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# Perverse Sheaves on Image Multiple Point Spaces

KEVIN HOUSTON
*School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K.. e-mail: khouston@amsta.leeds.ac.uk*

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**Abstract.** Using multiple point spaces some new examples of perverse sheaves on images of maps are described. Furthermore, suppose $f : X \to Y$ is a finite and proper map of complex analytic manifolds of dimension $n$ and $n+1$ such that every multiple point space is nonsingular and has the dimension expected of a generic map. Then we can describe the composition series for the constant sheaf on the image in the category of perverse sheaves.

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**Key words:** intersection cohomology, composition series, multiple point set.

## 1. Introduction

Suppose $f : X \to Y$ is a finite and proper complex analytic map, (here finite means that each fibre is a finite set). The closure of the set of points in the images with $k$ or more preimages is denoted $M_k$ and is called the $k$th multiple point set in the image (also called an image multiple point set).

In [7] it was shown that in certain cases the constant sheaf of rational numbers on $M_k$ was perverse, even though the spaces were not (obviously) complete intersections. (Recall that if $Z$ is a complex analytic space which is locally a complete intersection then it is well known that the rational constant sheaf, shifted by the dimension of $Z$, is a perverse sheaf. That is, the sheaf complex, defined by the constant sheaf in degree $-\dim_{\mathbb{C}} Z$ and zero elsewhere, denoted $\mathbb{Q}^{\bullet}_{Z}[\dim_{\mathbb{C}} Z]$ is a perverse sheaf).

In this paper (under some not too strict conditions on $f$) we find some nonconstant perverse sheaves on $M_k$. In certain good cases, these perverse sheaves are simple in the perverse category, i.e. they are quasi-isomorphic to twisted intersection cohomology sheaves. This has implications for the case in which the image of $f$ is a hypersurface. In this case the constant sheaf on the image is perverse and hence has a decomposition series in the perverse category such that the subquotients are twisted intersection cohomology sheaves. Using a resolution of the constant sheaf (described in [3]) we are able to define a filtration of this sheaf so that subquotients are precisely the sheaves on $M_k$ that we have described. More precisely, we have the following theorem:

THEOREM 1.1. *Suppose $f: X \to Y$ is a complex analytic map between complex manifolds of dimension $n$ and $n+1$. Suppose that the multiple point spaces $D^k(f)$ (see Section 2.1) are nonsingular and of dimension $n-k+1$. Then the perverse sheaf $\mathbb{Q}^\bullet_{f(X)}$ has a composition series constructed from the simple sheaves*

$$\mathbf{IC}^\bullet_{M_k^\alpha}(\mathcal{L}_k)[n-k+1],$$

*where $M_k^\alpha$ is an irreducible component of $M_k$ and $\mathcal{L}_k$ is a certain irreducible local system arising from the action of $S_k$ on $D^k$.*

We can apply this theorem to a number of examples. For instance, discriminants of stable corank 1 map-germs $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+r}, 0)$ where $r = 0$ or 1 and the hypersurface formed in the source by the double points of such maps are examples. Another example is the arrangement of complex hyperplanes in general position.

## 2. Perverse Sheaves on Multiple Point Spaces

### 2.1. NOTATION AND DEFINITIONS

Suppose $f: X \to Y$ is finite complex analytic map, (many of the following definitions will work for finite and continuous maps but we shall only concern ourselves with the complex analytic case).

DEFINITION 2.1. For $k \geqslant 1$ define the $k$th multiple point space of $f$, $D^k(f)$, to be

$$D^k(f) := \text{closure}\{(x_1, \dots x_k) \in X^k \mid f(x_1) = \dots = f(x_k) \text{ for } x_i \neq x_j, i \neq j\}.$$

DEFINITION 2.2. We say a map $f$ is dimensionally correct if when $D^k(f) \neq \emptyset$ we have $\dim_{\mathbb{C}} D^k(f) = nk - p(k-1)$.

Basically the dimensions of the multiple point spaces are what one would expect from a generic map. The condition is introduced so that we can say something about the vanishing of high cohomology groups of the multiple point spaces of a map.

DEFINITION 2.3. The diagonal of $D^k(f)$, denoted $\text{Diag}(D^k(f))$, is defined to be the subset of $D^k(f)$ for which $x_i = x_j$ for some $i$ and $j$ such that $i \neq j$.

The group of permutations on $k$ objects, $S_k$, act on $D^k(f)$ through permutation of the copies of $X$ in $X^k$.

DEFINITION 2.4. Let $D^k_\alpha(f)$ denote the orbit of a connected component of $D^k(f)$.

Thus we can see that $D^k(f) = \amalg_\alpha D^k_\alpha(f)$.

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There exist maps $\epsilon^k : D^k(f) \to Y$ given by $\epsilon^k(x_1, \dots, x_k) = f(x_i)$ for any $i$ and any $k \geqslant 1$. (Note that $\epsilon^1 = f$.) The restriction of this map to $D^k_\alpha(f)$ for some $\alpha$ is denoted $\epsilon^k_\alpha$.

DEFINITION 2.5. The $k$th multiple point space in the image, $M_k(f)$, is defined to be $\epsilon^k(D^k(f))$. It is the closure in the image of the set of points that have $k$ or more preimages.

There are also maps $\epsilon^{j,k}: D^k(f) \to D^{k-1}(f)$ given by

$$\epsilon^{j,k}(x_1, \dots, x_j, \dots, x_k) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots x_k)$$

for $1 \leqslant j \leqslant k$. That is, we omit one of the coordinates.

