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Nonabelian H^1 and the Étale Van Kampen Theorem

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Abstract. Generalized étale homotopy pro-groups $\pi_1^{\text{ét}}(\mathcal{C}, x)$ associated with pointed, connected, small Grothendieck sites $(\mathcal{C}, x)$ are defined, and their relationship to Galois theory and the theory of pointed torsors for discrete groups is explained.

Applications include new rigorous proofs of some folklore results around $\pi_1^{\text{ét}}(\text{ét}(X), x)$, a description of Grothendieck's short exact sequence for Galois descent in terms of pointed torsor trivializations, and a new étale van Kampen theorem that gives a simple statement about a pushout square of pro-groups that works for covering families that do not necessarily consist exclusively of monomorphisms. A corresponding van Kampen result for Grothendieck's profinite groups π_1^{Gal} immediately follows.

1 Introduction

The étale fundamental group $\pi_1^{\text{ét}}(X, x)$ of a pointed, connected, locally noetherian scheme was defined by Artin and Mazur in [AM69] by means of the cofiltered category of pointed representable hypercovers of X and pointed simplicial homotopy classes of maps between them. The significance of this object may be seen immediately in the case $X = \operatorname{Spec} k$, where k is a field. Fixing a geometric point $x\colon \operatorname{Spec} \Omega \to \operatorname{Spec} k$ associated with some separable closure Ω/k, one may directly compute that $\pi_1^{\text{ét}}(\operatorname{Spec} k, x) \cong \operatorname{Gal}(\Omega/k)$, the absolute Galois group of k with respect to Ω. As the definition of the term "hypercover" is independent of the underlying Grothendieck topology, one may generalize the definition of $\pi_1^{\text{ét}}$ to apply to the hypercovers of any pointed, connected, small Grothendieck site $(\mathcal{C}, x)$, where the "point" x is interpreted as a geometric morphism $x\colon \mathbf{Set} \to \mathbf{Shv}(\mathcal{C})$ of toposes. The object of this paper is to explain some of the basic properties of $\pi_1^{\text{ét}}(\mathcal{C}, x)$ at this level of generality, including its relationship with Grothendieck's Galois theory and discrete group torsors and their trivializations, and to prove a new van Kampen theorem for $\pi_1^{\text{ét}}(\text{ét}(X), x)$ (and some of its generalizations), which simplifies and extends previous work (in particular [Sti06, Zoo02]) in a new homotopy theoretic direction.

The principal technical device at work here is that of groupoids $H\text{-Tors}_x$ of *pointed H-torsors* for constant group sheaves $H := \Gamma^* H$ associated with discrete groups H. The characterization of these groupoids as homotopy fibres by Jardine in [Jar09b] allows one to give precise proofs of facts that previously (to the author's knowledge) had the status of folklore; in particular, Theorem 3.5 shows that the pro-groupe fondamental élargi G associated with the full subtopos $\mathbf{SLC}(\mathcal{C})$ of sums of locally con-

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stant objects of $\mathbf{Shv}(\mathcal{C})$ is pro-isomorphic to (a Čech hypercover version of) $\pi_1^{\text{ét}}(\mathcal{C}, x)$. The subtlety here lies in the identification

$$\pi_1^{\text{ét}}(\mathcal{C}, x) \cong \pi_1^{\text{ét}}(\mathbf{SLC}(\mathcal{C}), x),$$

as the latter pro-group is easily shown (Proposition 3.7) to be pro-isomorphic to G. Artin and Mazur apparently thought they proved this in [AM69, §10], but in fact those methods only achieve the identification of $\pi_1^{\text{ét}}(\mathcal{C}, x)$ as a representing pro-object in $\pi\mathbf{Grp}$, the category of groups and homotopy classes of homomorphisms between them, rather than $\mathbf{Grp}$ itself. The proof here is new and requires the homotopy theoretic characterization of pointed torsors. Pointed torsors may also be used to rigorously establish the bit of folklore that the profinite completion $\pi_1^{\text{ét}}(\text{ét}(X), x)^\frown$ of the nonfinite étale $\pi_1^{\text{ét}}$ of a connected locally noetherian scheme is pro-isomorphic to Grothendieck's profinite fundamental group $\pi_1^{\text{Gal}}(X, x)$ associated with the *finite* étale site $\mathbf{Finét}(X)$ based at x (Proposition 3.9). Proposition 3.8 establishes that the $\pi_1^{\text{ét}}(\mathcal{C}, x)$ constructed from arbitrary (not just Čech) pointed representable hypercovers agrees with the Čech variants thereof whenever the site $\mathcal{C}$ admits a "rigid diagram of hypercovers", as is the case for the small étale site of any connected locally noetherian scheme.

With these issues out of the way, one may study trivializations of pointed torsors and use them to show how Grothendieck's short exact sequence for Galois descent

$$1 \to \pi_1^{\text{Gal}}(\mathcal{C}/(Y, y), y) \to \pi_1^{\text{Gal}}(\mathcal{C}, x) \twoheadrightarrow G_Y \to 1$$

associated with connected Galois objects (Y, y) in pointed Galois categories $(\mathcal{C}, x, F_x)$ arises naturally from short exact sequences in pointed nonabelian $H_x^1(\mathcal{C}, -)$; see Corollary 5.5.

The final section of this paper uses the general methods established above to prove a new variant (Corollary 6.5) of étale van Kampen theorem which is both simple to state (it is just a pushout as in the usual topological van Kampen theorem) and does *not* require the covering family to consist exclusively of monomorphisms (as in the case of coverings by open subschemes or substacks; cf. [Zoo02]). Corollary 6.6 shows in particular how this result specializes to a statement about Grothendieck's profinite fundamental groups. The methods employed to prove these statements are conceptual, homotopy theoretic, and in fact give the presumably stronger Theorem 6.4 whose statement does not in any direct way depend upon the underlying topology.

2 Torsors and (Geometrically) Pointed Torsors

Étale homotopy theory begins with the observation that, for suitably nice Grothendieck sites $\mathcal{C}$, the canonical constant sheaf functor $\Gamma^* \colon \mathbf{Set} \to \mathbf{Shv}(\mathcal{C})$ has a left adjoint $\Pi \colon \mathbf{Shv}(\mathcal{C}) \to \mathbf{Set}$ called the *connected components* functor. Naturally, this functor has the geometric interpretation of sending a scheme to its set of scheme-theoretic connected components whenever one is working with some good enough site of schemes with a subcanonical topology (*i.e.,* all representable presheaves are sheaves).

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2.1 Connectedness and Local Connectedness

Recall that a sheaf F on a site $\mathcal{C}$ is called *connected* if, whenever there is a coproduct decomposition $F = F_1 \sqcup F_2$, either $F_1 = \varnothing$ or $F_2 = \varnothing$, where $\varnothing$ denotes the initial sheaf on $\mathcal{C}$. A site $\mathcal{C}$ is called *locally connected* if every sheaf on $\mathcal{C}$ splits uniquely (up to canonical isomorphism) as a coproduct of connected sheaves, and if representable sheaves similarly decompose as coproducts of connected representable sheaves. On the sites of interest here any connected scheme will represent a connected representable sheaf (see *e.g.*, [Zoo01, Lemma 3.3]). It is known that the étale sites of locally noetherian Deligne–Mumford stacks are locally connected ([Zoo01, 3.1]). Under these conditions, the aforementioned connected components functor π exists (defined by sending each sheaf to its set of connected components) and is easily shown to be left adjoint to Γ^*. A Grothendieck site $\mathcal{C}$ with terminal sheaf $*$ will be called *connected* if $*$ is connected. However the functor π may arise, the results below depend only upon its *existence*.

2.2 Closed Model Structures and Hypercovers

Say that a category $\mathcal{C}$ is *small* if its class of morphisms $\mathrm{Mor}(\mathcal{C})$ forms a set. Any small Grothendieck site $\mathcal{C}$ admits a closed model structure on the associated category of simplicial (pre)sheaves where the cofibrations are monomorphisms, the weak equivalences are the local weak equivalences, and the fibrations are what will be called here the *global fibrations*. These closed model structures are due to Joyal in the sheaf case (in Joyal's famous letter to Grothendieck) and Jardine in the presheaf case; the reader is encouraged to refer to [Jar87] for terms not defined here. These are known as the "injective" model structures, and will sometimes be used in what follows.

A morphism $f\colon X \to Y$ of simplicial (pre)sheaves will be called a *local fibration* if it has the local right lifting property with respect to all the standard inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ of k-horns into the standard n-simplices for $n \geq 0$ (cf. [Jar86, §1] for a definition and discussion of the local right lifting property). A morphism $f\colon X \to Y$ of simplicial (pre)sheaves will be called a *local trivial fibration* if it is simultaneously a local fibration and a local weak equivalence; by a theorem of Jardine ([Jar87, 1.12]) these are exactly the morphisms of simplicial (pre)sheaves having the local right lifting property with respect to the standard inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ of boundaries of the standard n-simplices for $n \geq 0$. By a simple adjointness argument beginning with the observation that $\partial\Delta^n \cong \mathrm{sk}_{n-1}\Delta^n$, one sees that this local lifting property is equivalent to the assertion that the (pre)sheaf morphisms

$$X_0 \to Y_0$$
$$X_n \to \mathrm{cosk}_{n-1}X_n \times_{\mathrm{cosk}_{n-1}Y_n} Y_n$$

are local epimorphisms for $n \geq 1$. This is true in particular when $Y = K(Z, 0)$, the constant (or "discrete") simplicial (pre)sheaf associated with a (pre)sheaf Z. When Z is a scheme and X is representable by a simplicial scheme this amounts to the classical definition of a hypercover $f\colon X \to Z$ (cf. [AM69]; these observations appeared in [Jar94]). For this reason and others it is now standard to call any local trivial fibration

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of simplicial (pre)sheaves on a small Grothendieck site $\mathcal{C}$ a *hypercover*. This is what is meant by the term"hypercover" in the remainder of this paper.

