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Holomorphic Mappings between Domains in $\mathbb{C}^2$

Rasul Shafikov and Kaushal Verma

Abstract. An extension theorem for holomorphic mappings between two domains in $\mathbb{C}^2$ is proved under purely local hypotheses.

1 Introduction

The question of determining the boundary behaviour of a biholomorphic or, more generally, a proper holomorphic mapping between two given domains in $\mathbb{C}^n$ $(n > 1)$ is well known. In particular, if $f: D \to D'$ is a proper holomorphic mapping between smoothly bounded domains in $\mathbb{C}^n$, the conjecture that f extends smoothly up to ∂D, the boundary of D, remains open in complete generality. We shall henceforth restrict ourselves to the case when the boundaries are smooth real analytic, where much progress has recently been made. The main result of this article is the following theorem.

Theorem 1.1 *Let D, D' be domains in $\mathbb{C}^2$, both possibly unbounded, and $f: D \to D'$ a holomorphic mapping. Let $M \subset \partial D$ and $M' \subset \partial D'$ be open pieces, which are smooth real analytic and of finite type, and fix $p \in M$. Suppose there is a neighbourhood U of p in $\mathbb{C}^2$ such that the cluster set of $U \cap M$ does not intersect D'. Then f extends holomorphically across p if one of the following conditions holds:*

(i) *p is a strongly pseudoconvex point, and the cluster set of p contains a point in M';*
(ii) *M is pseudoconvex near p, and the cluster set of p is bounded and contained in M';*
(iii) *D is bounded, $f: D \to D'$ is proper, and the cluster set of M is contained in M'.*

The following examples that have been borrowed from [11, 18] show that holomorphic extendability of f cannot be hoped for in the absence of the hypotheses considered above.

Example 1 Let

$$D = \{z \in \mathbb{C}^2 : 2\Re z_2 + |z_1|^2 < 0\},$$
$$D' = \{z \in \mathbb{C}^2 : 2\Re(z_2')^2 + |z_1'|^2 < 0\}.$$

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Note that $0 \in \partial D$ and $0' \in \partial D'$. Furthermore, $D \backsimeq \mathbb{B}^2$, the unit ball in $\mathbb{C}^2$, while D' has two connected components none of which has smooth boundary near the origin. However, $\partial D'$ is a real analytic set. Since $z_2 \neq 0$ in D, it is possible to choose a well-defined branch of $\sqrt{z_2}$ in D and, having made such a choice, let $f(z_1, z_2) = (z_1, \sqrt{z_2})$. Then f is a biholomorphism between D and a connected component say D'_1 of D'. Moreover $0' \in \mathrm{cl}_f(0)$. But f does not extend holomorphically across $0 \in \partial D$.

Example 2 Let

$$D = \{z \in \mathbb{C}^2 : 2\Re z_2 + |z_1|^2 < 0\},$$
$$D' = \{z \in \mathbb{C}^2 : 2\Re z'_2 + |z'_1|^2 |z'_2|^2 < 0\}$$

and $f(z_1, z_2) = (z_1/z_2, z_2)$. Again $0 \in \partial D, 0' \in \partial D'$ and $f\colon D \to D'$ is a biholomorphism. Both $\partial D, \partial D'$ are smooth real analytic but the curve $\zeta \mapsto (\zeta, 0), \zeta \in \mathbb{C}$, is contained in $\partial D'$. Consequently $\partial D'$ is not of finite type near $0'$. Choose a sequence of positive reals $\epsilon_j \to 0$ and observe that $f(0, -\epsilon_j) = (0, -\epsilon_j)$. This shows that $0' \in \mathrm{cl}_f(0)$, but f evidently does not have a holomorphic extension across $0 \in \partial D$. Other examples of biholomorphisms from D that are of a similar nature can be constructed by considering rational functions on $\mathbb{C}^2$ whose indeterminacy locus contains the origin.