Since $f$ is a proper complex analytic map then $\epsilon^{j,k}$ and $\epsilon^k$ are as well and $D^k(f)$ is a complex analytic space. It is possible to Whitney stratify $D^k(f)$ such that every fixed point set under the action of $S_k$ forms a substratification of $D^k(f)$ and that the stratification is $S_k$-invariant. Through the maps $\epsilon^{j,k+1}$ it is possible to refine the stratification of $D^k(f)$ so that $\epsilon^{j,k+1}$ is a stratified submersion. By generalising this we can find a Whitney stratification of $M_k(f)$ such that $\epsilon^k$ is a stratified submersion, and such that the image of $\epsilon^j$ for $j > k$ is a substratification.

One of the keys to the study of images of maps through multiple point spaces is alternating homology and cohomology. We begin the definition of these by the following.

Let $M$ be a $\mathbb{Q}$-vector space upon which acts $S_k$ and give $S_k$ the usual sign representation. Then define the idempotent functor Alt to be given by

$$\operatorname{Alt} M = \{m \in M \mid \sigma(m) = \operatorname{sign}(\sigma)m \text{ for all } \sigma \in S_k\}.$$

Note that we drop the reference to $k$ since it will be obvious in the paper which $k$ is needed. This definition is the same as taking Alt as the functor

$$\operatorname{Alt} = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\sigma.$$

There will be various types of cohomology used in this paper. We will denote ordinary cohomology with $H$, hypercohomology with $\mathbb{H}$ and cohomology of a complex of sheaves by $\mathbf{H}$.

The $i$th alternating cohomology group of an $S_k$-invariant space $X$ is defined to be the group $\operatorname{Alt} H^i(X; \mathbb{Q})$. Local cohomology groups are groups of the form $H^i(X, X - \{x\}; \mathbb{Q})$, for some $x \in X$. There is a similar definition for alternating local cohomology groups.

The derived category of constructible sheaf complexes on a space $X$ will be denoted $D^b_c(X)$. The sheaves we deal with will be obviously constructible and so we make no further mention of constructibility. The sheaf $\mathbb{Q}^\bullet_X$ will denote the constant sheaf on $X$ in dimension zero and trivial elswhere. We denote $\mathbf{B}^\bullet$

quasi-isomorphic to $\mathbf{C}^\bullet$ by $\mathbf{B}^\bullet \cong \mathbf{C}^\bullet$. For any complex $\mathbf{C}^\bullet$ denote the Verdier dual of $\mathbf{C}^\bullet$ by $\mathcal{D}\mathbf{C}^\bullet$.

A good reference for many of the necessary definitions needed for perverse sheaves is [10].

### 2.2. A FILTRATION ON $\mathbb{Q}^\bullet_{f(X)}$

In this section we describe a resolution of the constant sheaf $\mathbb{Q}^\bullet_{f(X)}$, introduced in [3]. An obvious filtration on this resolution allows us to filter $\mathbb{Q}^\bullet_{f(X)}$ in $D^b_c(X)$ as we shall see shortly.

We now outline the facts from [3] that are need to describe this filtration. By pushing forward the constant sheaf $\mathbb{Q}_{D^k}$ on $D^k(f)$ by $\epsilon^k$ for each $k$, we obtain a (non-exact) sequence

$$0 \to \mathbb{Q}_{f(X)} \to \epsilon^1_*(\mathbb{Q}_X) \to \epsilon^2_*(\mathbb{Q}_{D^2}) \to \epsilon^3_*(\mathbb{Q}_{D^3}) \to \dots$$

where the differential $\delta_k : \epsilon^k_*(\mathbb{Q}_{D^k}) \to \epsilon^{k+1}_*(\mathbb{Q}_{D^{k+1}})$ is equal to

$$\sum_{j=1}^{k+1}(-1)^{k+j}(\epsilon^{j,k+1})^*.$$

There is an action of $S_k$ on $\epsilon^k_*(\mathbb{Q}_{D^k})$ given in the obvious way: it comes from the action on $\Gamma((\epsilon^k)^{-1}(U); \mathbb{Q}_{D^k})$ for $U$ an open subset in $Y$.

We can then form the sheaf complex $\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})$ by taking Alt of the stalks of $R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})$. (The right derived functor for Alt is equivalent to Alt since Alt is exact.) Note that $\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k}) \cong \oplus_\alpha \mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha})$.

In contrast to the earlier sequence the sequence

$$0 \to \mathbb{Q}_{f(X)} \to \epsilon^1_*(\mathbb{Q}_X) \to \mathbf{Alt}\epsilon^2_*(\mathbb{Q}_{D^2}) \to \mathbf{Alt}\epsilon^3_*(\mathbb{Q}_{D^3}) \to \dots$$

is exact. It is also proved in [3] that if $\mathbf{I}^\bullet_k$ is an injective resolution of $\mathbb{Q}_{D^k}$ then $\mathbf{Alt}\, R\epsilon^k_*(\mathbf{I}^\bullet_k)$ is an injective resolution of $\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})$.

The complex given by $\mathbf{Alt}\, R\epsilon^\bullet_*(\mathbb{Q}_{D^\bullet})$ is thus quasi-isomorphic to $\mathbb{Q}^\bullet_{f(X)}$ and it can obviously be filtered by $\mathbf{F}^\bullet_r$ with

$$F^p_r = \begin{cases} \mathbf{Alt}R\epsilon^{p+1}_*(\mathbb{Q}^\bullet_{D^{p+1}}) & \text{if } r \geqslant p \\ 0 & \text{if } r < p. \end{cases}$$

The filtration has successive quotients with

$$\mathbf{F}^\bullet_r/\mathbf{F}^\bullet_{r+1} \cong \mathbf{Alt}R\epsilon^{r+1}_*(\mathbb{Q}^\bullet_{D^{r+1}})[-r]$$

in $D^b_c(f(X))$.