2.3 Torsors and Homotopy Theory

Recall that a *torsor* X for a sheaf of groups G on a small Grothendieck site $\mathcal{C}$ is a sheaf X with an G-action such that there is a sheaf epi $U \twoheadrightarrow *$ to the terminal sheaf $*$ of $\mathcal{C}$ and a G-equivariant sheaf isomorphism $X \times U \cong G \times U$ called a *trivializaton* of X along U. This implies that the sheaf-theoretic quotient X/G is isomorphic to the terminal sheaf $*$. The simplicial sheaf $EG \times_G X$ is defined in sections U as the nerves of the translation categories $E_{G(U)}(X(U))$ for the actions of $G(U)$ on $X(U)$; each such category is a groupoid, so $\widetilde{\pi}_n(EG \times_G X) \cong 0$ for $n \geq 2$ and, as the isotropy groups locally vanish and the action is locally transitive, there is a local weak equivalence $EG \times_G X \simeq *$ (here $\widetilde{\pi}_n$ denotes the sheaf of homotopy groups in degree n; for the definition of a translation category see [GJ99, 1.8, IV]). As there is an isomorphism of sheaves $\widetilde{\pi}_0(EG \times_G X) \cong X/G$ one sees that X is a G-torsor if and only if the map $EG \times_G X \to *$ of simplicial sheaves is a local weak equivalence (this is another observation of Jardine; cf. [Jar09a, 3.1]). The maps $X \to Y$ of G-torsors are G-equivariant maps of sheaves, induced as fibres of comparisons of local fibrations $EG \times_G X \to BG$ (resp. for Y), and hence are local weak equivalences of constant simplicial sheaves, and thus are isomorphisms (following the notes of Jardine). The category of G-torsors on $\mathcal{C}$ is therefore a groupoid denoted by G-Tors($\mathcal{C}$); its path component set is denoted $H^1(\mathcal{C}, G)$, and this is the definition of nonabelian H^1 of $\mathcal{C}$ with coefficients in G. This set is pointed by the isomorphism class of the trivial G-torsor represented by G itself.

It has been known at least since [AM69] appeared that the étale fundamental group $\pi_1^{\text{ét}}(X, x)$ based at some geometric point x determines $H^1(\mathcal{C}, H)$ for constant sheaves of discrete groups H where $\mathcal{C} := \text{ét}(X)$, the étale site of a connected locally noetherian scheme X pointed by x (*i.e.,* where the covering families are taken to be surjective sums of étale morphisms; here "étale" is not taken to include "finite"). The following is a quick homotopy-theoretic argument to establish this. It is based on an earlier argument of Jardine made in the setting of a Galois category in the sense of [SGA03, V]. First, we have a lemma.

Lemma 2.1 *For any connected, small, Grothendieck site $\mathcal{C}$ and any hypercover $U \to *$ of the terminal sheaf, one has $\pi_0(\Pi U) \cong *$.*

Proof The canonical map $U \to *$ induces a map backwards

$$[*, K(\Gamma^* S, 0)] \to [U, K(\Gamma^* S, 0)]$$

in the homotopy category for any set S; this is an isomorphism, since $U \to *$ is a hypercover. On the other hand U and $*$ are cofibrant and $K(\Gamma^* S, 0)$ is globally fibrant, so these determine isomorphisms

$$\pi(*, K(\Gamma^* S, 0)) \cong \pi(U, K(\Gamma^* S, 0)),$$

where π denotes taking simplicial homotopy classes of maps. One has $\Pi(*) = *$, since $\mathcal{C}$ is connected, but then by adjunction $\pi(*, K(S, 0)) \cong \pi(\Pi U, K(S, 0))$, so

$\mathrm{Hom}(*, S) \cong \mathrm{Hom}(\pi_0(\Pi U), S)$ for any set S. Setting $S = \{0, 1\}$ completes the argument. ■

One may paraphrase this by saying that any hypercover of the terminal sheaf on a connected site is automatically path-connected.

Proposition 2.2 *For any connected, pointed, small, Grothendieck site $\mathcal{C}$ there are bijections*

$$\pi_{\mathrm{cts}}(\pi_1^{\text{ét}}(\mathcal{C}), H) \cong H^1(\mathcal{C}, H)$$

natural in discrete groups H, where $\pi_1^{\text{ét}}(\mathcal{C})$ is the étale fundamental group à la Artin–Mazur constructed by means of pointed (not necessarily representable) hypercovers, and where π_{cts} is defined as in the proof.

Proof There is a sequence of identifications

$$\begin{aligned}
\pi_{\mathrm{cts}}(\pi_1^{\text{ét}}(\mathcal{C}), H) &:= \varinjlim_{(U,u)\in \mathrm{HR}_*(\mathcal{C})} \pi(\pi_1(\Pi U, u), H) \\
&\cong \varinjlim_{(U,u)\in \mathrm{HR}_*(\mathcal{C})} \pi(\Pi U, BH) \\
&\cong \varinjlim_{(U,u)\in \mathrm{HR}_*(\mathcal{C})} \pi(U, B\Gamma^* H) \\
&\cong [*, B\Gamma^* H] \\
&\cong \pi_0(H\text{-}\mathrm{Tors}(\mathcal{C})) := H^1(\mathcal{C}, H),
\end{aligned}$$

where the transition from simplicial homotopy classes of maps to homotopy classes of maps is the generalized Verdier hypercovering theorem applied to the locally fibrant objects U and $B\Gamma^* H$ (cf. [Jar09c, Theorem 3]); $\mathrm{HR}_*(\mathcal{C})$ is the category of (geometrically) pointed hypercovers of the terminal sheaf (implicitly over the basepoint of $\mathcal{C}$) and pointed simplicial homotopy classes of maps between them; B denotes the sectionwise nerve functor; and H-$\mathrm{Tors}(\mathcal{C})$ denotes the groupoid of H-torsors associated with the constant sheaf of groups $\Gamma^* H$. ■

Lemma 2.1 was used in a subtle way to conflate the fundamental groupoid of ΠU with the fundamental group of $(\Pi U, u)$, as they are homotopy equivalent (in the sense of taking their nerves) in this case. This argument applies to any topology, not just the étale, and so is valid for the analogues of $\pi_1^{\text{ét}}$ in any other setting where one can talk about hypercovers in a pointed connected site.

The converse question of whether the pointed sets $H^1(\mathcal{C}, H)$ together with their naturality in discrete groups H determine $\pi_1^{\text{ét}}(\mathcal{C})$ is more difficult to answer. Proposition 2.2 may be interpreted as establishing that the functor

$$H^1(\mathcal{C}, -)\colon \pi\mathbf{Grp} \to \mathbf{Set}$$

is pro-representable by $\pi_1^{\text{ét}}(\mathcal{C})$, where $\pi\mathbf{Grp}$ is the category of groups and *simplicial homotopy classes* of homomorphisms between them (known elsewhere as homomorphisms up to conjugacy or "exterior" homomorphisms). As representing pro-objects are unique up to (unique) pro-isomorphism ([SGA72, 8.2.4.8, I]), one knows that $H^1(\mathcal{C}, -)$ determines $\pi_1^{\text{ét}}(\mathcal{C})$ up to (unique) pro-isomorphism in $\pi\mathbf{Grp}$, but one does not know at this stage of the argument that it is determined up to (unique) pro-isomorphism in $\mathbf{Grp}$. In the case that H need only vary over abelian groups this problem does not arise since the conjugacy relation degenerates. *Geometrically pointed torsors* (or just "pointed torsors" for short) were introduced precisely to fix this problem wherever Galois theory (in the modern topos theoretic sense) is available.

3 Pointed Torsors and Galois Theory

3.1 Pointed Torsors

Geometrically pointed torsors may be motivated by the following simple example from the Galois theory of fields. Fix a base field k and choose a separably closed extension $x\colon k \hookrightarrow \Omega$. This determines a geometric point

$$x\colon \operatorname{Spec}\Omega \to \operatorname{Spec} k$$

of the associated scheme $\operatorname{Spec} k$. Considering any Galois extension $f\colon k \hookrightarrow L$ of k, the possible lifts y (*i.e.,* geometric points) making the associated diagram

$$\begin{tikzcd} & \operatorname{Spec} L \arrow[d, "f"] \\ \operatorname{Spec}\Omega \arrow[ur, "y"] \arrow[r, "x"'] & \operatorname{Spec} k \end{tikzcd}$$

commute coincide with the possible embeddings $L \hookrightarrow \Omega$ over k, which are of course permuted by the action of the associated Galois group $\operatorname{Gal}(L/k)$. The scheme $\operatorname{Spec} L$ represents a $\operatorname{Gal}(L/k)$-torsor trivializing over itself; however, if one requires that the Galois action respect a fixed choice of lift (*i.e.,* one makes a fixed choice of geometric point $y\colon \operatorname{Spec}\Omega \to \operatorname{Spec} L$ over x), then only the trivial action is possible, as the associated embedding has been fixed. This property of having no nontrivial pointed automorphisms means that groupoids of such pointed torsors over k and pointed equivariant maps between them are particularly simple from a homotopy theoretic point of view. They are disjoint unions of contractible path components.

Suppose $\mathcal{C}$ is some site of schemes having finite limits and finite coproducts that contains a separably closed field $\Omega := \operatorname{Spec}\Omega$. So long as the topology is defined by families of maps that are stable under pullback, Verdier's criterion from [SGA72, Exposé III], applies and determines a site morphism $x\colon \mathcal{C}/\Omega \to \mathcal{C}/X$ associated with any map $x\colon \operatorname{Spec}\Omega \to X$ to some object X. If $\mathcal{C}$ is a site with the étale topology then this further determines a geometric morphism

$$x^*\colon \mathbf{Shv}(\text{ét}(X)) \leftrightarrows \mathbf{Shv}(\text{ét}(\Omega)) \simeq \mathbf{Set} : x_*,$$

where one recognizes x^* as the functor sending any étale sheaf on X to its stalk at the geometric point x. One readily verifies that the choice of a global section $u\colon * \to$

$x^*(U)$ associated to a sheaf U represented by an object $U \to X$ of ét(X) is equivalent to the choice of a geometric point $u\colon \operatorname{Spec}\Omega \to U$ over x.

More generally, suppose that $x^*\colon \mathbf{Shv}(\mathcal{C}) \leftrightarrows \mathbf{Set} : x_*$ is a geometric morphism with direct image functor x_* for some small Grothendieck site $\mathcal{C}$; such a pair $(\mathcal{C}, x)$ will be called a *pointed site* with basepoint x. The functor x^* is an inverse image functor so is exact and thus preserves local weak equivalences: by exactness x^* commutes with Kan's Ex^∞ functor and thus it suffices to prove the statement for locally fibrant objects, but then one may factor the map as a local weak equivalence right inverse to a hypercover followed by a hypercover (cf. *e.g.*, [GJ99, 8.4, II]), and one may directly check that x^* preserves hypercovers by using their definition in terms of coskeleta. As x^* is exact it also preserves the Borel construction $EH \times_H X$ associated with any H-torsor X for any sheaf of groups H on $\mathcal{C}$, so in summary it preserves the local weak equivalence $EH \times_H X \xrightarrow{\simeq} *$, and thus sends H-torsors on $\mathcal{C}$ to x^*H-torsors in **Set**.