Example 3 Let $g(z)$ be an inner function on $\mathbb{B}^n$ $(n > 1)$. Then $g(z)$ has unimodular boundary values almost everywhere on $\partial\mathbb{B}^n$ by Fatou's theorem. However, it is known (see [19, Proposition 19.1.3] for example) that there is a dense G_δ-subset of $\partial\mathbb{B}^n$ such that the cluster set of any point from it must intersect the unit disc $\Delta \subset \mathbb{C}$. In fact the image of any radius in $\mathbb{B}^n$ that ends at a point on this G_δ under $g(z)$ is dense in Δ. Therefore, $f(z) = (g(z), 0, \dots, 0)$ is a holomorphic self map of $\mathbb{B}^n$ for which there does not exist a neighbourhood U of any given boundary point with the property that the cluster set of $U \cap \partial\mathbb{B}^n$ does not intersect $\mathbb{B}^n$. Evidently f fails to admit even a continuous extension to any $p \in \mathbb{B}^n$.

To provide a context for this theorem, recall the theorems of Diederich and Pinchuk from [9, 10]. In [9] it was shown that every proper holomorphic mapping between smoothly bounded real analytic domains in $\mathbb{C}^2$ extends holomorphically across the boundaries while [10] contains a similar statement for continuous CR mappings (that are *a priori* non-proper) between smooth real analytic finite type hypersurfaces in $\mathbb{C}^n$, $n > 2$. The theorem above shows that it is possible to study the boundary behaviour of f under purely local hypotheses. Motivated in part by [2, 13, 21], which deal with similar local theorems under convexity assumptions on the boundaries either of a geometric or a function theoretic nature, attempts to arrive at such a local statement were made in [23, 25], both of which were proved under additional hypotheses only on the mappings involved. It should be noted that local extension theorems for proper holomorphic mappings across pseudoconvex real analytic boundaries played a particularly useful role in the global theorem of [9]. Cases (i) and (ii) provide an instance of such extension theorems even *without properness* of f. Case (i) in particular shows that f can be completely localized near a strongly pseudoconvex point as soon as its cluster set contains a smooth real analytic finite type point. Thus, these statements cover the case of infinite sheeted coverings $f: D \to D'$. Second, in (ii) no assumptions are being made about the cluster set of points on M close to p. In particular, the possibility that the cluster set of a point $q \in M$ close to p contains the point at infinity in $\partial D'$ is *a priori* allowed; that this cannot happen will follow from the boundedness of the cluster set of p and will be explained later. Finally, a word about the various hypotheses being made in (i), (ii), and (iii) about the relative location of $p \in M$. While pseudoconvex points are taken care of by the first two cases, the content of (iii) lies in the fact that it addresses the remaining possibility that p is either on the border between the pseudoconvex and pseudoconcave points on M, or M is locally pseudoconcave near p. However, it is well known that f extends across pseudoconcave points. Therefore, it is sufficient to consider the case when p is on the border between the pseudoconvex and pseudoconcave points on M. The proofs of these theorems falls within the purview of the geometric reflection principle as developed in [6, 9, 10, 12]. Further refinements of these techniques from [23] and the ideas of analytic continuation of germs of holomorphic mappings along paths on smooth real analytic hypersurfaces from [22] are particularly useful.

The geometric reflection principle as developed in [6, 9, 12] studies the influence of a holomorphic mapping between two smoothly bounded real analytic domains on the Segre varieties that are associated with each boundary point. A quick review of what will be needed later is given below but detailed proofs that can be found in the aforementioned references have been skipped, the purpose of the exposition being solely to fix notation. For brevity, we work in $\mathbb{C}^2$, as the case for $n > 2$ is no different, and write $z = (z_1, z_2)$ for a point $z \in \mathbb{C}^2$. Let $\Omega \subset \mathbb{C}^2$ be open and $M \subset \Omega$ a closed, smooth real analytic real hypersurface. Pick $\zeta \in M$ and translate coordinates so that $\zeta = 0$. Let $r(z, \overline{z})$ be the defining function of M in a neighbourhood, say U of the origin and suppose that $\partial r / \partial z_2(0) \neq 0$. Let $U^{\pm} = \{z \in U : \pm r(z, \overline{z}) > 0\}$. If U is small enough, the complexification $r(z, \overline{w})$ of $r(z, \overline{z})$ is well defined by means of a convergent power series in $U \times U$. It may be noted that $r(z, \overline{w})$ is holomorphic in z and anti-holomorphic in w. For any $w \in U$, the associated Segre variety is