If $D^k(f)$ has a finite number of connected components for every $k$ then this filtration can then be refined to $\mathbf{F}'^\bullet$. This is done by removing the orbit of a connected component one at a time, thus filtering each $\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})$ in a natural manner.

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*Remark* 2.6. The resolution works in the general case that $f$ is finite and continuous. It gives rise to a very powerful spectral sequence which uses the alternating rational homology of multiple point spaces to calculate the rational homology of the image of $f$. See [3, 5–7].

To give a feel for the behaviour of alternating cohomology consider the following. Let $\phi_h$ and $\psi_h$ denote the vanishing and nearby cycles functor respectively for the complex analytic function $h: Y \to \mathbb{C}$.

THEOREM 2.7. *Suppose $f: X \to Y$ is a complex analytic map and $h: Y \to \mathbb{C}$ is a complex analytic function. Suppose that $y$ is a point of $f(X)$ with no more than $r$ preimages under $f$, then, for all $k > r$ the natural map*

$$\mathbf{H}^i(\psi_h \mathbf{Alt} R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k}))_y \to \mathbf{H}^i(\phi_h \mathbf{Alt} R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k}))_y$$

*is an isomorphism for all $i$.*

*Proof.* There is a distinguished triangle

$$\mathbf{Alt} R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})|_{h^{-1}(0)} \to \psi_h \mathbf{Alt} R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k}) \to \phi_h \mathbf{Alt} R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})$$

The functor $\mathbf{H}$ gives a long exact exact sequence which provides the map of the theorem at the stalk level. It is an isomorphism since $\mathbf{Alt} R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})$ has a trivial stalk at $y$ according to the proof of Proposition 2.1 of [3]. □

## 2.3. THE PERVERSE SHEAVES

We will show that in a quite general case the sheaves $\mathbf{Alt} R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})$ are perverse. The next lemma shows that the cosupport for these sheaves is determined by the local alternating cohomology of the multiple point space $D^k(f)$.

LEMMA 2.8. *Suppose $f: X \to Y$ is a finite and proper complex analytic map with $Y$ a manifold. Let $B_\varepsilon$ be a small open ball in $Y$ of radius $\varepsilon$ centered at $y \in f(X)$. Then for sufficiently small $\varepsilon$ we have the following isomorphism for any integers $s$ and $q$ and any index $\alpha$,*

$$\begin{aligned}&\mathbf{H}^q(\mathcal{D}(\mathbf{Alt}\, R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha})[s]))_y \\ &\quad \cong \mathrm{Alt}\, H^{-q+s}((\epsilon^k_\alpha)^{-1}(B_\varepsilon), (\epsilon^k_\alpha)^{-1}(B_\varepsilon - \{y\}); \mathbb{Q}).\end{aligned}$$

*The statement given by the omission of the $\alpha$s is also true.*

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*Proof.* The proof is by a sequence of isomorphisms:

$$\mathbf{H}^q(\mathcal{D}(\mathbf{Alt}\, R\epsilon_*^k(\mathbb{Q}^\bullet_{D^k_\alpha})[s]))_y$$

$$\cong \mathbb{H}^q(B_\varepsilon; \mathcal{D}(\mathbf{Alt}\, R\epsilon_*^k(\mathbb{Q}^\bullet_{D^k_\alpha})[s]))$$

$$\cong \mathbb{H}_c^{-q}(B_\varepsilon; \mathbf{Alt}\, R\epsilon_*^k(\mathbb{Q}^\bullet_{D^k_\alpha})[s])$$

$$\cong \mathbb{H}_c^{-q+s}(B_\varepsilon; \mathbf{Alt}\, R\epsilon_*^k(\mathbb{Q}^\bullet_{D^k_\alpha}))$$

$$\cong \mathbb{H}^{-q+s}(B_\varepsilon, B_\varepsilon - \{y\}; \mathbf{Alt}\, R\epsilon_*^k(\mathbb{Q}^\bullet_{D^k_\alpha}))$$

$$\cong \mathrm{Alt}\, \mathbb{H}^{-q+s}(B_\varepsilon, B_\varepsilon - \{y\}; R\epsilon_*^k(\mathbb{Q}^\bullet_{D^k_\alpha}))$$

$$\cong \mathrm{Alt}\, \mathbb{H}^{-q+s}((\epsilon_\alpha^k)^{-1}(B_\varepsilon), (\epsilon_\alpha^k)^{-1}(B_\varepsilon - \{y\}); \mathbb{Q}^\bullet_{D^k_\alpha})$$

$$\cong \mathrm{Alt}\, H^{-q+s}((\epsilon_\alpha^k)^{-1}(B_\varepsilon), (\epsilon_\alpha^k)^{-1}(B_\varepsilon - \{y\}); \mathbb{Q}).$$

The statements obviously remain true if one omits the $\alpha$s. □

The next theorem involves the rectified homological depth of a space. The definition of and the important theorems on this concept can be found in [4].