A *geometrically pointed H-torsor* on the pointed site $(\mathcal{C}, x)$ for a sheaf of groups H is an H-torsor Y on $\mathcal{C}$ together with a global section $y\colon * \to x^*(Y)$. A *morphism* of (geometrically) pointed H-torsors is a morphism of the underlying H-torsors that respects the points. Lemma 1 of [Jar09b] establishes in particular that the groupoid $H\text{-Tors}(\mathcal{C})_x$ of pointed H-torsors on $\mathcal{C}$ is the homotopy fibre of the map

$$H\text{-Tors}(\mathcal{C}) \to x^*(H)\text{-Tors}(\mathbf{Set}).$$

Suppose now that H is a *constant* sheaf of groups. Then every pointed H-torsor on $\mathcal{C}$ is locally constant on $\mathcal{C}$, so the groupoids $H\text{-Tors}(\mathcal{C})_x$ for variable discrete groups H all belong to the subtopos $\mathbf{SLC}(\mathcal{C}) \subset \mathbf{Shv}(\mathcal{C})$ of sums of locally constant objects of $\mathbf{Shv}(\mathcal{C})$. The inverse image $x^*\colon \mathbf{Shv}(\mathcal{C}) \to \mathbf{Set}$ then restricts (cf. [Moe95]) to an inverse image $x^*\colon \mathbf{SLC}(\mathcal{C}) \to \mathbf{Set}$.

3.2 Digression on Galois Theory

Now assume that $(\mathcal{C}, x)$ is also connected. In [Moe89], Moerdijk showed that there are then pointed topos equivalences

$$\mathbf{SLC}(\mathcal{C}) \simeq \mathbf{Shv}(\mathbf{G}/\mathbf{U}) \simeq \mathcal{B}G$$

analogous to that of classical Galois theory, where G is a prodiscrete localic group (pro-object in the category of localic groups, which themselves are group objects in the category of locales). Here one may conflate the diagram G with its limit object $\varprojlim G$ in localic groups (in a sense this is the reason for considering localic rather than topological groups in this context). Here, *prodiscrete* means that G may be interpeted as a cofiltered diagram of discrete localic groups, just as classical Galois theory deals with limits of discrete topological groups. The site $\mathbf{G}/\mathbf{U}$ is that of the right cosets G/U for localic open subgroups U of G (these are discrete), with the obvious G-action and all G-equivariant maps between them as G-sets, where a G-set Z is a set Z with a G-action in the localic sense. The topos $\mathcal{B}G$ is that of all discrete G-sets, and in particular the latter equivalence asserts that any discrete G-set of the form G/U for a localic open subgroup U represents a sheaf on the site $\mathbf{G}/\mathbf{U}$, so the topology is subcanonical. It is known that any prodiscrete localic group corresponds to a pro-group with surjective transition maps, and the canonical maps $G \to G_i$ are all also surjective ([Moe89, 1.4]).

Under Moerdijk's equivalence the inverse image $x^*\colon \mathbf{SLC}(\mathcal{C}) \to \mathbf{Set}$ goes to the functor $g^*\colon \mathcal{B}G \to \mathbf{Set}$ that forgets the G-action [Moe95]. This functor obviously reflects epis, so it is faithful and thus the topos $\mathcal{B}G$ has enough points, namely $\{g\}$. This point g restricts to a forgetful functor $u\colon \mathbf{G/U} \to \mathbf{Set}$. Recall that if $\mathbf{Set}$ is equipped with the standard topology where the covering families are surjections, then there is an equivalence $\mathbf{Shv}(\mathbf{Set}) \simeq \mathbf{Set}$ determined by $F \mapsto F(*)$ on the one hand and $X \mapsto \operatorname{Hom}(-,X)$ on the other. Following [Jar09b], the direct image $u_* \simeq g_*\colon \mathbf{Set} \to \mathbf{Shv}(\mathbf{G/U})$ sends any set X to the sheaf $\operatorname{Hom}(u(-),X)$ on $\mathbf{G/U}$, and the left adjoint $u^* \simeq g^*\colon \mathbf{Shv}(\mathbf{G/U}) \to \mathbf{Set}$ is the left Kan extension defined by

$$u^*(F)(*) := \varinjlim_{*/X} F(X),$$

where the index category $*/X$ has maps $* \to X$ as objects, where the X are sets of the form G/U for open localic subgroups U and morphisms are commutative triangles over morphisms in the category $\mathbf{G/U}$. The identity elements $e\colon * \to G_i$ represent the pro-group G itself in this index category, and this subcategory is cofinal as usual, so one arrives at the useful characterization

$$u^*(F) = \varinjlim_{i} F(G_i).$$

The pointed topos $(\mathbf{Shv}(\mathbf{G/U}), u)$ has enough points, since it is pointed equivalent to $(\mathcal{B}G, g)$, and in fact the list $\{u\}$ suffices, since equivalences of categories are faithful functors.

A word of explanation: intuitively, the subtopos $\mathbf{SLC}(\mathcal{C})$ captures and isolates the covering space theory of $\mathbf{Shv}(\mathcal{C})$. The category of locally constant *finite* sheaves on the étale site of a scheme X is known (by descent theory) to be equivalent to the *finite* étale site of X. More generally, any locally constant sheaf of sets on $\text{ét}(X)$ for a scheme X is represented by a (not necessarily finite) étale map (cf. [SGA72, 2.2, Exposé IX, III]), so the connection with geometry is closer than one might expect a priori.

3.3 Characterization of $\pi_1^{\text{ét}}$

Under Moerdijk's equivalence the groupoid $H\text{-Tors}(\mathcal{C})_x$ of pointed H-torsors over x for any *constant* sheaf of groups H on a connected site $\mathcal{C}$ goes to a groupoid $H\text{-Tors}(\mathbf{G/U})_u$ of pointed H-torsors over u. Equivalences of categories preserve path components and pointed torsors over x and u have no nontrivial (pointed) automorphisms since x^* and u^* are faithful on $\mathbf{SLC}(\mathcal{C})$, so there are induced weak equivalences of groupoids

$$H\text{-Tors}(\mathcal{C})_x \simeq H\text{-Tors}(\mathbf{G/U})_u$$

natural in H. It therefore suffices to study the latter class of groupoids for the purpose of computing $\pi_0(H\text{-Tors}(\mathcal{C})_x)$. By [Jar09b, Lemma 1] the groupoid $H\text{-Tors}(\mathbf{G/U})_u$ is the homotopy fibre of the canonical map

$$u^*\colon H\text{-Tors}(\mathbf{G}/\mathbf{U}) \to H\text{-Tors}(\mathbf{Set}).$$

Lemma 2 and Corollary 11 of [Jar09b] in this context determine bijections

$$\pi_0(H\text{-Tors}(\mathcal{C})_x) \cong \pi_0(H\text{-Tors}(\mathbf{G}/\mathbf{U})_u) \cong \varinjlim_{i\in I} \operatorname{Hom}(\check{C}(G_i), BH)$$

natural in H, where I is the indexing category for the prodiscrete localic group G, $\check{C}(G_i)$ is the simplicial Čech resolution associated with the G-equivariant epi $G_i \twoheadrightarrow *$, and all the elements of $\operatorname{Hom}(\check{C}(G_i), BH)$ are automatically pointed maps, since H is a constant sheaf of groups (rather than groupoids). Example 13 of [Jar09b] applies to give bijections

$$H^1_x(\mathcal{C}, H) := \pi_0(H\text{-Tors}(\mathcal{C})_x) \cong \varinjlim_{i} \operatorname{Hom}(G_i, H)$$

natural in H, and the latter may be further identified with the set $\operatorname{Hom}_{\text{loc.}}(G, H)$ of maps of prodiscrete localic groups from G to H to complete the analogy with the case of ordinary torsors discussed above.

The upshot of all this is summarized in the following proposition.

Proposition 3.1 *Suppose $(\mathcal{C}, x)$ is a connected (thus locally connected), small, Grothendieck site pointed by a choice of geometric morphism $x\colon \mathbf{Set} \to \mathbf{Shv}(\mathcal{C})$ ("point" in topos language). Then the pro-group G associated with the subtopos $\mathbf{SLC}(\mathcal{C})$ of $\mathbf{Shv}(\mathcal{C})$ is determined up to unique pro-isomorphism by the pointed nonabelian $H^1_x(\mathcal{C}, -)$ functor associated with x.*

Proof After the above discussion one must only note that $H^1_x(\mathcal{C}, -)$ is pro-representable by G and that representing pro-objects are unique up to unique pro-isomorphism. ■

The pro-group G here is (up to descent arguments for pointed H-torsors) what Grothendieck called the "pro-groupe fondamental élargi" based at x ([SGA70, p. 110, Book 2]). The expected explicit identification of G is given by $G \cong \pi_1^{\text{ét}}(\mathcal{C}, x)$, where the latter group is the fundamental pro-group of Artin–Mazur defined by means of pointed representable hypercovers. The standard reference for this identification is [AM69, §10], but unfortunately the argument there boils down to the characterization of unpointed nonabelian $H^1(|\Pi K|, H)$ for hypercovers K in classical topological covering space theory, so it cannot be considered to give a complete characterization of G. In other words, it runs into the same "homomorphisms up to simplicial homotopy" versus "actual homomorphisms" problem mentioned after Proposition 2.2.

Aspects of the statement $\pi_1^{\text{ét}}(\text{ét}(X), x) \cong G$ for small étale sites $\text{ét}(X)$ of connected locally noetherian schemes or DM stacks have also been considered in [SGA70, p. 111, Book 2]. There, Grothendieck considers pointed descent data for split pointed H-torsors, and shows that such sets are representable by fundamental groups of certain pointed simplicial sets. Unfortunately the argument there does not appear to address the issue of representability of descent data when H is infinite. The following argument is an alternative homotopy-theoretic approach to these matters.