$$Q_w = \{z \in U : r(z, \overline{w}) = 0\}.$$

By the implicit function theorem, Q_w can be written as a graph for each $w \in U$. In fact, it is possible to choose a pair of neighbourhoods U_1, U_2 of the origin with U_1 compactly contained in U_2 such that for any $w \in U_1$, $Q_w \subset U_2$ is a closed complex hypersurface and

$$Q_w = \{z = (z_1, z_2) \in U_2 : z_2 = h(z_1, \overline{w})\},$$

where $h(z_1, \overline{w})$ depends holomorphically on z and anti-holomorphically on w. Such neighbourhoods are usually called a *standard pair* of neighbourhoods, and for convenience they will be chosen to be polydiscs around the origin. As a set, Q_w is independent of the choice of $r(z, \overline{z})$, because any two local defining functions for M near the origin differ by a non-vanishing multiplicative factor, and this persists upon complexification as well. A similar argument shows that Segre varieties are invariantly attached to M in the following sense: let M' be another smooth real analytic hypersurface in $\mathbb{C}^2$. Pick p, p' on M, M', respectively, and open neighbourhoods $U_p, U'_{p'}$ containing them. If $f\colon U_p \to U'_{p'}$ is a holomorphic mapping such that $f(U_p \cap M) \subset U'_{p'} \cap M'$, then for all $w \in U_p$, $f(Q_w) \subset Q'_{f(w)}$, where $Q'_{f(w)}$ denotes the Segre variety associated with $f(w) \in U'_{p'}$. This will be referred to as the *invariance property* of Segre varieties. It forms the basis of constructing the complex analytic set that is used to extend holomorphic mappings across real analytic boundaries. The practice of distinguishing analogous objects in the target space by adding a prime will be followed in the sequel. For $\zeta \in Q_w$ the germ of Q_w at ζ will be denoted by ${}_\zeta Q_w$. Let $\mathcal{S} = \{Q_w : w \in U_1\}$ be the ensemble of all Segre varieties and let $\lambda\colon w \mapsto Q_w$ be the so-called Segre map. Then $\mathcal{S}$ admits the structure of a complex analytic set in a finite dimensional complex manifold. The fibres

$$I_w = \lambda^{-1}(\lambda(w)) = \{z : Q_z = Q_w\}$$

are then analytic subsets of U_1, and for $w \in M$ it can be shown that $I_w \subset M$. If M does not contain germs of positive dimensional complex analytic sets, *i.e.*, it is of finite type in the sense of D'Angelo, it follows that I_w is a finite collection of points. Other notions of finite type, such as in the sense of Bloom–Graham or essential finiteness, are discussed in more detail in [1]; in $\mathbb{C}^2$, however, all these notions are equivalent. Assuming now that M is of finite type, it follows that λ is proper in a neighbourhood of each point on M. Also note that $z \in Q_w$ is equivalent to $w \in Q_z$ and that $z \in Q_z$ if and only if $z \in M$, both of which are consequences of the reality of $r(z, \bar{z})$. The notion of a symmetric point from [9] will be useful here as well: for a fixed $w \in U_1$, the complex line l_w containing the real line through w and orthogonal to M intersects Q_w at a unique point. This is the symmetric point of w and will be denoted by ${}^s w$. If $w \in M$, then $w = {}^s w$ and it can be checked that for $w \in U^\pm$, the symmetric point ${}^s w \in U^\mp$. Finally if $w \in U^+$, the component of $Q_w \cap U^-$ that contains ${}^s w$ will be denoted by Q_w^c and referred to as the canonical component.