THEOREM 2.9. *Suppose $f : X \to Y$ is a dimensionally correct complex analytic map with* $\mathrm{rHd}(X; \mathbb{Q}) = \dim_{\mathbb{C}} X = n$ *and $Y$ a complex manifold of dimension $p > n$. Then* $\mathbf{Alt} R\epsilon_*^k(\mathbb{Q}^\bullet_{D^k_\alpha})[nk - p(k-1)]$ *is a perverse sheaf on $M_k(f)$ and hence on $f(X)$.*

*The statement given by the omission of the $\alpha$s is also true.*

*Proof.* The support condition is easily verified since

$$\mathbf{H}^{nk-p(k-1)}(\mathbf{Alt} R\epsilon_*^k(\mathbb{Q}^\bullet_{D^k_\alpha})[nk - p(k-1)])$$

is the only nontrivial cohomology sheaf and is supported on the top strata of $\epsilon^k(D^k) = M_k$, which have dimension $nk - p(k-1)$ as $f$ is dimensionally correct.

Verification of the the cosupport condition a bit more complicated and involves studying the support of $\mathbf{H}^q(\mathcal{D}(\mathbf{Alt} R\epsilon_*^k(\mathbb{Q}^\bullet_{D^k_\alpha})[nk - p(k-1)]))$. We know how to calculate the stalk of this sheaf at any point $y \in f(X)$ in terms of the alternating local cohomology of multiple point spaces by Lemma 2.8. Thus, we study

$$\mathrm{Alt}\, H^{-q+nk-p(k-1)}((\epsilon_\alpha^k)^{-1}(B_\varepsilon), (\epsilon_\alpha^k)^{-1}(B_\varepsilon - \{y\}); \mathbb{Q}).$$

Suppose $y \in M_k \subseteq Y$ and that we embed neighbourhoods of $(\epsilon^k)^{-1}(y)$ in a manifold. We stratify $M_k$ at $y$ by taking a stratification of $D^k$ so that the strata are $S_k$-invariant. Since $\epsilon^k$ is finite we can refine the stratifications on $D^k$ and $M_k$ so that the map is a stratified submersion and there is a local diffeomorphism of strata at every point.

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Suppose $y$ lies in the stratum $A$ of complex dimension $d$. Let $B_\varepsilon$ be an open ball of radius $\varepsilon$ about $y$ in $Y$, and let $N$ be a manifold in $Y$ transverse to the stratum $A$ with $N \cap A = \{y\}$. Take $\varepsilon > 0$ so that the boundary of $B_\varepsilon \cap N$ defines the real link $L$ of the stratum $A$ in $f(X)$.

Let $(U, V) = ((\epsilon_\alpha^k)^{-1}(B_\varepsilon), (\epsilon_\alpha^k)^{-1}(B_\varepsilon - \{y\}))$. The product structure of $M_k$ near the stratum of dimension $d$ gives a product structure on the various disjoint pieces of $(\epsilon_\alpha^k)^{-1}(B_\varepsilon)$ and so $(U, V)$ is $S_k$-homotopically equivalent to

$$\amalg_{x\in(\epsilon_\alpha^k)^{-1}(y)}(B^{2d}, \partial B^{2d}) \times (\mathrm{Cone}(L_x), L_x),$$

where the links of points in $D^k$ are denoted by $L_x$ and the cone over $L_x$ is denoted $\mathrm{Cone}(L_x)$. The set of links in $(\epsilon_\alpha^k)^{-1}(y)$ inherit the $S_k$-action, as does the cone over $L_x$ with the cone-points inheriting the action on $(\epsilon_\alpha^k)^{-1}(y)$. The standard ball of dimension $2d$ and its boundary, denoted $(B^{2d}, \partial D^{2d})$, are given the trivial action. (This makes the action on the product the correct one). Denoting the $S_k$-orbit of a space $Z$ by $\mathrm{Orbit}(Z)$, this gives us that

$$\begin{aligned}
&\mathrm{Alt}\, H^j(U, V; \mathbb{Q})\\
&\quad\cong \mathrm{Alt} \oplus_{x\in(\epsilon_\alpha^k)^{-1}(y)}\{\oplus_{a+b=j} H^a(B^{2d}, \partial B^{2d}; \mathbb{Q}) \otimes H^b(\mathrm{Cone}(L_x), L_x; \mathbb{Q})\}\\
&\quad\cong \mathrm{Alt} \oplus_{x\in(\epsilon_\alpha^k)^{-1}(y)} H^{j-2d}(\mathrm{Cone}(L_x), L_x; \mathbb{Q})\\
&\quad\cong \oplus\, \mathrm{Alt}\, H^{j-2d}(\mathrm{Orbit}(\mathrm{Cone}(L_x)), \mathrm{Orbit}(L_x); \mathbb{Q}).
\end{aligned}$$

Thus it remains to investigate the alternating cohomology of these orbits. To prove the vanishing of the stated groups we use Proposition 3.4 in [6].

Let $\tilde{f} = f|f^{-1}(N)$ and let $\tilde{\epsilon}^k$ be the natural map $D_\alpha^k(\tilde{f})$ to $f(X) \cap N$. Then for any set $Z$ in $Y$ we have an inclusion

$$(\tilde{\epsilon}_\alpha^k)^{-1}(Z \cap N) \hookrightarrow (\epsilon_\alpha^k)^{-1}(Z)$$

and the complement of the image of this map is exclusively in the set of non-regular orbits of $S_k$. It is not too difficult to prove that this implies that the alternating cohomology of the two spaces are equal, see Theorem 2.7 of [8] for an alternating homology version. We use this fact to show that the alternating local cohomology of $f$ can be calculated from that of $\tilde{f}$:

Since $N$ is defined by $d$ equations we get that $\mathrm{rHd}(X \cap f^{-1}(N)) \geqslant \mathrm{rHd}(X) - d$. The map $\tilde{f}$ is a dimensionally correct map into a manifold of dimension equal to $\dim_{\mathbb{C}} Y - d$ since $f$ is dimensionally correct and $N$ is transverse to $A$. So by Proposition 3.4 of [6] the alternating local cohomology of the orbit of $(\tilde{\epsilon}_\alpha^k)^{-1}(y)$ is trivial below $k\mathrm{rHd}(X) - p(k-1) - d$. This alternating cohomology is the same as for the links of original map as the spaces coincide outside the diagonal. Since

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$\mathrm{rHd}(X) = \dim_{\mathbb{C}} X = n$ we deduce that

$$\operatorname{Alt} H^j(U, V; \mathbb{Q}) = 0 \text{ for } j - 2d < nk - p(k-1) - d = \dim_{\mathbb{C}} D^k_\alpha - d,$$

i.e. for $j < \dim_{\mathbb{C}} D^k_\alpha + d$.