Fix a pointed, small, Grothendieck site $(\mathcal{C}, x)$, a constant sheaf of groups H on $\mathcal{C}$ with stalk also denoted H, and a pointed representable sheaf epi $(U, u) \twoheadrightarrow *$ of $\mathbf{Shv}(\mathcal{C})$. Let $?_{|U} \colon \mathbf{Shv}(\mathcal{C}) \to \mathbf{Shv}(\mathcal{C}/(U, u))$ denote the corresponding restriction functor; this functor is exact (right adjoint of a topos morphism and it preserves sheaf epis), so it sends pointed H-torsors on $\mathcal{C}$ (with respect to x) to pointed H-torsors (with respect to u) on $\mathcal{C}/(U, u)$, and morphisms of pointed H-torsors on $\mathcal{C}$ to morphisms of pointed H-torsors on $\mathcal{C}/(U, u)$. Let F_U denote the homotopy fibre of the induced map

$$H\text{-Tors}_x \xrightarrow{?_{|U}} H\text{-Tors}_u$$

of groupoids, where $H\text{-Tors}_x$ is the groupoid of pointed H-torsors on $\mathcal{C}$ with respect to x and $H\text{-Tors}_u$ is the groupoid of pointed H-torsors on $\mathcal{C}/(U, u)$ with respect to u. Then the objects of F_U are morphisms $H \to T_{|U}$ of *pointed* H-torsors (the trivial H-torsor H being pointed by $e \in H$), and the morphisms are commutative triangles

$$\begin{tikzcd} & T_{|U} \arrow[d, "m_{|U}"] \\ H \arrow[ur] \arrow[r] & T'_{|U} \end{tikzcd}$$

of morphisms of pointed H-torsors on $\mathcal{C}/(U, u)$, where $m \colon T \to T'$ is a morphism of pointed H-torsors on $\mathcal{C}$. The objects of F_U correspond exactly to pointed trivializations $\sigma \colon (U, u) \to T$ so that F_U is equivalent to the groupoid F_U whose objects are pointed trivializations of the form σ and whose morphisms are commutative triangles

$$\begin{tikzcd} & T \arrow[d, "m"] \\ (U, u) \arrow[ur, "\sigma"] \arrow[r, "\sigma'"] & T' \end{tikzcd}$$

of pointed maps in $\mathcal{C}$, where m is a morphism of pointed H-torsors. The map $F_U \to H\text{-Tors}_x$ is that which forgets the pointed trivializations.

Lemma 3.2 *With the definitions above, there are bijections*

$$\pi_0(F_U) \cong \mathrm{Hom}(\check{C}(U, u), BH),$$

natural in discrete groups H, where $\check{C}(U, u)$ is the pointed Čech resolution associated with the pointed sheaf epi $(U, u) \twoheadrightarrow *$ of $\mathcal{C}$.

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Proof By [Jar09b, Lemma 2] the path components of F_U correspond to the path components of the cocycle category $h_{\check{C}}(*, BH)_x$ of those pointed Čech cocycles under (U, u). Each pointed trivialization $\sigma\colon (U, u) \to T$ determines a pointed Čech cocycle

$$* \xleftarrow{\simeq} \check{C}(U, u) \xrightarrow{\sigma_*} \check{C}(T) \to BH.$$

Such cocycles are initial in their respective path components of $h_{\check{C}}(*, BH)_x$, so these path components correspond to certain maps $\check{C}(U, u) \to BH$. Any element in $\mathrm{Hom}(\check{C}(U, u), BH)$ determines a pointed H-torsor T equipped with a fixed pointed trivialization σ by (U, u), so this correspondence is surjective, hence bijective. ■

In order to study torsor trivializations on $\mathcal{C}$, it is convenient to introduce a well-ordered category E_x whose objects are representable pointed sheaf epimorphisms $U_i \twoheadrightarrow *$ on $\mathcal{C}$ to the terminal sheaf, whose morphisms are fixed choices of pointed epis $U_j \twoheadrightarrow U_i$ of objects for $j > i$, and with the property that any constant group torsor on $\mathcal{C}$ trivializes on some such object $U_i \twoheadrightarrow *$ (hence also on any object with a larger index). The key property of E_x is that it is (co)filtered.

Say that a pointed H-torsor (T, t) *admits a pointed trivialization* if there exists a pointed representable sheaf epi $(U, u) \twoheadrightarrow *$ and a pointed section $\sigma\colon (U, u) \to T$. Pointed trivializations exist in most cases of interest.

Lemma 3.3 *Suppose $(\mathcal{C}, x)$ is a pointed small Grothendieck site with a subcanonical topology (*i.e.*, representable presheaves are sheaves) and arbitrary small coproducts. Then any pointed H-torsor for a sheaf of groups H on $\mathcal{C}$ admits a pointed trivialization by a representable sheaf epi.*

Proof Any pointed H-torsor (T, t) admits some ordinary trivialization $\sigma\colon V \to T$ from a sheaf epi $V \twoheadrightarrow *$, but then the composite $V \to T \to *$ is an epi so that the canonical map $T \to *$ is also an epi. As a sheaf, T is a colimit of representable presheaves U_i for some small index category I, which are sheaves since the topology is subcanonical, and there is a sheaf epi $U := \sqcup_{i \in I} U_i \twoheadrightarrow T$ so that the composite $U \to T \to *$ is a sheaf epi. In particular the map $U \twoheadrightarrow T$ is a representable trivialization of T, and as it is a sheaf epi, any point $t \in x^*(T)$ lifts to some point $u \in x^*(U)$, so that $(U, u) \twoheadrightarrow T$ is a pointed trivialization of T. ■

This is true in particular for pointed sites of connected locally noetherian schemes and DM stacks with nonfinite étale topologies. The homotopy long exact sequence associated with the fibre sequence

$$F_U \to H\text{-Tors}_x \to H\text{-Tors}_u$$

in groupoids for a representable pointed sheaf epi $(U, u) \twoheadrightarrow *$ has the form

$$1 \to \pi_1(H\text{-Tors}_u) \to \pi_0(F_U) \to \pi_0(H\text{-Tors}_x) \to \pi_0(H\text{-Tors}_u)$$

for any *constant* sheaf of groups H on a pointed connected site $(\mathcal{C}, x)$, since the objects of $H\text{-Tors}_x$ have no pointed automorphisms. The trivial pointed torsor H on $\mathcal{C}/(U, u)$ is represented by $H \times U$ pointed by $e \times u$ over x with its canonical pointed projection $\mathrm{pr}_u \colon H \times U \to U$. The sheaf $H \times U$ is obviously locally constant, since H is constant, so it lives in $\mathbf{SLC}(\mathcal{C})$, but then it has no nontrivial pointed (U, u)-automorphisms (despite the fact that U may be disconnected) as any such must be the identity on x^*, and x^* is faithful for $\mathbf{SLC}(\mathcal{C})$ by Galois theory. Thus $\pi_1(H\text{-Tors}_u)$ vanishes at the basepoint given by the trivial H-torsor, so the sequence above always reduces to an exact sequence

$$1 \to \pi_0(F_U) \to \pi_0(H\text{-Tors}_x) \to \pi_0(H\text{-Tors}_u).$$

Lemma 3.4 *With the definitions above there are bijections*

$$\varinjlim_{(U,u)\in E_x} \pi_0(F_U) \cong \pi_0(H\text{-Tors}_x) := H^1_x(\mathcal{C}, H),$$

natural in discrete groups H, whenever pointed H-torsors on $(\mathcal{C}, x)$ admit pointed trivializations by representable sheaf epis for all constant sheaves of discrete groups H.

Proof Taking the filtered colimit over E_x of the homotopy short exact sequences of pointed sets above, one obtains a short exact sequence

$$1 \to \varinjlim_{E_x} \pi_0(F_U) \to \pi_0(H\text{-Tors}_x) \to *$$

of pointed sets, since one has $\varinjlim_{E_x} \pi_0(H\text{-Tors}_u) \cong *$ as every pointed H-torsor on any $\mathcal{C}/(U, u)$ pointed trivializes on a sufficiently fine choice of pointed sheaf epi $(V, v) \twoheadrightarrow *$ dominating (U, u). The middle map is therefore surjective by exactness. The filtered colimit of the maps $\pi_1(H\text{-Tors}_x) \to \pi_1(H\text{-Tors}_u)$ for any choice of basepoint of $H\text{-Tors}_x$ is surjective since all non-trivial torsors eventually pointed trivialize and one knows that the trivial H-torsor on any (U, u) has no nontrivial pointed automorphisms by the above discussion. Therefore the colimit map $\pi_0(\varinjlim_{E_x} F_U) \to \pi_0(H\text{-Tors}_x)$ must be injective by Lemma 4.1. The middle map must therefore be bijective, as was to be shown. ■

Given a category E_x of sheaf epis as above, one may define a "Čech variant" $\pi_1^{\text{ét}}(\mathcal{C}, x)_{E_x}$ of the usual $\pi_1^{\text{ét}}(\mathcal{C}, x)$ by sending any object $(U, u) \twoheadrightarrow *$ of E_x to $\pi_1(\Pi\check{C}(U, u))$. As E_x is cofiltered this gives a well-defined pro-group. The following Theorem gives a characterization of G by this Čech version $\pi_1^{\text{ét}}(\mathcal{C}, x)_{E_x}$ of $\pi_1^{\text{ét}}$.

