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ON MULTIPLY TRANSITIVE PERMUTATION GROUPS

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Abstract

We present a direct combinatorial proof of the characterization of the degree of transivity of a finite permutation group in terms of the Bell numbers.

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The kth Bell number B_k is the number of partitions of a set of k elements (Comtet, 1974, p. 210). It has been observed by Merris and Pierce (1971) that if G is a group of permutations of a set X, then G is k-fold transitive if and only if

$$\frac{1}{|G|}\sum_{g\in G}\pi(g)^k = B_k,$$

where $\pi(g)$ denotes the number of elements of X fixed by g. The proof given by Merris and Pierce is by induction on k and uses the recurrence relation $B_{k+1}=\sum_{j=0}^{k}\binom{k}{j}B_j$. In this note we give a proof based directly on the interpretation of B_k as the number of partitions of a set of k elements.

THEOREM. *G is k-fold transitive if and only if*

$$\frac{1}{|G|}\sum_{g\in G}\pi(g)^k = B_k.$$

PROOF. Let Y be the set of k-tuples of elements of X, and let G act on Y by setting $g(\langle x_1,\ldots,x_k\rangle)=\langle g(x_1),\ldots,g(x_k)\rangle$. Note that if $g\in G$, g fixes $\pi(g)^k$ elements of Y. It follows from a theorem of Burnside on the number of orbits of a permutation group (Huppert, 1968, p. 536) that if Y has N orbits under G, then

$$\frac{1}{|G|}\sum_{g\in G}\pi(g)^k = N.$$

Now, let $\mathscr{P}$ be a partition of the set $\{1,\ldots,k\}$. Let $Y_{\mathscr{P}}$ be the subset of Y consisting of those k-tuples $\langle x_1,\ldots,x_k\rangle$ such that $x_i=x_j$ if and only if i and j are both in the same block of $\mathscr{P}$. Clearly $Y_{\mathscr{P}}$ is a union of orbits. If G is k-fold transitive each $Y_{\mathscr{P}}$

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is in fact an orbit, so there are B_k orbits. If G is not k-fold transitive, consider $\mathscr{P}_0 = \{\{1\}, \ldots, \{k\}\}$. $Y_{\mathscr{P}_0}$ will be the union of more than one orbit, so altogether there will be more than B_k orbits.

Weaker forms of this theorem are discussed by Huppert (1968, p. 599) and van Lint (1974, p. 31).

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