Thus suppose $H^j(U, V; \mathbb{Q}) \neq 0$ for some $y$ in a stratum of dimension $d$ then $j - \dim_{\mathbb{C}} D^k_\alpha \geqslant d$, which implies that

$$\dim \operatorname{supp} \mathbf{H}^j(\mathcal{D}(\mathbf{Alt} R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha}))) \leqslant j - \dim_{\mathbb{C}} D^k_\alpha,$$

which is the cosupport condition. □

Thus we obtain some new examples of perverse sheaves on spaces which are not obviously set-theoretic complete intersections.

## 2.4. NONSINGULAR MULTIPLE POINT SPACES

In the case where $D^k(f)$ is nonsingular the map $\epsilon^k : D^k(f) \to Y$ acts in a similar way to a resolution of $M_k$. The next theorem in this section shows that the sheaf $\mathbf{Alt} R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})$ is semi-simple in the category of perverse sheaves on $M_k$ and is in fact simple if we take an irreducible component of $D^k(f)$.

Semi-simple perverse sheaves are intersection cohomology sheaves with coefficients in some local system. We shall begin with a description of the required one-dimensional local system on $M_k(f)$ and then show that for nonsingular $D^k(f)$ the $D^k_\alpha(f)$ give rise to the irreducible components of $M_k(f)$.

Let $A_k := D^k(f) \backslash \big(\epsilon^{j,k+1}(D^{k+1}(f)) \cup \operatorname{Diag}(D^k(f))\big)$ for some $j$, (it is not difficult to see that $A_k$ is independent of the chosen $j$). Let $P_k := \epsilon^k(A_k)$.

LEMMA 2.10. *Suppose that* $\dim D^{k+1}(f) < \dim D^k(f)$. *Then the map* $\epsilon^k|A_k : A_k \to P_k$ *is a* $k!$*-fold cover and the set* $A_k$*, resp.* $P_k$*, is open and dense in* $D^k$*, resp.* $M_k$.

*Proof.* The set $A_k$ is open and dense as $\epsilon^{j,k+1}(D^k)$ and $\operatorname{Diag}(D^k)$ are proper closed subsets of $D^k$. Similarly for $P_k$.

The map $\epsilon^k|A_k$ is a $k!$-fold cover because if $(x_1, \dots, x_k) \in A_k \subseteq D^k$ with $\epsilon(x_1, \dots, x_k) = y$ then the orbit of $(x_1, \dots, x_k) = (\epsilon^k)^{-1}(y)$. The orbit has $k!$ points since $(x_1, \dots, x_k) \notin \operatorname{Diag}(D^k)$. To see that $(\epsilon^k)^{-1}(y)$ consists only of the orbit of $(x_1, \dots, x_k)$ then assume that there exists $(\tilde{x}_1, \dots, \tilde{x}_k) \in (\epsilon^k)^{-1}(y)$ which is not in the orbit of $(x_1, \dots, x_k)$. Then there exists an $r$ such that $\tilde{x}_r \neq x_i$ for all $i$. Since $\tilde{x}_r \in f^{-1}(y)$ we have that $(x_1, \dots, x_{j-1}, \tilde{x}_r, x_j, \dots, x_k) \in D^{k+1}$ which implies that $(x_1, \dots, x_k) \in \epsilon^{j,k+1}(D^{k+1})$ as

$$\epsilon^{j,k+1}(x_1, \dots, x_{j-1}, \tilde{x}_r, x_j, \dots, x_k) = (x_1, \dots, x_k).$$

This is a contradiction. □

LEMMA 2.11. *The sheaf* $R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})|P_k$ *is a one-dimensional local system on* $P_k$.

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*Proof.* Let $y$ be any point of $P_k$ and let $\{z_s\}$, $s = 1, \dots, k!$, be the set of points in the preimage of $y$ by $\epsilon^k$. Then we have

$$(R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})|P_k)_y \cong \oplus_{s=1}^{k!}\left(\mathbb{Q}^\bullet_{D^k}\right)_y \cong \left(\mathbb{Q}^\bullet_y\right)^{k!}.$$

Applying the alternating functor we find

$$(\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k})|P_k)_y \cong (\mathbf{Alt}\oplus_{k!}\mathbb{Q}^\bullet_{D^k})_{z_i} \cong \mathbb{Q}.$$

This shows that the stalks are one-dimensional.