Theorem 3.5 *Suppose $(\mathcal{C}, x)$ is a connected, pointed, small, Grothendieck site with finite limits and arbitrary small coproducts, where pointed H-torsors admit pointed trivializations by representable sheaf epis for all constant sheaves of discrete groups H. Then the pro-groupe fondamental élargi G associated with the full subtopos* $\mathbf{SLC}(\mathcal{C})$ *of* $\mathbf{Shv}(\mathcal{C})$ *via Galois theory is (uniquely) pro-isomorphic to $\pi_1^{\text{ét}}(\mathcal{C}, x)_{E_x}$.*

Proof The strategy is to demonstrate that $\pi_1^{\text{ét}}(\mathcal{C}, x)_{E_x}$ pro-represents the functor $H^1_x(\mathcal{C}, -)$ so that it must be pro-isomorphic to G. By Lemmas 3.2 and 3.4 one has

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identifications

$$\begin{aligned}\operatorname{Hom}_{\mathrm{cts}}(\pi_1^{\text{ét}}(\mathcal{C},x)_{E_x},H) &\cong \varinjlim_{E_x}\operatorname{Hom}(\pi_1(\Pi\check{C}(U,u)),H)\cong \varinjlim_{E_x}\operatorname{Hom}(\check{C}(U,u),B\Gamma^*H)\\ &\cong \varinjlim_{E_x}\pi_0(F_U)\cong \pi_0(H\text{-}\mathrm{Tors}_x):=H^1_x(\mathcal{C},H),\end{aligned}$$

natural in discrete groups H, so the result follows. ∎

This theorem also establishes that $\pi_1^{\text{ét}}(\mathcal{C},x)_{E_x}$ does not depend (up to unique pro-isomorphism) on the choice of the category E_x; from now on this pro-group will therefore be denoted $\pi_1^{\text{ét}}(\mathcal{C},x)_{\check{C}}$. Note also the lack of any assumptions about the topology on $\mathcal{C}$ (despite the use of the word "étale" which is present here only for historical reasons) and the lack of any explicit descent theory. The following comparison result shows that it often suffices to use sheaf-theoretic hypercovers to study $\pi_1^{\text{ét}}(\mathcal{C},x)$:

Proposition 3.6 *Suppose $(\mathcal{C},x)$ is a pointed connected small Grothendieck site with a subcanonical topology (*i.e., *representable presheaves are sheaves), finite limits, and arbitrary small coproducts. Then there is a pro-isomorphism*

$$\pi_1^{\text{ét}}(\mathcal{C},x)\cong \pi_1^{\text{ét}}(\mathcal{C},x)^{\mathrm{rep}},$$

where the left-hand side is the fundamental pro-group defined by means of pointed hypercovers, and the right-hand side is the classical fundamental pro-group defined by means of pointed representable *hypercovers.*

Proof The essential fact here is that any hypercover $U\to *$ of the terminal sheaf $*$ may be dominated by a representable hypercover $V\to *$ (cf. [Jar94, Lemma 2.2]). A careful reading of the argument there shows that the representable simplicial scheme X at the base may be replaced by the terminal sheaf $*$ in the present context without losing representability of the split hypercover, even when the terminal sheaf $*$ itself is not representable (*e.g.*, on certain fibred sites), as it is a constant simplicial object.

The argument of [Jar94] is inductive and begins with the observation that any sheaf epi $U\twoheadrightarrow *$ to the terminal sheaf may be dominated by a representable sheaf epi: every presheaf (in particular every sheaf U) is a colimit of representable presheaves U_i (which are sheaves, since the topology is subcanonical) on some small index category I. This colimit appears as the coequalizer

$$\bigsqcup_{k\colon i\to j\in I}U_i\rightrightarrows\bigsqcup_{j\in I}U_j\twoheadrightarrow U,$$

so, in particular, the indicated map is always an epi, and thus it determines a commutative triangle

$$\begin{tikzcd} & U \arrow[d, two heads] \\ \bigsqcup_{j\in I}U_j \arrow[ur] \arrow[r, two heads] & * \end{tikzcd}$$

of epis whenever $U \to *$ is a sheaf epi. In particular, the map

$$V := \coprod_{j \in I} U_j \twoheadrightarrow U$$

is an epi in this case so that $x^*(V) \to x^*(U)$ is also an epi by exactness, thus any point $u_x \in x^*(U)$ lifts to a point $v_x \in x^*(V)$, and therefore any *pointed* sheaf epi may be dominated by a *pointed* representable sheaf epi.

Thus any pointed hypercover is dominated by some pointed *representable* hypercover. Now the category $\mathrm{HR}_*(\mathcal{C})$ of pointed hypercovers of the terminal sheaf of $\mathcal{C}$ and pointed simplicial homotopy classes of maps between them has products and equalizers, so the result follows by a cofinality argument. ∎

One may also wonder what happens if $\pi_1^{\text{ét}}(\mathbf{SLC}(\mathcal{C}), x)_{\check{C}}$ is computed directly instead of $\pi_1^{\text{ét}}(\mathcal{C}, x)_{\check{C}}$.

Proposition 3.7 *Suppose $(\mathcal{C}, x)$ is a pointed, connected, small, Grothendieck site having all finite limits, arbitrary coproducts, and a subcanonical topology. Then the pro-group G associated with $\mathbf{SLC}(\mathcal{C}, x)$ is (uniquely) pro-isomorphic to the pro-group $\pi_1^{\text{ét}}(\mathbf{SLC}(\mathcal{C}), x)_{\check{C}}$ defined by means of pointed representable Čech hypercovers in $\mathbf{SLC}(\mathcal{C})$.*

Proof Choosing a suitable category E_x of pointed representable sheaf epis for $\mathbf{SLC}(\mathcal{C})$, one directly calculates

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\text{ét}}(\mathbf{SLC}(\mathcal{C})_{\check{C}}, x), H) &\cong \varinjlim_{E_x} \mathrm{Hom}(\pi_1(\Pi\check{C}(U,u)), H) \\
&\cong \varinjlim_{E_x} \mathrm{Hom}(\check{C}(U,u), B\Gamma^* H) \\
&\cong \varinjlim_{(Y,y)\in \mathrm{Gal}(\mathbf{SLC}(\mathcal{C}))} \mathrm{Hom}(\check{C}(Y,y), B\Gamma^* H) \\
&\cong \varinjlim_{(G_Y,\, e_{G_Y})} \mathrm{Hom}(\check{C}(G_Y), B\Gamma^* H) \\
&\cong \varinjlim_{G_Y} \mathrm{Hom}(G_Y, H) := \mathrm{Hom}_{\mathrm{loc.}}(G, H),
\end{aligned}$$

where the $(U,u) \twoheadrightarrow *$ are pointed representable sheaf epis over x, the $(Y,y) \in \mathbf{SLC}(\mathcal{C})$ are pointed Galois objects for $(\mathbf{SLC}(\mathcal{C}), x)$ that are cofinal among such sheaf epis by Galois theory (cf. [Dub04, Prop. 3.1.1, Thm. 3.3.8]), the $G_Y := \mathrm{Gal}(Y)$ are the associated Galois groups regarded as discrete localic groups, and $\check{C}(U,u)$ is the Čech construction for any pointed sheaf epi $(U,u) \twoheadrightarrow *$. As these bijections are natural in H the result follows. ∎

A variation of Proposition 3.7 was previously given by Artin-Mazur ([AM69, §9]), who used a slightly different method.

The above results also apply to the version of $\pi_1^{\text{ét}}$ constructed from *arbitrary* pointed representable hypercovers whenever the site $\mathcal{C}$ admits a "rigid diagram of pointed hypercovers": a small diagram $\mathcal{S}$ of pointed representable hypercovers in $\mathcal{C}$ will be called a *rigid diagram of pointed hypercovers* if

(i) $\mathcal{S}$ is cofiltered (*i.e.*, without first passing to pointed simplicial homotopy classes of maps between the objects); and

(ii) the functor $\mathcal{S} \to \mathrm{HR}_*(\mathcal{C})$ obtained by passing to pointed simplicial homotopy classes of maps is cofinal such that for every object U of $\mathrm{HR}_*(\mathcal{C})$ there is an object S of $\mathcal{S}$ and a pointed map $S \to U$ that is surjective in degree 0.

The canonical example of such a diagram S is that of the *pointed rigid hypercovers* of the small étale site of a connected locally noetherian scheme X (cf. [Fri82, Proof of Prop. 4.5]).

Proposition 3.8 *With the assumptions of Theorem* 3.5*, assume that $\mathcal{C}$ admits a rigid diagram $\mathcal{S}$ of pointed hypercovers. Then the pro-groupe fondamental élargi G is (uniquely) pro-isomorphic to the étale fundamental pro-group $\pi_1^{\text{ét}}(\mathcal{C}, x)$ constructed by means of pointed representable hypercovers.*

Proof As $\mathcal{S}$ is cofinal in $\mathrm{HR}_*(\mathcal{C})$, one may reindex $\pi_1^{\text{ét}}(\mathcal{C}, x)$ by $\mathcal{S}$. Then one has identifications

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\text{ét}}(\mathcal{C}, x), H) &\cong \varinjlim_{(U,u)\in S} \mathrm{Hom}(\pi_1(\Pi(U,u)), H)\\
&\cong \varinjlim_{(U,u)\in S} \mathrm{Hom}((U,u), B\Gamma^* H)\\
&\cong \varinjlim_{(U,u)\in S} \mathrm{Hom}(\check{C}(U_0,u), B\Gamma^* H)\\
&\cong \varinjlim_{(U,u)\in S} \pi_0(F_{U_0}) \cong \pi_0(H\text{-}\mathrm{Tors}_x) := H_x^1(\mathcal{C}, H),
\end{aligned}$$

natural in discrete groups H, where the third bijection follows from the proof of [Jar89, Prop. 1.1], and the identification of the colimit of $\pi_0(F_{U_0})$ with $\pi_0(H\text{-}\mathrm{Tors}_x)$ follows by the same argument as Lemma 3.4 since $\mathcal{S}$ is cofiltered. ■

3.4 Application to Finite Étale Sites

As a consequence of [SGA03, V] and [Noo04, 4.2] one has pointed equivalences

$$\mathbf{Shv}(\mathbf{Fin\acute{e}t}(X), x) \simeq \mathbf{Shv}(\pi_1^{\mathrm{Gal}}(X,x)\text{-}\mathbf{Set}_{df}) \simeq \mathcal{B}\pi_1^{\mathrm{Gal}}(X,x)$$

whenever $(\mathbf{Fin\acute{e}t}(X), x)$ is the *finite* étale site of a connected locally noetherian scheme or DM stack with geometric point x, where $\pi_1^{\mathrm{Gal}}(X,x)$ denotes the profinite fundamental group of $(\mathbf{Fin\acute{e}t}(X), x)$ determined by Grothendieck's general theory of Galois categories. Techniques similar to the proof of Proposition 3.7 establish that $\pi_1^{\mathrm{Gal}}(X,x)$ pro-represents the functor

$$H_x^1(\mathbf{Fin\acute{e}t}(X), -)\colon \mathbf{FinGrp} \to \mathbf{Set},$$

where **FinGrp** is the category of finite groups and group homomorphisms between them. Any finite group torsor on $\text{ét}(X)$ automatically belongs to $\textbf{Finét}(X)$, so $H^1_x(\text{ét}(X), -)$ and $H^1_x(\textbf{Finét}(X), -)$ are isomorphic as functors on **FinGrp**. The following proposition establishes a relationship between the finite and nonfinite étale sites.

