r\ge 0$.

Proof. We only give the proof for the case $i=0$; the case $i=1$ is similar and we omit details to readers. Taking $y=1$ in (8.5) and using

$$\left|(\varepsilon|k|)^M\cdot e^{-\varepsilon|k|}\cdot\frac{e^{-(\gamma-\varepsilon|k|)}-1}{\gamma-\varepsilon|k|}\right| \le \frac{C}{\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}}, \tag{8.28}$$

we get

$$\int_0^t |(\varepsilon|k|)^M\phi^0_{bc,S}|_{y=1}|^2_{X^{r+\frac{1}{3}}}\,ds \le \frac{C}{\lambda}\int_0^t |h^0|^2_{X^r}\,ds.$$

However, we refer to (8.8) and take $y=1$ in it by noticing

$$\left|e^{-\mathbf{Re}(\gamma)}(\varepsilon|k|)^M\right| \le \left|e^{-\frac{1}{2}\varepsilon|k|}(\varepsilon|k|)^M e^{-\frac{1}{2}\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{6}}}\right| \le Ce^{-\frac{1}{2}\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}} \le \frac{C}{(\lambda\langle k\rangle^{\frac{2}{3}})^{N/2}},$$

for any $N\ge 0$, and combining with (8.28) to deduce

$$\int_0^t |(\varepsilon\partial_x)^M\partial_y\phi^0_{bc,S}|_{y=1}|^2_{X^{r+\frac{1}{3}}}\,ds \le \frac{C}{\lambda}\int_0^t |h^0|^2_{X^r}\,ds. \tag{8.29}$$

Thus, we finish our proof. □

At the end of this subsection, we give some weight estimates of vorticity $\omega^i_{bc,S}=\Delta_\varepsilon\phi^i_{bc,S}$. Denote

$$\varphi^0(y)=y,\quad \varphi^1(y)=1-y. \tag{8.30}$$

Proposition 8.5. *It holds that*

$$|\widehat{(\omega^i_{bc,S})}_\Phi(\zeta,k,y)|+|\varphi^i\partial_y\widehat{(\omega^i_{bc,S})}_\Phi(\zeta,k,y)|\le C(|\gamma|+\varepsilon|k|)e^{-\mathbf{Re}(\gamma)\varphi^i}|\widehat{h^i_\Phi}(\zeta,k)|. \tag{8.31}$$

As a consequence, we get for $\theta'\in[-\frac{1}{2},2]$

$$\int_0^t \|(\varphi^i)^{1+\theta'}\omega^i_{bc,S}\|^2_{X_i^{\frac{7}{3}+\frac{1}{3}(\theta'+\frac{1}{2})}}+\|(\varphi^i)^{2+\theta'}(\partial_y,\varepsilon|k|)\omega^i_{bc,S}\|^2_{X_i^{\frac{7}{3}+\frac{1}{3}(\theta'+\frac{1}{2})}}ds\le\frac{C}{\lambda^{\frac{1}{2}+\theta'}}\int_0^t|h^i|^2_{X^{\frac{7}{3}}}ds.$$

Proof. The result is obtained by using formula (8.6), (8.9) and (8.12), the Plancherel theorem and by observing that multiplier $\varphi^i(y)$ gains $\frac{1}{\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}}$. More precisely,

$$\|(\varphi^i)^{1+m}|\gamma|e^{-\mathbf{Re}(\gamma)\varphi^i}\|_{L^2_y(I_i)}\le\Big(\frac{C}{\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}}}\Big)^{m+\frac{1}{2}}.$$

Thus, we complete the proof. □

Based on the above proposition, we have more estimates on $\omega^i_{bc,S}$:

Proposition 8.6. *Let* $\theta\in[0,2]$. *It holds that*

$$\int_0^t\|\varphi^i\Delta_\varepsilon\phi^i_{bc,S}\|^2_{X_i^{\frac{7}{3}+\frac{1}{6}}}+\|(\varphi^i)^2\partial_y\Delta_\varepsilon\phi^i_{bc,S}\|^2_{X_i^{\frac{7}{3}+\frac{1}{6}}}ds\le\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t|h^i|^2_{X^{\frac{7}{3}}}ds, \tag{8.32}$$

$$\int_0^t\|\langle D_x\rangle^{\frac{\theta}{3}-\frac{1}{3}}(\varphi^i)^{\theta+\frac{3}{2}}(\partial_x\Delta_\varepsilon\phi^i_{bc,S})\|^2_{X_i^2}ds\le\frac{C}{\lambda^{\theta+1}}\int_0^t|h^i|^2_{X^{\frac{7}{3}}}ds, \tag{8.33}$$

$$\int_0^t\|\langle D_x\rangle^{\frac{\theta}{3}-\frac{1}{3}}(\varphi^i)^{\theta+\frac{3}{2}}(\partial_y\Delta_\varepsilon\phi^i_{bc,S})\|^2_{X_i^2}ds\le\frac{C}{\lambda^{\theta}}\int_0^t|h^i|^2_{X^{\frac{5}{3}}}ds. \tag{8.34}$$

Proof. (8.32) is a direct result of Lemma 8.5 by taking $\theta=0$. It is easy to check

$$\|\langle k\rangle^{\frac{\theta}{3}-\frac{1}{3}}(\varphi^i)^{\theta+\frac{3}{2}}k\Delta_\varepsilon\widehat{(\phi^i_{bc,S})}_\Phi\|_{L^2_y(I_i)}\le C\frac{\langle k\rangle^{\frac{2}{3}+\frac{\theta}{3}}}{(\lambda^{\frac{1}{2}}\langle k\rangle^{\frac{1}{3}})^{\theta+\frac{1}{2}+\frac{1}{2}}}|\widehat{h^i_\Phi}|=\frac{C\langle k\rangle^{\frac{1}{3}}}{\lambda^{\frac{1}{2}(\theta+1)}}|\widehat{h^i_\Phi}|$$

by taking $\theta'=\theta+\frac{1}{2}$ in Lemma 8.5 and completing (8.33). Similarly, we check

$$\|\langle k\rangle^{\frac{\theta}{3}-\frac{1}{3}}(\varphi^i)^{\theta+\frac{3}{2}}\partial_y\Delta_\varepsilon\widehat{(\phi^i_{bc,S})}_\Phi\|_{L^2_y(I_i)}\le\frac{C\langle k\rangle^{\frac{\theta}{3}-\frac{1}{3}}}{\lambda^{\frac{\theta}{2}}\langle k\rangle^{\frac{\theta}{3}}}|\widehat{h^i_\Phi}|\le\frac{C}{\lambda^{\frac{\theta}{2}}\langle k\rangle^{\frac{1}{3}}}|\widehat{h^i_\Phi}|$$

by taking $\theta'=\theta-\frac{1}{2}$ in Lemma 8.5 to completing (8.34). □

8.2. The estimates of $\phi_{bc,T}$: vorticity transport estimate.

$\phi_{bc,T}$ is defined by

$$\phi_{bc,T}=\phi^0_{bc,T}+\phi^1_{bc,T},$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

where $\phi^0_{bc,T}$ is defined by

$$\begin{cases} (\partial_t-\Delta_\varepsilon)\Delta_\varepsilon\phi^0_{bc,T}+u^p\partial_x\Delta_\varepsilon\phi^0_{bc,T}+v^p\partial_y\Delta_\varepsilon\phi^0_{bc,T} \\ \qquad =-u^p\partial_x\Delta_\varepsilon\phi^0_{bc,S}-v^p\partial_y\Delta_\varepsilon\phi^0_{bc,S}\overset{\text{def}}{=}H^0, \quad (x,y)\in\mathbb{T}\times(0,+\infty) \\ \phi^0_{bc,T}|_{y=0}=0, \quad \Delta_\varepsilon\phi^0_{bc,T}|_{y=0}=0, \quad \phi^0_{bc,T}|_{t=0}=0, \end{cases} \tag{8.35}$$

and $\phi^1_{bc,T}$ is defined by

$$\begin{cases} (\partial_t-\Delta_\varepsilon)\Delta_\varepsilon\phi^1_{bc,T}+u^p\partial_x\Delta_\varepsilon\phi^1_{bc,T}+v^p\partial_y\Delta_\varepsilon\phi^1_{bc,T} \\ \qquad =-u^p\partial_x\Delta_\varepsilon\phi^1_{bc,S}-v^p\partial_y\Delta_\varepsilon\phi^1_{bc,S}\overset{\text{def}}{=}H^1, \quad (x,y)\in\mathbb{T}\times(-\infty,1) \\ \phi^1_{bc,T}|_{y=1}=0, \quad \Delta_\varepsilon\phi^1_{bc,T}|_{y=1}=0, \quad \phi^1_{bc,T}|_{t=0}=0. \end{cases} \tag{8.36}$$

We need to emphasize that we extend (u^p,v^p) to $y\in\mathbb{R}$ by zero, which means that $(u^p,v^p)=0$ when $y\in\mathbb{R}\setminus[0,1]$.