Since $\epsilon^k|A_k$ is finite and proper then for any $y$ there exists a contractible open set $U \subseteq P_k$ with $U$ such that the system is trivialisable over $U$. To see this one need only take a neighbourhood $U$, such that the boundary defines the link of $U$ and it is taken so small that there exists an open set $V \subseteq (\epsilon^k)^{-1}(U)$ with $\epsilon^k(V) = U$ and $\sigma(V) \cap V = \emptyset$ for $\sigma \in S_k - \{\mathrm{id}\}$. □

*Remark* 2.12. Note that the system is not necessarily equal to the constant sheaf on $P_k$. To see this consider $f\colon \mathbb{C}^2 \to \mathbb{C}^3$ given by $f(u, v) = (u, uv, v^2)$. This map has double point space equal to the complex plane and after a suitable choice of coordinates the map $\epsilon^2\colon D^2 \to \mathbb{C}^3$ is given by $\epsilon^2(z) = (0, 0, z^2)$. (The action of $S_2$ in these coordinates is simply $\sigma(z) = -z$.) This gives a two-fold cover over the complement of the origin in the image of $\epsilon^2$. For $y \in \mathbb{R}^+ \subset \mathbb{C} = \epsilon^2(\mathbb{C})$ we have $(\epsilon^2_*\mathbb{Q}_\mathbb{C})_y \cong \mathbb{Q}_{\sqrt{y}} \oplus \mathbb{Q}_{-\sqrt{y}}$. Take the element $(b, -b)$ in $\mathrm{Alt}(\epsilon^2_*\mathbb{Q}_\mathbb{C})_y$ for some non-zero $b$. Then a loop around the origin in $\mathbb{C} = \epsilon^2(\mathbb{C})$ lifts to a path from $\sqrt{y}$ to $-\sqrt{y}$ in $D^2 = \mathbb{C}$. The natural lift arising from the double cover given by $\epsilon^2$ means that $(b, -b)$ is transported to $(-b, b)$ by the loop in $\epsilon^2(\mathbb{C})$.

LEMMA 2.13. *Suppose $D^k(f)$ is nonsingular and* $\dim D^{k+1}(f) < \dim D^k(f)$. *Then $\epsilon^k(D^k_\alpha(f))$ is an irreducible component of $M_k(f)$ and every irreducible component has this form.*

*Proof.* We can assume that the dimension of $M_k$ is greater than zero since the statements are trivial in the dimension equals zero case.

Since $f$ is a finite complex analytic map we can stratify it so that it is a stratified submersion, with the images under $\epsilon^k$ of each $D^k(f)$ and its diagonal form substratifications of $M_k$.

Let $\Sigma$ denote the 'singular set' of this stratification of $M_k$ and let $M'_k = M_k \backslash \Sigma$. Since $\dim D^{k+1} < \dim D^k$ and the diagonal forms a substratification of $D^k(f)$ then $M'_k \subseteq P_k$, for $P_k$ from Lemma 2.10. (Presumably $M'_k = P_k$ but this equality is unnecessary for later arguments.) Define $D^{k\prime} = (\epsilon^k)^{-1}(M'_k)$ and $D^{k\prime}_\alpha = D^{k\prime} \cap D^k_\alpha$. Through restriction we see that the local system on $P_k$ forms a system on $M'_k$ and we note that

$$\mathrm{closure}(\epsilon^k(D^{k\prime}_\alpha)) = \epsilon^k(\mathrm{closure}(D^{k\prime}_\alpha)) = \epsilon^k(D^k_\alpha).$$

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First it is claimed that $\epsilon^k(D_\alpha^{k\prime})$ is connected. Suppose that $\tilde{D}_\alpha^k$ is a connected component of $D^k$. Then let $\tilde{D}_\alpha^{k\prime} = \tilde{D}_\alpha^k \backslash (\epsilon^k)^{-1}(\Sigma)$. The set $\tilde{D}_\alpha^{k\prime}$ is connected because $(\epsilon^k)^{-1}(\Sigma)$ is a proper complex analytic subspace of the manifold $\tilde{D}_\alpha^k$ and hence any path in the manifold can be assumed to be smooth and by transversality can be assumed to miss the subspace. It is obviously true that

$$\epsilon^k(\tilde{D}_\alpha^{k\prime}) = \epsilon^k(\mathrm{Orbit}(\tilde{D}_\alpha^{k\prime})) = \epsilon^k(D_\alpha^{k\prime}).$$

Thus $\epsilon^k(D_\alpha^{k\prime})$ is connected as claimed.

Suppose that $M_k^\alpha$ is an irreducible component of $M_k$. Then $M_k^{\alpha\prime} := M_k^\alpha \backslash \Sigma$ is by definition a connected component of $M_k'$.

The fact that $\epsilon^k$ forms a $k!$-fold cover of $M_k'$ implies that the connected components of $M_k'$ are in a one-to-one correspondence with orbits of connected components in $(\epsilon^k)^{-1}(M_k')$. Thus there exists a $Y \subseteq (\epsilon^k)^{-1}(M_k')$ such that $\epsilon^k(\mathrm{Orbit}(Y)) = M_k^{\alpha\prime}$ and $Y$ is a connected component.

Then $Y \subseteq \tilde{D}_\alpha^{k\prime}$ for some $\alpha$ and

$$M_k^{\alpha\prime} = \epsilon^k(\mathrm{Orbit}(Y)) \subseteq \epsilon^k(\mathrm{Orbit}(\tilde{D}_\alpha^{k\prime})) = \epsilon^k(D_\alpha^{k\prime}).$$

That is, the connected component $M_k^{\alpha\prime}$ is a subset of a connected set and hence must be equal to that set. Through taking the closure of both sides of the equation above we see that $M_k^\alpha = \epsilon^k(D_\alpha^k)$.