Proposition 3.9 *Suppose $(\text{ét}(X), x)$ is the pointed (nonfinite) étale site of a connected locally noetherian scheme. Then the profinite Grothendieck fundamental group $\pi_1^{\text{Gal}}(X, x)$ associated with $\textbf{Finét}(X)$ is (uniquely) pro-isomorphic to $\pi_1^{\text{ét}}(\text{ét}(X), x)^\frown$, the profinite completion of the classical étale homotopy pro-group $\pi_1^{\text{ét}}(\text{ét}(X), x)$ constructed by means of pointed representable hypercovers.*

Proof As above $\pi_1^{\text{Gal}}(X, x)$ pro-represents the functor

$$H^1_x(\text{ét}(X), -)\colon \textbf{FinGrp} \to \textbf{Set},$$

and on the other hand $\pi_1^{\text{ét}}(\text{ét}(X), x)$ is known to pro-represent

$$H^1_x(\text{ét}(X), -)\colon \textbf{Grp} \to \textbf{Set},$$

by Theorem 3.5 and Proposition 3.8. Therefore, $\pi_1^{\text{ét}}(\text{ét}(X), x)^\frown$ also pro-represents the former functor, so it must be pro-isomorphic to $\pi_1^{\text{Gal}}(X, x)$. ■

4 Homotopy Pullbacks of Groupoids and π_0

The results below will require some basic facts about the path components functor π_0 as it pertains to pullbacks of groupoids. The following Lemma exemplifies the utility of representing a homotopy fibre sequence by a diagram that commutes on the nose.

Lemma 4.1 *Suppose $f\colon G \to H$ is a map of groupoids inducing surjections $\pi_1(G, x) \twoheadrightarrow \pi_1(H, f(x))$ on all fundamental groups; y is an object of H, and F_y is the homotopy fibre of f over y. Then the induced map $\pi_0(f)\colon \pi_0(F_y) \to \pi_0(G)$ is an injection.*

Proof Form the pullback square

$$\begin{tikzcd}
G \times_H H^I \arrow[r, "f_*"] \arrow[d, "d_{1*}"] & H^I \arrow[d, "d_1"] \\
G \arrow[r, "f"'] \arrow[u, bend left, "s"] & H
\end{tikzcd}$$

and note the section s (thus weak equivalence) of d_{1*} defined by sending any object x of G to $1\colon f(x) \to f(x)$ in $G \times_H H^I$. Generally, an object of $G \times_H H^I$ is an object

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x of G together with a morphism $\omega\colon f(x) \to z$ of H, and a morphism is a pair of morphisms (m, n) making the square

$$\begin{tikzcd} f(x) \arrow[r, "\omega"] \arrow[d, "f(m)"'] & z \arrow[d, "n"] \\ f(x') \arrow[r, "\omega'"] & z' \end{tikzcd}$$

commute in H. Then $\pi = d_0 f_*\colon G \times_H H^I \to H$ is a fibration (functorially) replacing f and the pullback

$$\begin{tikzcd} F_y \arrow[r] \arrow[d] & G \times_H H^I \arrow[d, "\pi"] \\ * \arrow[r, "y"] & H \end{tikzcd}$$

defines a model F_y for the homotopy fibre over y. Since $\pi s = f$, the objects of F_y look like $\omega\colon f(x) \to y$, and the morphisms are triangles

$$\begin{tikzcd} f(x) \arrow[rr, "f(m)"] \arrow[dr, "\omega"'] & & f(x') \arrow[dl, "\omega'"] \\ & y & \end{tikzcd}$$

that commute on the nose. Suppose $\omega\colon f(x) \to y, \omega'\colon f(x') \to y$ are two objects of F_y in the same path component of $G \times_H H^I$. Then there is a commutative square

$$\begin{tikzcd} f(x) \arrow[r, "\omega"] \arrow[d, "f(m)"'] & y \arrow[d, "n"] \\ f(x') \arrow[r, "\omega'"] & y \end{tikzcd}$$

in H, but the map $\pi\colon G \times_H H^I \to H$ is surjective on π_1, so n has a lift

$$\begin{tikzcd} f(x) \arrow[r, "\omega"] \arrow[d, "f(p)"'] & y \arrow[d, "n"] \\ f(x) \arrow[r, "\omega"] & y \end{tikzcd},$$

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where p is an automorphism of x in G. But then there is a commutative triangle

$$\begin{tikzcd} f(x) \arrow[rr, "f(mp^{-1})"] \arrow[dr, "\omega"'] & & f(x') \arrow[dl, "\omega'"] \\ & y & \end{tikzcd} ,$$

so ω and ω' are in the same path component of F_y. ■

To apply the above lemma to pullbacks of groupoids, it is helpful to know the following slightly unusual characterization of pullback diagrams in sets.

Lemma 4.2 *A square diagram of sets*

$$\begin{tikzcd} A \arrow[r, "g_*"] \arrow[d, "f_*"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is a pullback if and only if for each $c \in C$, the fibre A_c of f_* over c is isomorphic to the fibre B_{gc} of f over $gc \in D$ via the induced map $i_c \colon A_c \to B_{gc}$.

Proof Assume the induced map $i_c \colon A_{c'} \to B_{gc'}$ is an isomorphism for each $c' \in C$ and suppose $(c, d, b) \in C \times_D B$. Then $g(c) = d = f(b)$, so $b \in B_{gc}$ and there is a unique element $a_b \in A_c$ such that $i_c(a_b) = b$. To see that $(c, d, b) \mapsto a_b := i_c^{-1}(b)$ is the desired assignment, observe that $f_*(a_b) = c$, since $a_b \in A_c$ and $g_*(a_b) = i_c(a_b) = b$. The collection of such assignments determines a unique map $C \times_D B \to A$, so A is a pullback of the diagram. The converse is trivial. ■

Proposition 4.3 *Suppose $f \colon G \to H$ is a map of groupoids inducing surjections $\pi_1(G, x) \twoheadrightarrow \pi_1(H, f(x))$ on all fundamental groups. Then any homotopy pullback P along a map $g \colon N \to H$ of groupoids induces a pullback*

$$\begin{tikzcd} \pi_0(P) \arrow[r] \arrow[d] & \pi_0(G) \arrow[d] \\ \pi_0(N) \arrow[r] & \pi_0(H) \end{tikzcd}$$

of path component sets.

Proof Replace f by a fibration. For any object y of N, consider the diagram

$$\begin{tikzcd} F_y \arrow[r, "y_*"] \arrow[d] & P \arrow[r, "g_*"] \arrow[d, "f_*"] & G \arrow[d, "f"] \\ * \arrow[r, "y"'] & N \arrow[r, "g"'] & H \end{tikzcd}$$

of pullbacks and the induced comparison

$$\begin{tikzcd} \pi_0(F_y) \arrow[r, "y_*"] \arrow[d, "\cong"'] & \pi_0(P) \arrow[r, "f_*"] \arrow[d, "g_*"] & \pi_0(N) \arrow[d, "g"] \\ \pi_0(F_{gy}) \arrow[r, "(gy)_*"'] & \pi_0(G) \arrow[r, "f"'] & \pi_0(H) \end{tikzcd}$$

of long exact sequences of fibrations in degree 0. By Lemma 4.1 the map $(gy)_*$ is monic, so y_* is monic as well. Thus $\pi_0(F_y)$ (resp. $\pi_0(F_{gy})$) is the set-theoretic fibre of the map $\pi_0(f_*)$ (resp. $\pi_0(f)$) for any such choice of y. Apply the previous lemma. ■

5 Short Exact Sequences Associated with Torsor Trivializations

The purpose of this section is to illustrate how Grothendieck's short exact sequences

$$1 \to \pi_1^{\mathrm{Gal}}(X_Y) \to \pi_1^{\mathrm{Gal}}(X) \to G_Y \to 1$$

associated with Galois objects $Y \to X$ of $(\text{ét}(X), x)$ arise (at least in part) from analogous sequences in (pointed) nonabelian H^1, and to explain exactly how using pointed nonabelian H^1_x gives different results than unpointed nonabelian H^1 in this context. Grothendieck's proof of these sequences made use of his theory of base change for fundamental functors in Galois categories $(\mathcal{C}, F)$ ([SGA03, §6, V]); the methods presented here give a different interpretation directly in terms of torsors that may be of independent interest.

Suppose $\mathcal{C}$ is a small Grothendieck site; H is any sheaf of groups on $\mathcal{C}$, and $U \twoheadrightarrow *$ is any representable sheaf epi of $\mathcal{C}$. Consider the sequence of groupoids

$$H\text{-Tors}(\mathcal{C})_U \hookrightarrow H\text{-Tors}(\mathcal{C}) \xrightarrow{?_{|U}} H\text{-Tors}(\mathcal{C}/U),$$

where $H\text{-Tors}(\mathcal{C})_U$ is the full subgroupoid of ordinary H-torsors on $\mathcal{C}$ trivializing upon restriction $?_{|U}$ to U.

Proposition 5.1 *For any sheaf of groups H and representable sheaf epi $U \twoheadrightarrow *$ on a small Grothendieck site $\mathcal{C}$, there is an exact sequence of pointed sets*

$$1 \to \pi_0(H\text{-Tors}(\mathcal{C})_U) \hookrightarrow H^1(\mathcal{C}, H) \xrightarrow{\pi_0(?_{|U})} H^1(\mathcal{C}/U, H),$$

where the indicated map is injective.

Proof An inclusion of a full subgroupoid is always injective on path components, so the indicated map is injective. Suppose $[t] \in \pi_0(H\text{-Tors}(\mathcal{C}))$ and $\pi_0(?_{|U})([t]) = [*_{\mathcal{C}/U}]$ so that $[t] \in \ker(\pi_0(?_{|U}))$. Then $?_{|U}(t) = t_{|U}$ is in the path component of $*_{\mathcal{C}/U}$, so there is an isomorphism $t_{|U} \cong *_{\mathcal{C}/U}$ that says exactly that t trivializes upon restriction. Therefore, t is an object of $H\text{-Tors}(\mathcal{C})_U$, and so is any other object $x \in [t]$, since by definition there is then an isomorphism $x \cong t$ that induces an isomorphism upon restriction. Conversely, if t trivializes upon restriction then by definition there will be an H-equivariant isomorphism $t_{|U} \cong *_{|U}$ so that $[t] \in \ker \pi_0(?_{|U})$. Therefore $\ker \pi_0(?_{|U}) = \pi_0(H\text{-Tors}(\mathcal{C})_U)$, so the sequence is exact as was to be shown. ∎

In particular $(\mathcal{C}, x)$ may be the pointed *finite* étale site $\mathbf{Fin\acute{e}t}(X)$ for some connected locally noetherian scheme or DM stack X and $U = Y \twoheadrightarrow *$ a Galois object of $\mathcal{C}$. The following justifies a certain notational identification:

Lemma 5.2 *For any connected Galois object $Y \twoheadrightarrow *$ of a Galois category $(\mathcal{C}, F)$ that is also a connected site, there are bijections of pointed sets*

$$\pi_0(H\text{-Tors}(\mathcal{C})_Y) \cong H^1(G_Y, H),$$

natural in constant sheaves of discrete groups $H := \Gamma^* H$, where the latter pointed set is the nonabelian H^1 of Serre (cf. [Ser94]*) for the trivial action of the group G_Y on H. There are also bijections*

$$H^1(G_Y, H) \cong [BG_Y, BH],$$

natural in discrete groups H, where the right-hand side is pointed by the class of the trivial homomorphism.