Before we give the estimates of $\phi^i_{bc,T}$, we use Proposition 8.6 and $(u^p,v^p)|_{y=0,1}=0$ to get that

$$\begin{aligned} \int_0^t \|(\varphi^i)^{\frac12+\theta}H^i\|^2_{X_i^{\frac53+\frac\theta3}}ds &\le C\int_0^t \|(\varphi^i)^{\frac32+\theta}(\partial_x\Delta_\varepsilon\phi^i_{bc,S}+\partial_y\Delta_\varepsilon\phi^i_{bc,S})\|^2_{X_i^{\frac53+\frac\theta3}}ds \\ &\le C\int_0^t |h^i|^2_{X^{\frac73}}ds, \end{aligned} \tag{8.37}$$

where $\theta=0,1,2$.

We are in the position to give the estimates of $\phi^i_{bc,T}$:

Proposition 8.7. *Let $\theta=0,1,2$ and $i=0,1$. There exists $\lambda_0>1$ and $0<T<\min\{T_p,\frac{1}{2\lambda}\}$ such that for all $t\in[0,T]$, $\lambda\ge\lambda_0$, it holds that*

$$\begin{aligned} &\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac\theta3}}+\lambda\int_0^t\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{13}{6}+\frac\theta3}}ds \\ &\quad+\int_0^t\|(\varphi^i)^\theta\nabla_\varepsilon\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac\theta3}}ds\le\frac{C}{\lambda^{\frac12}}\int_0^t|h^i|^2_{X^{\frac73}}ds, \end{aligned}$$

where $\Delta_\varepsilon\phi^i_{bc,T}=\omega^i_{bc,T}$ and φ^i is given in (8.30).

Proof. Acting $e^{\Phi(t,D_x)}$ on the first equation of (8.35), we obtain

$$\begin{aligned} &(\partial_t+\lambda\langle D_x\rangle^{\frac23}-\Delta_\varepsilon)(\omega^i_{bc,T})_\Phi+u^p\partial_x(\omega^i_{bc,T})_\Phi+v^p\partial_y(\omega^i_{bc,T})_\Phi \\ &\quad+\Big((u^p\partial_x\omega^i_{bc,T})_\Phi-u^p\partial_x(\omega^i_{bc,T})_\Phi\Big)+\Big((v^p\partial_y\omega^i_{bc,T})_\Phi-v^p\partial_y(\omega^i_{bc,T})_\Phi\Big)=H^i_\Phi. \end{aligned} \tag{8.38}$$

Then, taking the $L^2_y(I_i;H_x^{\frac{11}{6}+\frac\theta3})$ inner product with $(\varphi^i)^{2\theta}(\omega^i_{bc,T})_\Phi$, we get by using $\omega^i_{bc,T}|_{y=i}=0$, $\partial_xu^p+\partial_yv^p=0$ and integrating by parts that

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}+\lambda\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{13}{6}+\frac{\theta}{3}}}+\|(\varphi^i)^\theta\nabla_\varepsilon\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}\\
&=-\int_{\mathcal{S}_i}(\varphi^i)^{2\theta}[\langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}},u^p\partial_x+v^p\partial_y](\omega^i_{bc,T})_\Phi\ \langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}(\omega^i_{bc,T})_\Phi dxdy\\
&\quad+\frac{1}{2}\int_{\mathcal{S}_i}\partial_y\big((\varphi^i)^{2\theta}\big)v^p|\langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}(\omega^i_{bc,T})_\Phi|^2dxdy\\
&\quad-\int_{\mathcal{S}_i}\langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}\Big((u^p\partial_x\omega^i_{bc,T})_\Phi-u^p\partial_x(\omega^i_{bc,T})_\Phi\Big)\ \langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}(\omega^i_{bc,T})_\Phi(\varphi^i)^{2\theta}dxdy\\
&\quad-\int_{\mathcal{S}_i}\langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}\Big((v^p\partial_y\omega^i_{bc,T})_\Phi-v^p\partial_y(\omega^i_{bc,T})_\Phi\Big)\ \langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}(\omega^i_{bc,T})_\Phi(\varphi^i)^{2\theta}dxdy\\
&\quad-\int_{\mathcal{S}_i}\partial_y\big((\varphi^i)^{2\theta}\big)\langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}\partial_y(\omega^i_{bc,T})_\Phi\ \langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}(\omega^i_{bc,T})_\Phi dxdy\\
&\quad+\int_{\mathcal{S}_i}\langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}H^i_\Phi\ \langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}(\omega^i_{bc,T})_\Phi(\varphi^i)^{2\theta}dxdy\\
&=I^i_1+\cdots+I^i_6,
\end{aligned}
$$

where $\mathcal{S}_i=\mathbb{T}\times I_i$. Integrating on $[0,t)$ with $t\le T$, we obtain

$$
\begin{aligned}
\|(\varphi^i)^\theta\omega^i_{bc,T}(t)\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}&+2\lambda\int_0^t\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{13}{6}+\frac{\theta}{3}}}ds+2\int_0^t\|(\varphi^i)^\theta\nabla_\varepsilon\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}ds\\
&\le 2\int_0^t|I^i_1|+\cdots+|I^i_6|ds.
\end{aligned}
\tag{8.39}
$$

Now, we estimate I^i_j, $j=1,\cdots,6$ term by term.

Estimate of I^i_1. It follows from Lemma 2.1 that

$$
\|[\langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}},u^p\partial_x+v^p\partial_y](\omega^i_{bc,T})_\Phi\|_{L^2_x}\le C(\|(\omega^i_{bc,T})_\Phi\|_{H_x^{\frac{11}{6}+\frac{\theta}{3}}}+\|\partial_y(\omega^i_{bc,T})_\Phi\|_{H_x^{\frac{11}{6}+\frac{\theta}{3}}}),
$$

which deduces that

$$
\begin{aligned}
|I^i_1|&\le C(\|(\varphi^i)^\theta\omega^i_{bc,T}\|_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}+\|(\varphi^i)^\theta\partial_y\omega^i_{bc,T}\|_{X_i^{\frac{11}{6}+\frac{\theta}{3}}})\|(\varphi^i)^\theta\omega^i_{bc,T}\|_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}\\
&\le\frac{1}{10}\|(\varphi^i)^\theta\partial_y\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}+C\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}.
\end{aligned}
$$

Estimate of I^i_2. Thanks to

$$
|\partial_y\big((\varphi^i)^{2\theta}\big)v^p|=|2\theta(\varphi^i)'(\varphi^i)^{2\theta-1}v^p|\le C\theta(\varphi^i)^{2\theta}|\frac{v^p}{\varphi^i}|\le C\theta(\varphi^i)^{2\theta},\tag{8.40}
$$

by using $v^p|_{y=i}=0$, it is obvious to see

$$
|I^i_2|\le C\theta\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}.
$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

Estimate of I_3^i. Applying Lemma 2.3, we find

$$\begin{aligned}|I_3^i| \le & \|(\varphi^i)^\theta \langle D_x\rangle^{\frac{9}{6}+\frac{\theta}{3}}\Big((u^p\partial_x\omega^i_{bc,T})_\Phi - u^p\partial_x(\omega^i_{bc,T})_\Phi\Big)\|_{L^2_y(I_i;L^2_x)}\|(\varphi^i)^\theta\omega^i_{bc,T}\|_{X_i^{\frac{13}{6}+\frac{\theta}{3}}}\\ &\le C\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{13}{6}+\frac{\theta}{3}}}.\end{aligned}$$