Conversely, if $\epsilon^k(D_\alpha^k) \subseteq M_k^\beta$ for some $\beta$, then

$$\epsilon^k(D_\alpha^{k\prime}) \subseteq M_k^{\beta\prime} = \epsilon^k(D_\beta^{k\prime})$$

so $D_\alpha^{k\prime} \subseteq D_\beta^{k\prime}$. Since both are assumed to be connected components then they must be equal. So $\epsilon^k(D_\alpha^k) = M_k^\beta$. □

THEOREM 2.14. *Suppose $f: X \to Y$ is a finite and proper complex analytic map where $X$ and $Y$ are complex analytic spaces of dimensions $n$ and $p$ respectively. If $D^k(f)$ is a complex analytic manifold of dimension $nk - p(k-1)$ and* $\dim D^{k+1}(f) < \dim D^k(f)$ *then*

$$\mathbf{Alt}R\epsilon_*^k(\mathbb{Q}_{D_\alpha^k}^\bullet) \cong \mathbf{IC}_{M_k^\alpha}^\bullet(\mathcal{L}_k)$$

*where $\mathcal{L}_k$ is the one-dimensional local system of Lemma 2.11 arising from the action of $S_k$ on $D^k$.*

*The statement given by the omission of the $\alpha$s is also true.*

*Proof.* To prove this we use the local system version of the axioms [AX2] in [1]. These conditions are called normalisation, lower bound, support and cosupport.

To lighten notation we shall ignore the shift by $nk - p(k-1)$ that is required to put everything into the perverse category. Effectively we are using the notion of positively perverse from [10].

As shown in Lemma 2.11 $\mathbf{Alt}R\epsilon_*^k(\mathbb{Q}_{D_\alpha^k}^\bullet)|P_k$ is a local system on $P_k$ with one-dimensional stalks and so the complex satisfies the normalisation condition.

The lower bound condition,

$$\mathbf{H}^i(\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha})) = 0 \text{ for } i < 0,$$

is obviously satisfied.

The support condition

$$\dim \operatorname{supp} \mathbf{H}^{-i}(\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha})[nk - p(k-1)]) < i \text{ for all } i < nk - p(k-1),$$

is also trivially satisfied as the only nontrivial group occurs when $i = nk - p(k-1)$.

The cosupport follows from the fact that $D^k$ is nonsingular and hence

$$\operatorname{Alt} H^{nk-p(k-1)+i}((\epsilon^k_\alpha)^{-1}(B_\varepsilon), (\epsilon^k_\alpha)^{-1}(B_\varepsilon - \{y\}); \mathbb{Q})$$

is non trivial only when $nk - p(k-1) + i = 2(nk - p(k-1))$. By Lemma 2.8 we deduce that the sheaf

$$\mathbf{H}^{-i}(\mathcal{D}(\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha}))[nk - p(k-1)])$$

has support only possible when $i = nk - p(k-1)$, thus satisfying the cosupport condition.

If we set $\mathcal{L}_k = \mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha})|P_k$ then we can use the uniqueness theorem of intersection cohomology sheaves to show

$$\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha}) \cong \mathbf{IC}^\bullet_{\overline{M^{\alpha\prime}_k}}(\mathcal{L}_k) \cong \mathbf{IC}^\bullet_{M^\alpha_k}(\mathcal{L}_k). \qquad \square$$

*Remark* 2.15. For the case $k = 1$ we have $M_1 = f(X)$ and $\mathcal{L}_1$ is the constant sheaf on the nonsingular part of $f(X)$ and so $Rf_*(\mathbb{Q}^\bullet_X) \cong \mathbf{IC}^\bullet_{f(X)}$. This result is already well known, see Section 6.2 of [1].

COROLLARY 2.16. *Suppose $f$ is dimensionally correct and every $D^k(f)$ is nonsingular then there exists a spectral sequence*

$$E_1^{p,q} = IH^q(M_{p+1}; \mathcal{L}_{p+1}) \Rightarrow H^*(f(X); \mathbb{Q}).$$

*That is, the rational cohomology of the image can be calculated using twisted intersection cohomology sheaves on the image multiple point spaces.*

*Proof.* The sequence of Proposition 2.3 of [3] has $E_1^{p,q} \cong \operatorname{Alt} H^q(D^{p+1}; \mathbb{Q})$ and converges to the rational cohomology of the image of $f$. But

$$\operatorname{Alt} H^q(D^{p+1}; \mathbb{Q}) \cong \mathbb{H}^q(Y; \mathbf{Alt}R\epsilon^{p+1}_*(\mathbb{Q}^\bullet_{D^{p+1}})),$$

which by the theorem is the same as

$$\mathbb{H}^q(Y; \mathbf{IC}^\bullet_{M_{p+1}}(\mathcal{L}_{p+1})) \cong IH(M_{p+1}, \mathcal{L}_{p+1})$$

and this is the required group. $\square$

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## 3. The Image Hypersurface Case

If the image of $f$ is a hypersurface then the constant sheaf $\mathbb{Q}^\bullet_{f(X)}$ shifted by dimension is a perverse sheaf. In this section we provide conditions which imply that the complexes in the filtration of the resolution of Section 2.2 are perverse. We also show that when in addition every multiple point space is nonsingular and of the right dimension then we can use the filtration to describe the composition series for the perverse constant sheaf on the image of $f$.

THEOREM 3.1. *Suppose $f\colon X \to Y$ is a proper and finite dimensionally correct complex analytic map with* $\mathrm{rHd}(X;\mathbb{Q}) = \dim_{\mathbb{C}} X = n$ *and $Y$ a manifold of dimension $n+1$. Then $\mathbf{F}^\bullet_r[n]$ is a perverse sheaf for all $r$.*

*Proof.* The main point is that

$$\mathbf{F}^\bullet_{r+1}[n]/\mathbf{F}^\bullet_r[n] \cong (\mathbf{F}^\bullet_{r+1}/\mathbf{F}^\bullet_r)[n] \cong \mathbf{Alt}R\epsilon^{r+1}_*(\mathbb{Q}^\bullet_{D^{r+1}})[n-r]$$

and the latter is perverse by the theorems above.