Proof By [Jar09a, Proposition 1] there is a bijection

$$\pi_0 BH(*, BH)_Y \cong \pi_0 BH\text{-Tors}(\mathcal{C})_Y,$$

where $H(*, BH)_Y$ is the union of all path components of the cocycle category $H(*, BH)$ containing cocycles of the form $* \xleftarrow{\simeq} \check{C}(Y) \to BH$. The latter set is bijective to $\pi(\check{C}(Y), B\Gamma^* H)$ by ([Jar09a, Lemma 4]). As $H = \Gamma^* H$ is a constant sheaf of groups there are bijections

$$\pi(\check{C}(Y), B\Gamma^* H) \cong \pi(\Pi\check{C}(Y), BH) \cong \pi(BG_Y, BH) \cong H^1(G_Y, H),$$

where $\Pi\check{C}(Y) \cong BG_Y$ because $Y \times Y \cong \sqcup_{G_Y} Y$, since Y is connected Galois. The latter assertion follows from the bijection $\pi(BG_Y, BH) \cong [BG_Y, BH]$, which exists because BH is fibrant and all simplicial sets are cofibrant. ∎

Proposition 5.3 *Suppose $(\mathcal{C}, F)$ is a Galois category that is also a connected Grothendieck site; $Y \twoheadrightarrow *$ is a connected Galois object of $\mathcal{C}$ with group G_Y, and H is a constant sheaf of discrete groups on* $\text{ét}(X)$*. Then there is an exact sequence of pointed sets*

$$1 \to H^1(G_Y, H) \hookrightarrow H^1(X, H) \to H^1(Y, H),$$

natural in H.

Proof Proposition 5.1 applies. Combine the resulting short exact sequence with the identification $\pi_0(H\text{-Tors}(\mathcal{C})_Y) \cong H^1(G_Y, H)$ of the previous lemma. ∎

This may be considered to be the unpointed torsor theoretic analogue of Grothendieck's short exact sequence for Galois descent. Analogous arguments work for pointed torsors. Let F_U denote the homotopy fibre of the restriction map

$$H\text{-Tors}_x \xrightarrow{?_{|U}} H\text{-Tors}_u$$

for any *pointed* representable sheaf epi $(U, u) \twoheadrightarrow *$ of a pointed, connected, small, Grothendieck site $(\mathcal{C}, x)$ and *constant* sheaf of discrete groups $H := \Gamma^* H$ as in Subsection 3.3. Then there is a bijection $\pi_0(F_U) \cong \operatorname{Hom}(\check{C}(U, u), BH)$ by Lemma 3.2 and the long exact sequence in homotopy groups associated with $?_{|U}$ degenerates to the exact sequence of pointed sets

$$1 \to \operatorname{Hom}(\check{C}(U, u), BH) \to \pi_0(H\text{-Tors}_x) \to \pi_0(H\text{-Tors}_u).$$

Proposition 5.4 *Suppose $(\mathcal{C}, x)$ is a pointed, connected, small, Grothendieck site; $(U, u) \twoheadrightarrow *$ is a pointed representable sheaf epi of $(\mathcal{C}, x)$, and $H := \Gamma^* H$ is a constant sheaf of discrete groups on $\mathcal{C}$. Then there are exact sequences of pointed sets*

$$1 \to \operatorname{Hom}(\check{C}(U, u), BH) \hookrightarrow H^1_x(\mathcal{C}, H) \to H^1_u(\mathcal{C}/(U, u), H),$$

natural in discrete groups H, where the indicated map is injective if (U, u) is connected.

Proof By definition $H^1_x(\mathcal{C}, H) = \pi_0(H\text{-Tors}_x)$ and $H^1_u(\mathcal{C}/(U, u), H) = \pi_0(H\text{-Tors}_u)$. The indicated map is injective if (U, u) is connected, since the site $\mathcal{C}/(U, u)$ is connected so that $H\text{-Tors}_u$ has no nontrivial automorphisms at any of its basepoints by Galois theory Now apply lemma 4.1. ∎

Corollary 5.5 *With the hypotheses of Proposition 5.4, if $\mathcal{C}$ is furthermore a Galois category for the fibre functor F_x, then there are exact sequences*

$$1 \to \operatorname{Hom}(G_Y, H) \hookrightarrow H^1_x(\mathcal{C}, H) \to H^1_y(\mathcal{C}/(Y, y), H),$$

natural in discrete groups H, associated with any connected Galois object $(Y, y) \twoheadrightarrow *$ of $\mathcal{C}$.

Proof By observation, $\pi\check{C}(Y, y) \cong BG_Y$. ∎

As the functors $\operatorname{Hom}(G_Y, -)$, $H^1_x(\mathcal{C}, -)$, and $H^1_y(\mathcal{C}/(Y, y), -)$ restricted to **FinGrp** are pro-representable by G_Y, $\pi_1^{\mathrm{Gal}}(\mathcal{C}, x)$, and $\pi_1^{\mathrm{Gal}}(\mathcal{C}/(Y, y), y)$, respectively, there are induced exact sequences of profinite groups

$$\pi_1^{\mathrm{Gal}}(\mathcal{C}/(Y, y), y) \to \pi_1^{\mathrm{Gal}}(\mathcal{C}, x) \twoheadrightarrow G_Y \to 1,$$

where the indicated map is an epimorphism. The injectivity of the first map may then be established by the classical observation that $\pi_1^{\mathrm{Gal}}(\mathcal{C}/(Y, y), y)$ corresponds to the subgroup of $\operatorname{Aut}(F_x)$ of automorphisms of F_x fixing (Y, y) ([SGA03, 6.13, V]).

6 Étale Van Kampen Theorems

Étale van Kampen theorems have previously been developed with the goal of expressing the pro-groupe fondamental élargi G of any pointed, connected, small, Grothendieck site $(\mathcal{C}, x)$ in terms of the corresponding pro-groups of hopefully "simpler" spaces. Past approaches include [Sti06], [SGA03, §5, IX] upon which the arguments of [Sti06] are based, and [Zoo02]. The first two works listed here are based on rather intricate constructions coming from descent theory and only treat the case of profinite π_1^{Gal}, whereas the third addresses the pro-groupoid fondamental élargi ([Zoo02, 4.8]) but only with respect to covers by Zariski open substacks.

The approach described here makes use of homotopy theory to simultaneously hide the descent theoretic aspects of the former two constructions while generalizing the results of [Zoo02] to deal with certain non-Zariski coverings. The resulting van Kampen theorem here is simpler to state than [Sti06, 5.3, 5.4], does not require the covering to consist exclusively of monomorphisms, and does not depend on specific properties of the étale topology. On the other hand it is a statement about pro-groups rather than pro-groupoids, so in the latter case one must still defer to the methods of [Zoo02] that effectively assume that the covering is given by monomorphisms.

Given any sheaf of groups H on a small Grothendieck site $\mathcal{C}$, the restrictions $H_{|U}$ of H to the sites $\mathcal{C}/U$ for objects U of $\mathcal{C}$ have associated cocycle categories $H(*, BH_{|U})$, and these determine a simplicial presheaf $B\mathbf{H}(*, BH)$ defined in sections U by

$$B\mathbf{H}(*, BH)(U) := BH(*, BH_{|U}).$$

This simplicial presheaf satisfies descent in the sense that it is sectionwise equivalent to a globally fibrant model (*i.e.,* a fibrant object for the injective closed model structure) and admits a local weak equivalence $BH \xrightarrow{\simeq} B\mathbf{H}(*, BH)$, and so is a model for the classifying stack of H ([Jar06, Corollary 2.4]). There is a sectionwise weak equivalence $B\mathbf{H}(*, BH) \simeq B(H_{|?}\text{-Tors})$, where the latter is the simplicial presheaf of nerves of groupoids of $H_{|U}$-torsors for restrictions of H to the various sites $\mathcal{C}/U$ (by remarks above), so the latter also satisfies descent.

Suppose now that $(\mathcal{C}, x)$ is a pointed small Grothendieck site with a subcanonical topology and finite products and that (U, u) and (V, v) are pointed objects of $\mathcal{C}$ such that the pullback square S

$$\begin{tikzcd} U \times V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & * \end{tikzcd}$$

is a pushout of pointed sheaves on $\mathcal{C}$, where $U \times V$ is pointed by $u \times v$. By Quillen's axiom SM7 (see [GJ99, 11.5, I] for a definition and [Jar87, 3.1] for a proof), this pushout determines a homotopy cartesian square H_S

$$\begin{tikzcd} H\text{-Tors}(\mathcal{C}) \arrow[r] \arrow[d] & H\text{-Tors}(U) \arrow[d] \\ H\text{-Tors}(V) \arrow[r] & H\text{-Tors}(U \times V) \end{tikzcd}$$

of groupoids of H-torsors naturally in H for any sheaf of groups H on $\mathcal{C}$ whenever $U \to *$ or $V \to *$ is monic (*i.e.,* a *cofibration* for the injective closed model structure), or more generally whenever S is homotopy cocartesian for the injective closed model structure.

Lemma 6.1 *Suppose a commutative square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

is homotopy cartesian over an object B in some right proper closed model category, and suppose the structure maps from the objects of this square to B are fibrations. Let $c\colon A \to B$ be a weak equivalence. Then the pullback of this square along c is also homotopy cartesian.