Estimate of I_4^i. Applying Lemma 2.2, we get

$$\begin{aligned}|I_4^i| &\le C\|(\varphi^i)^\theta\partial_y\omega^i_{bc,T}\|_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}\|(\varphi^i)^\theta\omega^i_{bc,T}\|_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}\\ &\le \frac{1}{10}\|(\varphi^i)^\theta\partial_y\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}} + C\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}.\end{aligned}$$

Estimate of I_5^i. By the fact

$$\partial_y\big((\varphi^i)^{2\theta}\big) = 2\theta(\varphi^i)^{2\theta-1},$$

we have

$$\begin{aligned}|I_5^i| &\le C\theta\|(\varphi^i)^\theta\partial_y\omega^i_{bc,T}\|_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}\|(\varphi^i)^{\theta-1}\omega^i_{bc,T}\|_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}\\ &\le \frac{1}{10}\|(\varphi^i)^\theta\partial_y\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}} + C\theta^2\|(\varphi^i)^{\theta-1}\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}.\end{aligned}$$

Estimate of I_6^i. It follows from

$$\begin{aligned}&\|\langle D_x\rangle^{2+\frac{\theta}{3}}(\omega^i_{bc,T})_\Phi(\varphi^i)^{\theta-\frac{1}{2}}\|_{L^2_y(I_i;L^2_x)}\\ &\quad\le \|\langle D_x\rangle^{\frac{13}{6}+\frac{\theta}{3}}(\omega^i_{bc,T})_\Phi(\varphi^i)^\theta\|^{\frac{1}{2}}_{L^2_y(I_i;L^2_x)}\|\langle D_x\rangle^{\frac{11}{6}+\frac{\theta}{3}}(\omega^i_{bc,T})_\Phi(\varphi^i)^{\theta-1}\|^{\frac{1}{2}}_{L^2_y(I_i;L^2_x)}\\ &\quad\le \|(\varphi^i)^\theta\omega^i_{bc,T}\|^{\frac{1}{2}}_{X_i^{\frac{13}{6}+\frac{\theta}{3}}}\|(\varphi^i)^{\theta-1}\omega^i_{bc,T}\|^{\frac{1}{2}}_{X_i^{\frac{11}{6}+\frac{\theta}{3}}},\end{aligned}$$

for $\theta = 1{,}2$ and

$$\begin{aligned}\|\langle D_x\rangle^2(\omega^i_{bc,T})_\Phi(\varphi^i)^{-\frac{1}{2}}\|_{L^2_y(I_i;L^2_x)} &\le \|\langle D_x\rangle^{\frac{13}{6}}(\omega^i_{bc,T})_\Phi\|^{\frac{1}{2}}_{L^2_y(I_i;L^2_x)}\|\langle D_x\rangle^{\frac{11}{6}}(\omega^i_{bc,T})_\Phi(\varphi^i)^{-1}\|^{\frac{1}{2}}_{L^2_y(I_i;L^2_x)}\\ &\le C\|\omega^i_{bc,T}\|^{\frac{1}{2}}_{X_i^{\frac{13}{6}}}\|\partial_y\omega^i_{bc,T}\|^{\frac{1}{2}}_{X_i^{\frac{11}{6}}},\end{aligned}$$

by using Hardy inequality for $\theta = 0$. Therefore, we get for $\theta = 0{,}1{,}2$ that

$$\begin{aligned}|I_6^i| &\le C\|(\varphi^i)^{\frac{1}{2}+\theta}H^i\|_{X_i^{\frac{5}{3}+\frac{\theta}{3}}}\times\|(\varphi^i)^\theta\omega^i_{bc,T}\|^{\frac{1}{2}}_{X_i^{\frac{13}{6}+\frac{\theta}{3}}}\times\begin{cases}\|(\varphi^i)^{\theta-1}\omega^i_{bc,T}\|^{\frac{1}{2}}_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}, & \theta = 1{,}2,\\ \|\partial_y\omega^i_{bc,T}\|^{\frac{1}{2}}_{X_i^{\frac{11}{6}}}, & \theta = 0,\end{cases}\\ &\le \frac{1}{10}\|\partial_y\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}}} + \frac{\lambda}{4}\Big(\|(\varphi^i)^\theta\omega^i_{bc,T}\|^2_{X_i^{\frac{13}{6}+\frac{\theta}{3}}} + \frac{\theta^2}{2}\|(\varphi^i)^{\theta-1}\omega^i_{bc,T}\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}\Big) + \frac{C}{\lambda^{\frac{1}{2}}}\|(\varphi^i)^{\frac{1}{2}+\theta}H^i\|^2_{X_i^{\frac{5}{3}+\frac{\theta}{3}}}.\end{aligned}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

Putting $I_1^i - I_6^i$ together, we have

$$|I_1^i| + \cdots + |I_6^i| \le \frac{1}{2}\|(\varphi^i)^\theta \partial_y \omega_{bc,T}^i\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}} + (C + \frac{\lambda}{4})\|(\varphi^i)^\theta \omega_{bc,T}^i\|^2_{X_i^{\frac{13}{6}+\frac{\theta}{3}}} + \frac{\lambda}{8}\theta^2\|(\varphi^i)^{\theta-1}\omega_{bc,T}^i\|^2_{X_i^{\frac{11}{6}+\frac{\theta}{3}}}$$
$$+ \frac{C}{\lambda^{\frac{1}{2}}}\|(\varphi^i)^{\frac{1}{2}+\theta} H^i\|^2_{X_i^{\frac{5}{3}+\frac{\theta}{3}}}.$$

Then we insert them into (8.39) and take λ large enough to obtain

$$\begin{aligned}
\|(\varphi^i)^\theta \omega_{bc,T}^i(t)\|^2_{X^{\frac{11}{6}+\frac{\theta}{3}}} + \frac{3}{2}\lambda \int_0^t \|(\varphi^i)^\theta \omega_{bc,T}^i\|^2_{X^{\frac{13}{6}+\frac{\theta}{3}}}\,ds + \int_0^t \|(\varphi^i)^\theta \nabla_\varepsilon \omega_{bc,T}^i\|^2_{X^{\frac{11}{6}+\frac{\theta}{3}}}\,ds \\
\le \frac{C}{\lambda^{\frac{1}{2}}}\int_0^t \|(\varphi^i)^{\frac{1}{2}+\theta} H^i\|^2_{X^{\frac{5}{3}+\frac{\theta}{3}}} + \frac{\lambda}{8}\int_0^t \theta^2 \|(\varphi^i)^{\theta-1}\omega_{bc,T}^i\|^2_{X^{\frac{11}{6}+\frac{\theta}{3}}}\,ds \\
\le \frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |h^i|^2_{X^{\frac{7}{3}}}\,ds + \frac{\lambda}{8}\int_0^t \theta^2 \|(\varphi^i)^{\theta-1}\omega_{bc,T}^i\|^2_{X^{\frac{11}{6}+\frac{\theta}{3}}}\,ds,
\end{aligned}$$

where we use (8.37) in the last step.

All we have left to do is to control the last term of the above inequality. For that, we rewrite it as follows:

$$\begin{aligned}
\frac{\lambda}{8}\int_0^t \theta^2 \|(\varphi^i)^{\theta-1}\omega_{bc,T}^i\|^2_{X^{\frac{11}{6}+\frac{\theta}{3}}}\,ds &\le \frac{\lambda}{2}\sum_{\theta=1}^{2}\int_0^t \|(\varphi^i)^{\theta-1}\omega_{bc,T}^i\|^2_{X^{\frac{13}{6}+\frac{\theta-1}{3}}}\,ds \\
&= \frac{\lambda}{2}\sum_{\theta=0}^{1}\int_0^t \|(\varphi^i)^{\theta}\omega_{bc,T}^i\|^2_{X^{\frac{13}{6}+\frac{\theta}{3}}}\,ds.
\end{aligned}$$

Combing all the above estimates, we get the desired results. □

Based on estimates of $\omega_{bc,T}^i$, we use the elliptic equation to get the estimates of $\phi_{bc,T}^i$.