We proceed by decreasing induction on $r$. We have the following distinguished triangle in $D^b_c(f(X))$,

$$\mathbf{F}^\bullet_{r+1}[n] \to \mathbf{F}^\bullet_r[n] \to \mathbf{Alt}R\epsilon^{r+1}_*(\mathbb{Q}^\bullet_{D^{r+1}})[n-r].$$

By the induction hypothesis and Theorem 2.9 the outer terms are perverse, so the middle one is too. The initial condition is statisfied because there exists an integer $s$ such that

$$\mathbf{F}^\bullet_s[n] \cong \mathbf{Alt}R\epsilon^{s+1}_*(\mathbb{Q}^\bullet_{D^{s+1}})[n-s]. \qquad \square$$

If, in addition to the assumptions of this theorem, each $D^k(f)$ nonsingular (note that this means $X = D^1$ is nonsingular and hence $\mathrm{rHd}(X) = n$) and has a finite number of components then we can identify the composition series in the perverse category for the constant sheaf.

THEOREM 3.2. *Suppose $f : X \to Y$ is a dimensionally correct map, with* $\dim X = n$ *and* $\dim Y = n+1$, *and $Y$ and all multiple point spaces are nonsingular with a finite number of connected components. Then the filtration of the perverse sheaf $\mathbb{Q}^\bullet_{f(X)[n]}$ by $\mathbf{F}'^\bullet$ given in Section 2.2 is such that $\mathbf{F}'^\bullet_r[n]$ is perverse for all $r$. Successive quotient terms are quasi-isomorphic to $\mathbf{IC}^\bullet_{M^\alpha_k}(\mathcal{L}_k)[n-k+1]$, for some $k$ and $\alpha$.*

*Proof.* As the multiple point spaces have a finite number of connected components then the filtration is finite and we can apply induction in the same way as Theorem 3.1 to show that $\mathbf{F}'^\bullet_r$ is perverse.

It is then obvious that $\mathbf{F}'^\bullet_{r+1}[n]/\mathbf{F}'^\bullet_r[n] \cong \mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha})[n-k+1]$ for some $k$ and $\alpha$. The result then follows from Theorem 2.14.

*Remark* 3.3. Note that the irreducible components of $\mathbf{IC}^\bullet_{f(X)}$ appear in the composition series as parts of $Rf(\mathbb{Q}^\bullet_X)$.

We have a number of interesting examples for which the multiple point spaces are nonsingular.

EXAMPLE 3.4. Suppose $f: X \to Y$ is such that at all points it is locally $\mathcal{A}$-equivalent to an immersion or the trivial extension of a corank 1 stable map-germ. By Marar and Mond's description of multiple point spaces in Theorem 2.14 of [9] the multiple point space $D^k$ is a complex analytic manifold of dimension $n-k+1$, (hence $f$ is a dimensionally correct map). So the corollary applies to such maps when the number of components of every $D^k(f)$ is finite.

EXAMPLE 3.5. Suppose $F: W \to Y$ is a map between two manifolds of dimension $n+1$ which is locally $\mathcal{A}$-equivalent to an immersion or a trivial extension of a corank 1 stable map-germ. Locally the latter are given as extensions of

$$(x_1, \dots, x_{m-1}, y) \mapsto (x_1, \dots x_{m-1}, y^{m+1} + x_{m-1}y^{m-1} + x_{m-2}y^{m-2} + \dots + x_1 y).$$

Then, by Theorem 4.1.1 of [2], the multiple point spaces of the map $f: X \to Y$, where $X$ is the critical locus of $F$, are nonsingular of dimension $n-k+1$. The image of $f$ is the discriminant of $F$.

EXAMPLE 3.6. In the two examples above the map $\epsilon^{j,k}: D^k(f) \to D^{k-1}(f)$ given by discarding one coordinate from $X^k$ has multiple point spaces equal to multiple point spaces of $f$. Hence, we can apply the corollary again. In particular the image of $\epsilon^{1,2}: D^2(f) \to X$ is the set of double points in the source of $f$ and we can describe the composition series for such a hypersurface.

EXAMPLE 3.7. Suppose $H = \cup_i H_i$ is a finite hyperplane arrangement in general position in $\mathbb{C}^n$. Then $H$ is the image of the obvious finite map given by inclusions

$$f: \amalg_i H_i \hookrightarrow \mathbb{C}^n.$$

Since the planes are in general position the map is a finite-dimensionally correct map and the multiple point spaces are obviously nonsingular. Hence the corollary applies.

*Remark* 3.8. Suppose that $f: X \to Y$ is a dimensionally correct corank 1 map between complex analytic manifolds of dimension $n$ and $n+1$. Then $D^k(f)$ is a local complete intersection and thus the constant sheaf upon it is perverse. It seems likely that the compostion series for this can be used to filter the perverse sheaf $\mathbf{Alt}R\epsilon^k_*(\mathbb{Q}^\bullet_{D^k_\alpha})$ so that the compostion series for $\mathbb{Q}^\bullet_{f(X)}$ can be described.

Dropping the corank 1 condition means that $D^k(f)$ is not a local complete intersection. However, with respect to alternating cohomology these multiple point spaces behave as though they were local complete intersections. Thus, it may even be possible to give a filtration of the perverse sheaf $\mathbb{Q}^\bullet_{f(X)}$ in this case as well.

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