Proof Factor the map $W \to Z$ as a trivial cofibration $q\colon W \to Q$ followed by a fibration $g\colon Q \to Z$, and let q_c (resp. g_c) denote the base change of q (resp. g) along c. Then the composite $Q \to Z \to B$ is a fibration, since $Z \to B$ is a fibration and $W \to B$ is also a fibration and so q_c is a weak equivalence by right properness, and one sees directly that g_c is a fibration. Pulling back $g_c q_c$ along the base change along c of $Y \to Z$ determines a fibration $(Q \times_Z Y) \times_B A \to Y \times_B A$, and it suffices to show that the induced map $X \times_B A \to (Q \times_Z Y) \times_B A$ is a weak equivalence. This map is the base change along c of the induced map $X \to Q \times_Z Y$ that is a weak equivalence, since the original square was homotopy cartesian. The composite $Q \times_Z Y \to Y \to B$ is a fibration, since $Y \to B$ is a fibration, and $X \to B$ is also a fibration, so the result follows by right properness. ■

Proposition 6.2 *For any constant sheaf of groups $H := \Gamma^* H$ on a pointed small Grothendieck site $(\mathcal{C}, x)$ with a subcanonical topology and finite products and any square S of pointed maps as above such that either one of the sheaf maps $U \to *$ or $V \to *$ is monic or S itself is homotopy cocartesian for the injective closed model structure on* **sShv**$(\mathcal{C})$, *there is a homotopy cartesian diagram of groupoids of pointed torsors*

$$\begin{tikzcd} H\text{-Tors}_x(\mathcal{C}) \arrow[r] \arrow[d] & H\text{-Tors}_u(U) \arrow[d] \\ H\text{-Tors}_v(V) \arrow[r] & H\text{-Tors}_{u\times v}(U \times V) \end{tikzcd}$$

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natural in H.

Proof There is a diagram

$$\begin{tikzcd} x^*(H)\text{-Tors}(\mathbf{Set}) \arrow[r, "?_{|U}"] \arrow[d, "?_{|V}"] & u^*(H)\text{-Tors}(\mathbf{Set}) \arrow[d, "?_{|U\times V}"] \\ v^*(H)\text{-Tors}(\mathbf{Set}) \arrow[r, "?_{|U\times V}"] & (u\times v)^*(H)\text{-Tors}(\mathbf{Set}) \end{tikzcd}$$

under H_S induced by S which commutes, since the maps of S are *pointed*. In fact, $H = x^*(H) = u^*(H_{|U})$ and similarly for the other points, so that all of the groupoids in this diagram are identical to the groupoid H-Tors(**Set**), and all of the restriction maps are equalities. One may therefore unambiguously take the homotopy fibre of H_S over the trivial torsor H of the groupoid H-Tors(**Set**), and this of course results in the desired commutative square H_* of groupoids of *pointed* H-torsors over x by [Jar09b, Lemma 1].

It remains to be shown that H_* is also homotopy cartesian. Let

$$h\colon * \to H\text{-Tors}(\mathbf{Set})$$

denote the unique map picking out the trivial H-torsor in **Set** and factor it as a trivial cofibration $c\colon * \to C$ followed by a fibration $f\colon C \to H$-Tors(**Set**) (this may be understood in terms of the nerves of the groupoids and the closed model structure on simplicial sets). A direct calculation shows that pulling back along fibrations sends homotopy cartesian squares to homotopy cartesian squares, so it suffices to show that pulling back along c preserves homotopy cartesian squares. Suppose one has a homotopy cartesian square H_C over C, and functorially replace all the maps to C by fibrations; the resulting objects of the new commutative square H'_C are weakly (even homotopy) equivalent to the old ones, and nerves of groupoids are Kan complexes so that H'_C is also homotopy cartesian. Now apply Lemma 6.1. ■

Lemma 6.3 *Suppose that $D\colon J \to \mathbf{Set}^{\mathcal{C}}$ is a finite loop-free diagram of pro-representable functors on a category $\mathcal{C}$ with all finite limits and colimits. Then $\varprojlim_J D$ is a functor that is pro-representable by the colimit on J^{op} of the representing pro-objects.*

Proof The J^{op}-diagram of representing pro-objects is also loop-free, so by the uniform approximation theorem of [AM69, 3.3, appendix] it may be replaced by a cofiltered levelwise diagram of the same shape that is pro-isomorphic to the original. The finite colimits computed levelwise represent the colimit of the representing pro-objects by [AM69, 4.1, appendix]. Moreover, at each object i of the new index category I the functor represented by corresponding levelwise colimit represents the limit over J of the corresponding diagram of representable functors at i. Therefore, the colimits determine a levelwise pro-representation of the limit of the J-diagram D. ■

Finally, the main theorem may be proven.

Theorem 6.4 *Suppose that $(\mathcal{C}, x)$ is a pointed, connected, small, Grothendieck site with a subcanonical topology, finite limits, and arbitrary small coproducts, and that (U, u) and (V, v) are pointed connected objects of $\mathcal{C}$ such that $U \times V$ is also connected and the pullback square S*

$$\begin{tikzcd} U \times V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & * \end{tikzcd}$$

is a pushout of pointed sheaves on $\mathcal{C}$, where $U \times V$ is pointed by $u \times v$. Furthermore, suppose that either one of the sheaf maps $U \to *$ or $V \to *$ is monic or that S itself is homotopy cocartesian for the injective closed model structure on **sShv**$(\mathcal{C})$. *Then there is a pushout diagram*

$$\begin{tikzcd} \pi_1^{\text{ét}}(\mathcal{C}/U \times V, u \times v)_{\check{C}} \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(\mathcal{C}/U, u)_{\check{C}} \arrow[d] \\ \pi_1^{\text{ét}}(\mathcal{C}/V, v)_{\check{C}} \arrow[r] & \pi_1^{\text{ét}}(\mathcal{C})_{\check{C}} \end{tikzcd}$$

of "étale" homotopy pro-groups, each defined in the usual way by means of pointed representable Čech hypercovers on its respective site.

Proof The homotopy cartesian squares of Proposition 6.2 are clearly natural in H by construction. As U, V, and $U \times V$ are connected, the sites $\mathcal{C}/U$, $\mathcal{C}/V$, and $\mathcal{C}/U \times V$ are also connected so that the respective groupoids of pointed torsors for any constant sheaf of groups H have trivial fundamental groups. Then Proposition 4.3 implies that these homotopy cartesian squares together determine a pullback diagram

$$\begin{tikzcd} H^1_x(\mathcal{C}, -) \arrow[r] \arrow[d] & H^1_u(\mathcal{C}/U, -) \arrow[d] \\ H^1_v(\mathcal{C}/V, -) \arrow[r] & H^1_{u \times v}(\mathcal{C}/U \times V, -) \end{tikzcd}$$

in the functor category $\mathbf{Set}^{\mathbf{Grp}}$. Each of these functors is pro-representable in **Grp** by Theorem 3.5, since all pointed H-torsors admit pointed trivializations, so Lemma 6.3 applies to give the desired pushout square. ■

The word "étale" is in quotes in the statement of the theorem because one need not assume that the topology on $\mathcal{C}$ is the étale topology: the statement still makes sense in any pointed connected site $(\mathcal{C}, x)$, since the pro-groups $\pi_1^{\text{ét}}(-)$ (and $\pi_1^{\text{ét}}(-)_{\check{C}}$) are defined by means of pointed connected hypercovers, and the definition of hypercovers is independent of any particular Grothendieck topology. This theorem requires no assumptions regarding effective descent associated with the cover by U and V

(cf. [Sti06]) nor is it restricted exclusively to covers by monomorphisms (as in the case of covers by open substacks; cf. [Zoo02]). Theorem 3.5 immediately applies to give the same statement in terms of the pro-groups coming from Galois theory, thus recovering and generalizing the pro-group variant of the results of [Zoo02].

Of course, in the case where one is actually working in an étale site where the "points" are defined by geometric points, one may conclude with a more geometric statement. Here is an example.

Corollary 6.5 *Suppose that X is a connected locally noetherian scheme or DM stack; $\text{ét}(X)$ is the étale site of X as defined in [Zoo02]; $x\colon \Omega \to X$ is a geometric point of X, (U,u), and (V,v) pointed (over x) and connected objects of $\text{ét}(X)$ such that $U \times_X V$ is also pointed (by $u \times v$) and connected, and suppose that the pullback square S*

$$\begin{tikzcd} U \times V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & * \end{tikzcd}$$

is a pushout of sheaves on $\text{ét}(X)$. Then, if either one of the sheaf maps $U \to *$ or $V \to *$ is monic or S itself is homotopy cocartesian for the injective closed model structure on $\mathbf{sShv}(\text{ét}(X))$, and a rigid diagram of pointed hypercovers exists (this is true at least in the scheme case), then there is a pushout diagram

$$\begin{tikzcd} \pi_1^{\text{ét}}(U \times V, u \times v) \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(U,u) \arrow[d] \\ \pi_1^{\text{ét}}(V,v) \arrow[r] & \pi_1^{\text{ét}}(X,x) \end{tikzcd}$$

of étale homotopy pro-groups, each defined in the usual way by means of pointed representable hypercovers on its respective subsite.

Proof The only condition to check is that pointed H-torsors admit representable pointed trivializations for all discrete groups H, but this follows from Lemma 3.3. Apply Theorem 6.4 and Proposition 3.8 to finish. ■

The corresponding statement about the usual profinite Grothendieck fundamental groups immediately follows:

Corollary 6.6 *With the hypotheses and notation of Corollary 6.5, there is a pushout square*

$$\begin{tikzcd} \pi_1^{\text{Gal}}(U \times V, u \times v) \arrow[r] \arrow[d] & \pi_1^{\text{Gal}}(U,u) \arrow[d] \\ \pi_1^{\text{Gal}}(V,v) \arrow[r] & \pi_1^{\text{Gal}}(X,x) \end{tikzcd}$$

of profinite Grothendieck fundamental groups.

Proof The profinite completion functor $\widehat{\ }$ is a left adjoint to the inclusion functor

$$i\colon \text{pro-}\mathbf{FinGrp} \hookrightarrow \text{pro-}\mathbf{Grp}$$

so, in particular, it preserves pushouts. Therefore, by Corollary 6.5 one gets a pushout square of the profinite completions of étale fundamental groups, but these are identified with the respective Grothendieck fundamental groups by Proposition 3.9. ■

Importantly, this result is valid even when the pointed covering square S is constructed in ét(X) rather than **Finét**(X).

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