Corollary 8.8. *There exists $\lambda_0 > 1$ and $0 < T < \min\{T_p, \frac{1}{2\lambda}\}$ such that for all $t \in [0,T]$, $\lambda \ge \lambda_0$, it holds that*

$$\begin{aligned}
&\int_0^t \Big(\|\nabla_\varepsilon \phi_{bc,T}^i\|^2_{X^{\frac{7}{3}+\frac{1}{6}}} + |\partial_y \phi_{bc,T}^i|_{y=0,1}|^2_{X^{\frac{7}{3}}} + \|\partial_x \phi_{bc,T}^i\|^2_{X^{\frac{5}{3}}} + |\phi_{bc,T}^i|_{y=1-i}|^2_{X^{\frac{8}{3}}}\Big)\,ds \\
&\le \frac{C}{\lambda}\int_0^t |h^i|^2_{X^{\frac{7}{3}}}\,ds.
\end{aligned} \tag{8.41}$$

Proof. Here, we only give the proof of the case $i = 0$. The case $i = 1$ is the same.

We recall the elliptic equation

$$\Delta_\varepsilon \phi_{bc,T}^0 = \omega_{bc,T}^0, \quad \phi_{bc,T}^0|_{y=0} = 0, \tag{8.42}$$

for $y > 0$. Then we take the X_0^r inner product with $\phi_{bc,T}^0$ and use the Hardy inequality to get

$$\|\nabla_\varepsilon \phi_{bc,T}^0\|^2_{X_0^r} \le \|\varphi^0 \omega_{bc,T}^0\|_{X_0^r}\|\frac{\phi_{bc,T}^0}{\varphi^0}\|_{X_0^r} \le C\|\varphi^0\omega_{bc,T}^0\|_{X_0^r}\|\partial_y \phi_{bc,T}^0\|_{X_0^r},$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

which implies

$$\|\nabla_\varepsilon \phi^0_{bc,T}\|_{X^r_0} \le C\|\varphi^0\omega^0_{bc,T}\|_{X^r_0}, \tag{8.43}$$

for $r \ge 0$. By Proposition 8.7, we get

$$\int_0^t \|\nabla_\varepsilon \phi^0_{bc,T}\|^2_{X_0^{\frac{5}{2}}} ds \le \int_0^t \|\varphi^0\omega^0_{bc,T}\|^2_{X_0^{\frac{5}{2}}} ds \le \frac{C}{\lambda^{\frac{3}{2}}}\int_0^t |h^0|^2_{X^{\frac{7}{3}}} ds. \tag{8.44}$$

For the boundary term, using the interpolation inequality, we get

$$\begin{aligned}|\partial_y \phi^0_{bc,T}|_{y=0,1}|_{X^{\frac{7}{3}}} &\le C\|\partial_y\phi^0_{bc,T}\|^{\frac{1}{2}}_{X_0^{\frac{5}{2}}}\|\partial_y^2\phi^0_{bc,T}\|^{\frac{1}{2}}_{X_0^{\frac{13}{6}}}\\ &\le C\|\varphi^0\omega^0_{bc,T}\|^{\frac{1}{2}}_{X_0^{\frac{5}{2}}}\|\omega^0_{bc,T}\|^{\frac{1}{2}}_{X_0^{\frac{13}{6}}},\end{aligned}$$

where we use (8.43) and the Calderon-Zygmund inequality in the last step. Along with Proposition 8.7, we arrive at

$$\int_0^t |\partial_y \phi^i_{bc,T}|_{y=0,1}|^2_{X^{\frac{7}{3}}} ds \le \frac{C}{\lambda}\int_0^t |h^i|^2_{X^{\frac{7}{3}}} ds. \tag{8.45}$$

Next, we deal with the term $\|\partial_x\phi^0_{bc,T}\|_{X^{\frac{5}{3}}}$. Taking the Fourier transform in x to (8.42), we write the solution

$$\widehat{\phi}^0_{bc,T}(k,y) = \int_0^y e^{-\varepsilon|k|(y-y')}\int_{y'}^{+\infty} e^{-\varepsilon|k|(y''-y')}\widehat{\omega}^0_{bc,T}(k,y'')dy''dy'. \tag{8.46}$$

Then we have

$$|\widehat{\phi}^0_{bc,T}(k,y)| \le \int_0^y\int_{y'}^{+\infty}|\widehat{\omega}^0_{bc,T}(k,y'')|dy''dy'.$$

Decomposing the integral $\int_0^y$ into $\int_0^{\min\{y,\langle k\rangle^{-\frac{1}{3}}\}}$ and $\int_{\min\{y,\langle k\rangle^{-\frac{1}{3}}\}}^y$, it follows from the Hölder inequality that

$$\sup_{y\ge 0}|\widehat{\phi}^0_{bc,T}(k,y)| \le C\langle k\rangle^{-\frac{1}{6}}\|y\widehat{\omega}^0_{bc,T}\|_{L^2_y(I_0)} + C\langle k\rangle^{\frac{1}{6}}\|y^2\widehat{\omega}^0_{bc,T}\|_{L^2_y(I_0)}.$$

We take summation $\sum_{k\in\mathbb{Z}}$ and use the Plancherel theorem to deduce

$$\sup_{y\ge 0}\|\phi^0_{bc,T}(\cdot,y)\|_{L^2_x} \le C\|\langle D_x\rangle^{-\frac{1}{6}}y\omega^0_{bc,T}\|_{L^2_y(I_0;L^2_x)} + \|\langle D_x\rangle^{\frac{1}{6}}y^2\omega^0_{bc,T}\|_{L^2_y(I_0;L^2_x)}. \tag{8.47}$$

Thus, we get that

$$\begin{aligned}\int_0^t\Big(\|\partial_x\phi^i_{bc,T}\|^2_{X^{\frac{5}{3}}} + |\phi^i_{bc,T}|_{y=1-i}|^2_{X^{\frac{8}{3}}}\Big)ds &\le C\int_0^t \|\varphi^i\omega^i_{bc,T}\|^2_{X_i^{\frac{5}{2}}} ds + C\int_0^t \|(\varphi^i)^2\omega^i_{bc,T}\|^2_{X_i^{\frac{17}{6}}} ds\\ &\le \frac{C}{\lambda}\int_0^t |h^i|^2_{X^{\frac{7}{3}}} ds.\end{aligned} \tag{8.48}$$

Collecting (8.44), (8.45) and (8.48) together, we get the corollary proved. □

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

8.3. The estimates of $\phi_{bc,R}$: full construction of boundary corrector

All we have left is the term $\phi_{bc,R}$. Like the previous argument, we define

$$\phi_{bc,R} = \phi^0_{bc,R} + \phi^1_{bc,R},$$

where $\phi^i_{bc,R}$ satisfies that

$$\begin{cases} (\partial_t - \Delta_\varepsilon)\Delta_\varepsilon \phi^0_{bc,R} + u^p \partial_x \Delta_\varepsilon \phi^0_{bc,R} + v^p \partial_y \Delta_\varepsilon \phi^0_{bc,R} + \partial_y \phi^0_{bc,R} \partial_x \omega^p - \partial_x \phi^0_{bc,R} \partial_y \omega^p \\ \qquad = -\partial_y(\phi^0_{bc,S} + \phi^0_{bc,T})\partial_x \omega^p + \partial_x(\phi^0_{bc,S} + \phi^0_{bc,T})\partial_y \omega^p, \quad t>0,\ x\in\mathbb{T},\ y\in(0,1), \\ \qquad \stackrel{\text{def}}{=} G^0, \\ \phi^0_{bc,R}|_{y=0} = 0, \quad \phi^0_{bc,R}|_{y=1} = -(\phi^0_{bc,S} + \phi^0_{bc,T})|_{y=1}, \quad \Delta_\varepsilon \phi^0_{bc,R}|_{y=0,1} = 0, \\ \phi^0_{bc,R}|_{t=0} = 0. \end{cases} \tag{8.49}$$

and

$$\begin{cases} (\partial_t \Delta_\varepsilon)\Delta_\varepsilon \phi^1_{bc,R} + u^p \partial_x \Delta_\varepsilon \phi^1_{bc,R} + v^p \partial_y \Delta_\varepsilon \phi^1_{bc,R} + \partial_y \phi^1_{bc,R} \partial_x \omega^p - \partial_x \phi^1_{bc,R} \partial_y \omega^p \\ \qquad = -\partial_y(\phi^1_{bc,S} + \phi^1_{bc,T})\partial_x \omega^p + \partial_x(\phi^1_{bc,S} + \phi^1_{bc,T})\partial_y \omega^p, \quad t>0,\ x\in\mathbb{T},\ y\in(0,1), \\ \qquad \stackrel{\text{def}}{=} G^1, \\ \phi^1_{bc,R}|_{y=0} = -(\phi^1_{bc,S} + \phi^1_{bc,T})|_{y=0}, \quad \phi^1_{bc,R}|_{y=1} = 0, \quad \Delta_\varepsilon \phi^1_{bc,R}|_{y=0,1} = 0, \\ \phi^1_{bc,R}|_{t=0} = 0. \end{cases} \tag{8.50}$$

For simplicity, denote $\omega^i_{bc,R} = \Delta_\varepsilon \phi^i_{bc,R}$, which has the following relationship:

$$\begin{cases} \Delta_\varepsilon \phi^0_{bc,R} = \omega^0_{bc,R}, \\ \phi^0_{bc,R}|_{y=0} = 0, \quad \phi^0_{bc,R}|_{y=1} = f^0, \end{cases} \tag{8.51}$$

and

$$\begin{cases} \Delta_\varepsilon \phi^1_{bc,R} = \omega^1_{bc,R}, \\ \phi^1_{bc,R}|_{y=0} = f^1, \quad \phi^1_{bc,R}|_{y=1} = 0, \end{cases} \tag{8.52}$$

where

$$f^0 = f^0(t,x) = -(\phi^0_{bc,S} + \phi^0_{bc,T})|_{y=1}, \tag{8.53}$$

$$f^1 = f^1(t,x) = -(\phi^1_{bc,S} + \phi^1_{bc,R})|_{y=0}. \tag{8.54}$$

To homogenize boundary conditions, we introduce

$$\widetilde{\phi}^0_{bc,R} = \phi^0_{bc,R} + g^0, \quad g^0 = y(\phi^0_{bc,S} + \phi^0_{bc,T}), \tag{8.55}$$

where $\widetilde{\phi}^0_{bc,R}$ satisfies

$$\begin{cases} \Delta_\varepsilon \widetilde{\phi}^0_{bc,R} = \omega^0_{bc,R} + \Delta_\varepsilon g^0, \\ \widetilde{\phi}^0_{bc,R}|_{y=0,1} = 0, \end{cases} \tag{8.56}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

where

$$\Delta_\varepsilon g^0 = y(\Delta_\varepsilon \phi^0_{bc,S} + \Delta_\varepsilon \phi^0_{bc,T}) + 2(\partial_y \phi^0_{bc,S} + \partial_y \phi^0_{bc,T}). \tag{8.57}$$

Similarly, we introduce

$$\widetilde{\phi}^1_{bc,R} = \phi^1_{bc,R} + g^1, \quad g^1 = (1-y)(\phi^1_{bc,S} + \phi^1_{bc,T}), \tag{8.58}$$

and $\widetilde{\phi}^1_2$ satisfies

$$\begin{cases} \Delta_\varepsilon \widetilde{\phi}^1_{bc,R} = \omega^1_{bc,R} + \Delta_\varepsilon g^1, \\ \widetilde{\phi}^1_{bc,R}|_{y=0,1} = 0, \end{cases} \tag{8.59}$$

where

$$\Delta_\varepsilon g^1 = (1-y)(\Delta_\varepsilon \phi^1_{bc,S} + \Delta_\varepsilon \phi^1_{bc,T}) - 2(\partial_y \phi^1_{bc,S} + \partial_y \phi^1_{bc,T}). \tag{8.60}$$

First, we give some elliptic estimates.

Lemma 8.9. *Let (f^0, f^1), (g^0, g^1) – introduced in (8.53)–(8.54), (8.55) and (8.58). It holds that*

$$\int_0^t \left(|f^i|^2_{X^{\frac{8}{3}}} + \|\nabla_\varepsilon g^i\|^2_{X^{\frac{5}{2}}} + \|\Delta_\varepsilon g^i\|^2_{X^{\frac{5}{2}}} \right) ds \le \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^i|^2_{X^{\frac{7}{3}}} ds \tag{8.61}$$

for $i = 0,1$.

Moreover, $\phi^i_{bc,R}$ $(i = 0,1)$ has the following estimates:

$$\int_0^t \|\nabla_\varepsilon \phi^i_{bc,R}\|^2_{X^{\frac{7}{3}}} ds \le \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^i|^2_{X^{\frac{7}{3}}} ds + C \int_0^t \|\omega^i_{bc,R}\|^2_{X^{\frac{7}{3}}} ds. \tag{8.62}$$

Proof. Here we only prove the case $i = 0$. The case $i = 1$ is almost the same, and we omit details to readers. We first give the proof for f^0. By the definition of f^0, we get

$$\int_0^t |f^0|^2_{X^{\frac{8}{3}}} ds \le \int_0^t |\phi^0_{bc,S}|_{y=1}|^2_{X^{\frac{8}{3}}} ds + \int_0^t |\phi^0_{bc,T}|_{y=1}|^2_{X^{\frac{8}{3}}} ds \le \frac{C}{\lambda} \int_0^t |h^0|^2_{X^{\frac{7}{3}}} ds,$$

where we used Lemma 8.4 and Corollary 8.8.

For g^0, by Corollary 8.8, Proposition 8.3, we have

$$\begin{aligned} \int_0^t \|\nabla_\varepsilon g^0\|^2_{X^{\frac{5}{2}}} ds &\le \int_0^t \left(\|\nabla_\varepsilon \phi^0_{bc,S}\|_{X^{\frac{5}{2}}} + \|\nabla_\varepsilon \phi^0_{bc,T}\|_{X^{\frac{5}{2}}} + \|\phi^0_{bc,S}\|_{X^{\frac{8}{3}}} + \|\phi^0_{bc,T}\|_{X^{\frac{8}{3}}} \right) ds \\ &\le \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^0|^2_{X^{\frac{7}{3}}} ds. \end{aligned}$$

On one hand, using Proposition 8.3, Proposition 8.5, Corollary 8.8 and Proposition 8.7, we get

$$\int_0^t \left(\|y\Delta_\varepsilon \phi^0_{bc,S}\|^2_{X^{\frac{5}{2}}} + \|\partial_y \phi^0_{bc,S}\|^2_{X^{\frac{5}{2}}} + \|y\Delta_\varepsilon \phi^0_{bc,T}\|^2_{X^{\frac{5}{2}}} + \|\partial_y \phi^0_{bc,T}\|^2_{X^{\frac{5}{2}}} \right) ds \le \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^0|^2_{X^{\frac{7}{3}}} ds,$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

which implies that

$$\int_0^t \|\Delta_\varepsilon g^0\|^2_{X^{\frac{5}{2}}} ds \le \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^0|^2_{X^{\frac{7}{3}}} ds.$$

At last, we prove (8.62). Taking the $X^{\frac{7}{3}}$ inner product with $\widetilde{\phi}^0_{bc,R}$ toward (8.56), we use integration by parts and then integrate time from 0 to t that

$$\int_0^t \|\nabla_\varepsilon \widetilde{\phi}^0_{bc,R}\|^2_{X^{\frac{7}{3}}} ds = -\int_0^t \langle \omega^0_{bc,R}, \widetilde{\phi}^0_{bc,R}\rangle_{X^{\frac{7}{3}}} ds + \int_0^t \langle \Delta_\varepsilon g^0, \widetilde{\phi}^0_{bc,R}\rangle_{X^{\frac{7}{3}}} ds.$$

Due to $\widetilde{\phi}^0_{bc,R}|_{y=0,1} = 0$, we use the Poincaré inequality to imply

$$\int_0^t \langle \omega^0_{bc,R}, \widetilde{\phi}^0_{bc,R}\rangle_{X^{\frac{7}{3}}} ds \le \frac{1}{10} \int_0^t \|\partial_y \widetilde{\phi}^0_{bc,R}\|_{X^{\frac{7}{3}}} ds + C\int_0^t \|\omega^0_{bc,R}\|^2_{X^{\frac{7}{3}}} ds. \tag{8.63}$$

According to (8.61), we get

$$\begin{aligned}\int_0^t \langle \Delta_\varepsilon g^0, \widetilde{\phi}^0_{bc,R}\rangle_{X^{\frac{7}{3}}} ds &\le \int_0^t \|\Delta_\varepsilon g^0\|_{X^{\frac{7}{3}}} \|\widetilde{\phi}^0_{bc,R}\|_{X^{\frac{7}{3}}} ds \\ &\le \frac{1}{10}\int_0^t \|\partial_y \widetilde{\phi}^0_{bc,R}\|^2_{X^{\frac{7}{3}}} ds + \frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |h^0|^2_{X^{\frac{7}{3}}} ds.\end{aligned} \tag{8.64}$$

Combining (8.63) and (8.64), we deduce

$$\int_0^t \|\nabla_\varepsilon \widetilde{\phi}^0_{bc,R}\|^2_{X^{\frac{7}{3}}} ds \le \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^0|^2_{X^{\frac{7}{3}}} ds + C\int_0^t \|\omega^0_{bc,R}\|^2_{X^{\frac{7}{3}}} ds.$$

Bringing $\nabla_\varepsilon \phi^0_{bc,R} = \nabla_\varepsilon \widetilde{\phi}^0_{bc,R} - \nabla_\varepsilon g^0$ into the above inequality, we obtain

$$\begin{aligned}\int_0^t \|\nabla_\varepsilon \phi^0_{bc,R}\|^2_{X^{\frac{7}{3}}} ds &\le \int_0^t \|\nabla_\varepsilon \widetilde{\phi}^0_{bc,R}\|^2_{X^{\frac{7}{3}}} ds + \int_0^t \|\nabla_\varepsilon g^0\|^2_{X^{\frac{7}{3}}} ds \\ &\le \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^0|^2_{X^{\frac{7}{3}}} ds + C\int_0^t \|\omega^0_{bc,R}\|^2_{X^{\frac{7}{3}}} ds.\end{aligned}$$

By now, we finish the proof. □

In order to estimate the right-hand side of (8.49) and (8.50) and the boundary term, we need the following results:

Lemma 8.10. *For $i = 0,1$, we have that*

$$\int_0^t \left(\|\partial_x(\phi^i_{bc,S} + \phi^i_{bc,T})\|^2_{X^{\frac{5}{3}}} + \|\partial_y(\phi^i_{bc,S} + \phi^i_{bc,T})\|^2_{X^{\frac{5}{3}}}\right) ds \le \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^i|^2_{X^{\frac{7}{3}}} ds, \tag{8.65}$$

$$\int_0^t |\partial_y \phi^i_{bc,R}|_{y=0,1}|^2_{X^{\frac{7}{3}}} ds \le C\int_0^t \|\omega^i_{bc,R}\|^2_{X^{\frac{7}{3}}} ds + \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^i|^2_{X^{\frac{7}{3}}} ds, \tag{8.66}$$

$$\int_0^t \left\langle \partial_x(\phi^i_{bc,R})_\Phi, \partial_y(\phi^i_{bc,R})_\Phi \right\rangle_{H^2_x}\Big|_{y=0}^{y=1} ds \le C\int_0^t \|\omega^i_{bc,R}\|^2_{X^{\frac{7}{3}}} ds + \frac{C}{\lambda^{\frac{1}{2}}} \int_0^t |h^i|^2_{X^{\frac{7}{3}}} ds. \tag{8.67}$$

Proof. Here, we only prove the case $i=0$. The case $i=1$ is almost the same, and we omit details to readers.

By Proposition 8.3 and Corollary 8.8, we get (8.65) proved.

Next, we deal with the boundary term. A direct calculation gives that

$$\begin{aligned}\partial_y\phi^0_{bc,R}|_{y=1} &= (\partial_y\widetilde{\phi}^0_{bc,R}-\partial_y g^0)|_{y=1}\\ &= \partial_y\widetilde{\phi}^0_{bc,R}|_{y=1}-(\partial_y\phi^0_{bc,S}+\partial_y\phi^0_{bc,T})|_{y=1}-(\phi^0_{bc,S}+\phi^0_{bc,T})|_{y=1},\end{aligned}$$

and

$$\partial_y\phi^0_{bc,R}|_{y=0}=(\partial_y\widetilde{\phi}^0_{bc,R}-\partial_y g^0)|_{y=0}=\partial_y\widetilde{\phi}^0_{bc,R}|_{y=0},$$

due to $\phi^0_{bc,S}|_{y=0}=\phi^0_{bc,T}|_{y=0}=0$.

By Corollary 8.8, we get

$$\begin{aligned}\int_0^t|\partial_y\widetilde{\phi}^0_{bc,R}|_{y=0,1}|^2_{X^{\frac{7}{3}}}ds &\le \frac{C}{\lambda^{\frac12}}\int_0^t|h^0|^2_{X^{\frac{7}{3}}}ds+C\int_0^t(\|\omega^0_{bc,R}\|^2_{X^{\frac{7}{3}}}+\|\Delta_\varepsilon g^0\|^2_{X^{\frac{7}{3}}})ds\\ &\le C\int_0^t\|\omega^0_{bc,R}\|^2_{X^{\frac{7}{3}}}ds+\frac{C}{\lambda^{\frac12}}\int_0^t|h^0|^2_{X^{\frac{7}{3}}}ds,\end{aligned}$$

where we use an elliptic estimate and the Calderon-Zygmund inequality

$$\|\partial_y\widetilde{\phi}^0_{bc,R}\|_{X^{\frac{7}{3}}}+\|\partial_y^2\widetilde{\phi}^0_{bc,R}\|_{X^{\frac{7}{3}}}\le C\|\omega^0_{bc,R}\|_{X^{\frac{7}{3}}}+C\|\Delta_\varepsilon g^0\|_{X^{\frac{7}{3}}}.$$

For the last estimate, we use (8.66) and (8.61) to imply

$$\begin{aligned}\int_0^t\Big\langle\partial_x(\phi^0_{bc,R})_\Phi,\partial_y(\phi^0_{bc,R})_\Phi\Big\rangle_{H^2_x}\Big|_{y=0}^{y=1}ds &\le C\int_0^t|(\phi^0_{bc,S}+\phi^0_{bc,T})|_{y=1}|_{X^{\frac{8}{3}}}|\partial_y\phi^0_{bc,R}|_{y=1}|_{X^{\frac{7}{3}}}ds\\ &\le C\int_0^t|f^0|_{X^{\frac{8}{3}}}|\partial_y\phi^0_{bc,R}|_{y=1}|_{X^{\frac{7}{3}}}ds\\ &\le C\int_0^t\|\omega^0_{bc,R}\|^2_{X^{\frac{7}{3}}}ds+\frac{C}{\lambda^{\frac12}}\int_0^t|h^0|^2_{X^{\frac{7}{3}}}ds.\end{aligned}$$

Here, we complete this lemma. □

We are coming to the main part of this section. We shall give the estimate for the system (8.49) and (8.50).

Proposition 8.11. *Let $\phi^0_{bc,R}$ and $\phi^1_{bc,R}$ be the solution of (8.49) and (8.50), respectively, and $\omega^i_{bc,R}=\Delta_\varepsilon\phi^i_{bc,R}$ for $i=0,1$. Then, for every $i=0,1$, it holds that*

$$\begin{aligned}\|\omega^i_{bc,R}(t)\|^2_{X^2}+\lambda\int_0^t\|\omega^i_{bc,R}\|^2_{X^{\frac{7}{3}}}ds+\int_0^t(\|\nabla_\varepsilon\phi^i_{bc,R}\|^2_{X^{\frac{7}{3}}}+|\partial_y\phi^i_{bc,R}|_{y=0,1}|^2_{X^{\frac{7}{3}}})ds+\int_0^t\|\nabla_\varepsilon\omega^i_{bc,R}\|^2_{X^2}ds\\ \le\frac{C}{\lambda^{\frac12}}\int_0^t|h^i|^2_{X^{\frac{7}{3}}}ds,\qquad t\in[0,T],\end{aligned}$$

where $0<T<\min\{T_p,\frac{1}{2\lambda}\}$.

Proof. The result mainly comes from the process of Proposition 7.1. Here we take $(\mathcal{N}_u,\varepsilon\mathcal{N}_v)=0$ and $\varepsilon^2 f_1+f_2-C(t)\partial_x\omega^p$ is replaced by $G^i=-\partial_y(\phi^i_{bc,S}+\phi^i_{bc,T})\partial_x\omega^p+$

$\partial_x(\phi^i_{bc,S}+\phi^i_{bc,T})\partial_y\omega^p$ for $i=0,1$. In order to estimate the source term $\int_0^t \|G^i\|^2_{X^{\frac{5}{3}}}ds$, using Lemma 8.10 and product estimate in Lemma 2.2, we get

$$\begin{aligned}\int_0^t \|G^i\|^2_{X^{\frac{5}{3}}}ds &\le C\int_0^t \|\partial_y(\phi^i_{bc,S}+\phi^i_{bc,T})\|^2_{X^{\frac{5}{3}}}ds + C\int_0^t \|\partial_x(\phi^i_{bc,S}+\phi^i_{bc,T})\|^2_{X^{\frac{5}{3}}}ds\\ &\le \frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |h^i|^2_{X^{\frac{7}{3}}}ds.\end{aligned}$$

The only difference comes from the boundary conditions

$$\phi^i_{bc,R}|_{y=i}=0,\quad \phi^i_{bc,R}|_{y=1-i}=-(\phi^i_{bc,S}+\phi^i_{bc,T})|_{y=1-i},$$

which are not zero compared with equation (6.2). We review T^5 in Propoposition 7.1. After integration by parts, the boundary term is left. More precisely, we need to estimate $\int_0^t \left\langle \partial_x(\phi^i_{bc,R})_\Phi,\partial_y(\phi^i_{bc,R})_\Phi\right\rangle_{H^2_x}\Big|_{y=0}^{y=1}ds$. According to Lemma 8.10, we have

$$\int_0^t \left\langle \partial_x(\phi^i_{bc,R})_\Phi,\partial_y(\phi^i_{bc,R})_\Phi\right\rangle_{H^2_x}\Big|_{y=0}^{y=1}ds \le C\int_0^t \|\omega^i_{bc,R}\|^2_{X^{\frac{7}{3}}}ds + \frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |h^i|^2_{X^{\frac{7}{3}}}ds.$$

Here, we take λ large enough to complete the proof. □

8.4. Proof of Proposition 6.2

In this subsection, we combine all the above estimates to finish the proof of Proposition 6.2. Recalling the definition of ϕ_{bc}:

$$\phi_{bc}=\phi_{bc,S}+\phi_{bc,T}+\phi_{bc,R}, \tag{8.68}$$

we get that

$$\begin{cases}(\partial_t-\Delta_\varepsilon)\Delta_\varepsilon\phi_{bc}+u^p\partial_x\Delta_\varepsilon\phi_{bc}+v^p\partial_y\Delta_\varepsilon\phi_{bc}+\partial_y\phi_{bc}\partial_x\omega^p-\partial_x\phi_{bc}\partial_y\omega^p=0,\\ \phi_{bc}|_{y=0,1}=0,\quad \partial_y\phi_{bc}|_{y=0}=h^0+R^{00}_{bc}+R^{01}_{bc},\quad \partial_y\phi_{bc}|_{y=1}=h^1+R^{10}_{bc}+R^{11}_{bc},\\ \phi_{bc}|_{t=0}=0.\end{cases} \tag{8.69}$$

Here, R^{ji}_{bc} $(j=0,1,\ i=0,1)$ are linear operators and are defined by

$$\begin{aligned}R^{00}_{bc}&=\big(\partial_y\phi^0_{bc,T}+\partial_y\phi^0_{bc,R}\big)|_{y=0},\\ R^{01}_{bc}&=\big(\partial_y\phi^1_{bc,S}+\partial_y\phi^1_{bc,T}+\partial_y\phi^1_{bc,R}\big)|_{y=0},\\ R^{10}_{bc}&=\big(\partial_y\phi^0_{bc,S}+\partial_y\phi^0_{bc,T}+\partial_y\phi^0_{bc,R}\big)|_{y=1},\\ R^{11}_{bc}&=\big(\partial_y\phi^1_{bc,T}+\partial_y\phi^1_{bc,R}\big)|_{y=1}.\end{aligned}$$

Compared with the system (8.1), we need to find (h^0,h^1) such that

$$\begin{cases}h^0+R^{00}_{bc}+R^{01}_{bc}=-\partial_y\phi_{slip}|_{y=0}+C(t),\\ h^1+R^{10}_{bc}+R^{01}_{bc}=-\partial_y\phi_{slip}|_{y=1}+C(t)\end{cases} \tag{8.70}$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

hold. To do that, we define an operator $R_{bc}[h^0,h^1]$, which is defined by

$$R_{bc}[h^0,h^1]=\begin{pmatrix} R_{bc}^{00} & R_{bc}^{01} \\ R_{bc}^{10} & R_{bc}^{11} \end{pmatrix}, \tag{8.71}$$

which is a 2×2 matrix operator and is well-defined on the Banach space

$$Z_{bc}=\{(h^0,h^1)\in L^2(0,t;L^2)|\int_0^t |(h^0,h^1)|^2_{X^{\frac{7}{3}}}ds<+\infty\}. \tag{8.72}$$

Proposition 8.12. *There exists $\lambda_0\geq 1$ such that if $\lambda\geq\lambda_0$, the map $R_{bc}:Z_{bc}\to Z_{bc}$ defined by (8.71) satisfies*

$$\int_0^t \left|R_{bc}[h^0,h^1]\right|^2_{X^{\frac{7}{3}}}ds\leq\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |(h^0,h^1)|^2_{X^{\frac{7}{3}}}ds. \tag{8.73}$$

Hence, the operator $I+R_{bc}$ is invertible in Z_{bc}. Moreover, there exists $(h_0,h_1)\in Z_{bc}$ such that (8.70) holds and (h_0,h_1) is defined by

$$(h_0,h_1)=(I+R_{bc})^{-1}(-\partial_y\phi_{slip}|_{y=0}+C(t),-\partial_y\phi_{slip}|_{y=1}+C(t)).$$

Proof. First, by Lemma 8.4, Proposition 8.8, Corollary 8.7 and Proposition 8.11, it is easy to get

$$\int_0^t \left|R_{bc}[h^0,h^1]\right|^2_{X^{\frac{7}{3}}}ds\leq\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |(h^0,h^1)|^2_{X^{\frac{7}{3}}}ds.$$

Taking λ large enough, we get that the operator $I+R_{bc}$ is invertible in Z_{bc}. Thus, there exists $(h_0,h_1)\in Z_{bc}$ such that (8.70) holds. □

Let's continue to prove Proposition 6.2. According to Proposition 8.3, Proposition 8.6, Corollary 8.8 and Proposition 8.11, we get by (8.68) that

$$\begin{aligned}\int_0^t \|\nabla_\varepsilon\phi_{bc}\|^2_{X^{\frac{7}{3}}}ds&\leq\int_0^t \|\nabla_\varepsilon\phi_{bc,S}\|^2_{X^{\frac{5}{2}}}ds+\int_0^t \|\nabla_\varepsilon\phi_{bc,T}\|^2_{X^{\frac{5}{2}}}ds+\int_0^t \|\nabla_\varepsilon\phi_{bc,R}\|^2_{X^{\frac{7}{3}}}ds\\ &\leq\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |(h^0,h^1)|^2_{X^{\frac{7}{3}}}ds,\end{aligned}$$

and

$$\begin{aligned}\int_0^t \|\varphi\Delta_\varepsilon\phi_{bc}\|^2_{X^2}ds&\leq\int_0^t \|\varphi\Delta_\varepsilon\phi_{bc,S}\|^2_{X^{\frac{5}{2}}}ds+\int_0^t \|\varphi\Delta_\varepsilon\phi_{bc,T}\|^2_{X^{\frac{5}{2}}}ds+\int_0^t \|\Delta_\varepsilon\phi_{bc,R}\|^2_{X^{\frac{7}{3}}}ds\\ &\leq\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |(h^0,h^1)|^2_{X^{\frac{7}{3}}}ds,\end{aligned}$$

which imply

$$\int_0^t \|\nabla_\varepsilon\phi_{bc}\|^2_{X^{\frac{7}{3}}}+\|\varphi\Delta_\varepsilon\phi_{bc}\|^2_{X^2}ds\leq\frac{C}{\lambda^{\frac{1}{2}}}\int_0^t |(h^0,h^1)|^2_{X^{\frac{7}{3}}}ds.$$

https://doi.org/10.1017/S1474748023000282 Published online by Cambridge University Press

Due to Proposition 8.12 and taking $\mathcal{A} = (I + R_{bc})^{-1}$, we know $\mathcal{A}$ is a zero-order bounded operator in Z_{bc} and obtain

$$\begin{aligned}\int_0^t |(h^0,h^1)|^2_{X^{\frac{7}{3}}}\,ds &= \int_0^t |\mathcal{A}(-\partial_y\phi_{slip}|_{y=0} + C(s), -\partial_y\phi_{slip}|_{y=1} + C(s))|^2_{X^{\frac{7}{3}}}\,ds\\ &\le C\int_0^t \Big(|\nabla_\varepsilon\phi_{slip}|_{y=0,1}|^2_{X^{\frac{7}{3}}} + |C(s)|^2\Big)ds,\end{aligned}$$

which finishes this proposition.

Acknowledgements. The authors would like to thank Professor Zhifei Zhang for the valuable discussions and suggestions. C. Wang is partially supported by the NSF of China under Grant 12071008. Y. Wang is partially supported by NSF of China under Grant 12101431 and the Sichuan Youth Science and Technology Foundation No. 21CXTD0076.

Competing Interests. The author declares none.

References

[1] R. Alexandre, Y. Wang, C.-J. Xu and T. Yang, Well-posedness of the Prandtl Equation in Sobolev Spaces, *J. Amer. Math. Soc.* **28** (2015), 745–784.

[2] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations* (Grundlehren der mathematischen Wissenschaften) vol. 343 (Springer-Verlag Berlin Heidelberg, 2011).

[3] Y. Brenier, Homogeneous hydrostatic flows with convex velocity profiles, *Nonlinearity* **12**(3) (1999), 495–512.

[4] Y. Brenier, Remarks on the derivation of the hydrostatic Euler equations, *Bull. Sci. Math.* **127**(7) (2003), 585–595.

[5] C. Cao, S. Ibrahim, K. Nakanishi and E. Titi, Finite-time blowup for the inviscid Primitive equations of oceanic and atmospheric dynamics, *Comm. Math. Phys.* **337**(2) (2015), 473–482.

[6] D. Chen, Y. Wang and Z. Zhang, Well-posedness of the linearized Prandtl equation around a nonmonotonic shear flow, *Ann. Inst. H. Poincare Anal. Non Lineaire* **35** (2018), 1119–1142.

[7] Q. Chen, D. Wu and Z. Zhang, On the L^∞ stability of Prandtl expansions in Gevrey class, *Sci. China Math.* **65** (2022), 2521—2562.

[8] H. Dietert and Gérard-Varet, Well-posedness of the Prandtl equation without any structural assumption, Ann. PDE **5** (2019), 1–51.

[9] M. Fei, T. Tao and Z. Zhang, On the zero-viscosity limit of the Navier-Stokes equations in R3+ without analyticity, J. Math. Pures Appl. **112**(9) (2018), 170–229.

[10] D. Gérard-Varet and E. Dormy, On the ill-posedness of the Prandtl equation, *J. Amer. Math. Soc.* **23** (2010), 591–609.

[11] D. Gérard-Varet, S. Iyer and Y. Maekawa, Improved well-posedness for the Triple-Deck and related models via concavity, Preprint, arXiv:2205.15829.

[12] D. Gérard-Varet, Y. Maekawa and N. Masmoudi, Gevrey stability of Prandtl expansions for 2-dimensional Navier-Stokes flows, *Duke Math. J.* **167** (2018), 2531–2631.

[13] D. Gérard-Varet, Y. Maekawa and N. Masmoudi, Optimal Prandtl expansion around concave boundary layer, *Preprint*, arXiv:2005.05022.

[14] D. Gérard-Varet and N. Masmoudi, Well-posedness for the Prandtl system without analyticity or monotonicity, *Ann. Sci. Ec. Norm. Super.* **48** (2015), 1273–1325.

[15] D. Gérard-Varet, N. Masmoudi and V. Vicol, Well-posedness of the hydrostatic Navier-Stokes equations, *Anal. PDE* **13** (2020), 1417–1455.

[16] E. Grenier, *On the derivation of homogeneous hydrostatic equations,. M2AN Math Model. Numer. Anal.* **33**(5) (1999), 965–970.

[17] E. Grenier, Y. Guo and T. Nguyen, Spectral instability of general symmetric shear flows in a two-dimensional channel, *Adv. Math.* **292** (2016), 52–110.

[18] E. Grenier and T. Nguyen, L^∞ instability of Prandtl layers, *Ann. PDE* **5** (2019), 1–36.

[19] I. Kukavica, N. Masmoudi, V. Vicol and T. Wong, On the local well-posedness of the Prandtl and the hydrostatic Euler equations with multiple monotonicity regions, *SIAM J. Math. Anal.* **46**(6) (2014), 3865–3890.

[20] I. Kukavica, R. Temam, V. Vicol and M. Ziane, Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain, *J. Differential Equations* **250**(3) (2011), 1719–1746.

[21] I. Kukavica, V. Vicol and F. Wang, The inviscid limit for the Navier-Stokes equations with data analytic only near the boundary. *Arch. Ration. Mech. Anal.* **237** (2020), 779–827.

[22] P.-Y. Lagrée and S. Lorthois, The RNS/Prandtl equations and their link with other asymptotic descriptions: application to the wall shear stress scaling in a constricted pipe, *Int. J. Eng. Sci.* **43** (2005), 352–378.

[23] W. Li and T. Yang, Well-posedness in Gevrey space for the Prandtl equations with non-degenerate critical points, *J. Eur. Math. Soc.* **22** (2020), 717–775.

[24] M. C. Lombardo, M. Cannone and M. Sammartino, Well-posedness of the boundary layer equations, *SIAM J. Math. Anal.* **35** (2003), 987–1004.

[25] Y. Maekawa, On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane, *Comm. Pure Appl. Math.* **67** (2014), 1045–1128.

[26] N. Masmoudi and T. Wong, On the H^s theory of hydrostatic Euler equations, *Arch. Ration. Mech. Anal.* **204**(1) (2012), 231–271.

[27] N. Masmoudi and T. Wong, Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods, *Comm. Pure Appl. Math.* **68** (2015), 1683–1741.

[28] T. Nguyen and T. Nguyen, The inviscid limit of Navier-Stokes equations for analytic data on the halfspace, *Arch. Ration. Mech. Anal.* **230** (2018), 1103–1129.

[29] O. Oleinik, On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid, *J. Appl. Math. Mech.* **30** (1966), 951–974.

[30] M. Paicu, P. Zhang and Z. Zhang, On the hydrostatic approximate of the Navier-Stokes equations in a thin strip, *Adv. Math.* **372** (2020), 1–42.

[31] M. Renardy, Ill-posedness of the hydrostatic Euler and Navier-Stokes equations, *Arch. Ration. Mech. Anal.* **194** (2009), 877–886.

[32] M. Sammartino and R. E. Caflisch, Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, *Comm. Math. Phys.* **192** (1998), 433–461.

[33] M. Sammartino and R. E. Caflisch, Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution, *Comm. Math. Phys.* **192** (1998), 463–491.

[34] C. Wang, Y. Wang and Z. Zhang, Zero-viscosity limit of the Navier-Stokes equations in the analytic setting, *Arch. Ration. Mech. Anal.* **224** (2017), 555–595.

[35] C. Wang, Y. Wang and Z. Zhang, Gevrey stability of hydrostatic approximate for the Navier-Stokes equations in a thin domain, *Nonlinearity* **34** (2021), 7185–7226.

[36] L. Wang, Z. Xin and A. Zang, Vanishing viscous limits for 3D Navier-Stokes equations with a Navier slip boundary condition, *J. Math. Fluid Mech.* **14** (2012), 791–825.

[37] X. Wang, A Kato type theorem on zero viscosity limit of Navier-Stokes flows, *Indiana Univ. Math. J.* **50** (2001), 223–241.

[38] T. Wong, Blowup of solutions of the hydrostatic Euler equations, *Proc. Amer. Math. Soc.* **143**(3) (2015), 1119–1125.

[39] Y. Xiao and Z. Xin, On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition, *Comm. Pure Appl. Math.* **60** (2007), 1027–1055.

[40] Z. Xin and L. Zhang, On the global existence of solutions to the Prandtl system, *Adv. Math.* **181** (2004), 88–133.