2 Remarks on the Proof of the Theorem

By shrinking the neighbourhood U of p whose existence is assumed in the theorem, we may suppose that $U \cap M$ is a smooth real analytic hypersurface of finite type. The standard pair of neighbourhoods $U_1 \subset U_2$ of p needed to define the Segre varieties associated with points on $U \cap \partial D$ near p will then be chosen to be compactly contained in U. Let p' be a point on M' that lies in $\mathrm{cl}_f(p)$, the cluster set of p. Likewise, fix a neighbourhood U' of p' so that $U' \cap \partial M'$ is again a smooth real analytic hypersurface of finite type and then fix a standard pair $U'_1 \subset U'_2$ around p'. Abusing notation, we shall denote $U \cap M$ and $U' \cap \partial M'$ by M and M' respectively. The following stratification of M from [9] will be needed: let T be the set of points on M where its Levi form vanishes. Then T admits a semianalytic stratification as $T = T_0 \cup T_1 \cup T_2$, where T_k is a locally finite union of smooth real analytic submanifolds of dimension $k = 0, 1, 2$, respectively. Denote by $M_s^\pm$ the set of strongly pseudoconvex (resp. strongly pseudoconcave) points on M. Let $M^\pm$ be the relative

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interior, taken with respect to the relative topology on M, of the closure of $M_s^{\pm}$. Then $M^{\pm}$ is the set of weakly pseudoconvex (resp. weakly pseudoconcave) points of M and the border $M \setminus (M^+ \cup M^-) \subset T$ separates M^+ and M^-. It was shown in [8, 9] that this stratification of T can be refined in such a way that the two-dimensional strata become maximally totally real manifolds. Retaining the same notation $T_k, k = 0, 1, 2$ for the various strata in the refined stratification, let $T_k^+ = M^+ \cap T_k$ for all k. Then T_2^+ is the maximally totally real stratum near which M is weakly pseudoconvex. It was shown in [7] that $(M \setminus (M^+ \cup M^-)) \cap T_2 \subset M \cap \hat{D}$, where $\hat{D}$ is the holomorphic hull of the domain D. Evidently f holomorphically extends to a neighbourhood of each point on $(M \setminus (M^+ \cup M^-)) \cap T_2$. Its complement in T is

$$M_e = (M \setminus (M^+ \cup M^-)) \cap (T_1 \cup T_0),$$

which will be called the *exceptional set*. Observe that each of M_e and $T_1^+ \cup T_0^+$ is a locally finite union of real analytic arcs and points. The point p can then lie in either M_s^+, $T_2^+ \cup T_1^+ \cup T_0^+$, M_e, or $(M \setminus (M^+ \cup M^-)) \cap T_2$, and the same possibilities hold for p' by considering the corresponding strata on M'.

The first thing to prove in cases (i) and (ii) of the theorem is that the restriction $f \colon U^- \to D'$ has discrete fibres. This will guarantee that the set defined by

$$X_f = \{(w, w') \in U_1^+ \times U_1'^+ : f(Q_w \cap D) \supset {}_{s_{w'}}Q'_{w'}\}, \tag{2.1}$$

if non-empty, is a locally complex analytic set. In the setting of cases (i) and (ii), there are negative plurisubharmonic barriers at boundary points near p (see the next section for details) and thus by [21], f is known to have discrete fibres. In Proposition 3.2 we show this near any smooth finite type boundary point. Let π, π' denote the projections from $U \times U'$ onto the first and second factor, respectively. Since λ, λ' are proper near p, p', respectively, the fact that f has discrete fibres near p forces both projections $\pi \colon X_f \to U_1^+$ and $\pi' \colon X_f \to U_1'^+$ to have discrete fibres as well. Now no assumptions on the global cluster set of p are being made and hence it is not possible to directly prove that X_f is contained in a closed analytic subset of $U_1 \times U_1'$ with proper projection onto U_1. If this were possible, f would then extend as a holomorphic correspondence and hence as a mapping by [9, Theorem 7.4].

To illustrate the salient features of the proof of case (i), note that $p \in M_s^+$ and $p' \in M'^+, T'$ or M'^-. The case when $p' \in M'^+$ is well understood, for there are local plurisubharmonic peak functions near p, p' that lead to the continuity of f near p. When $p' \in M_s'^-$, the local plurisubharmonic peak function near p (which can be continuously extended to all of D) can be pushed forward by f to get a negative plurisubharmonic function on D'. The strong pseudoconcavity of M' near p' implies the existence of complex discs in D' near p' whose boundaries are uniformly compactly contained in D'. The restriction of this negative plurisubharmonic function on D' to these discs is shown to violate the maximum principle. When $p' \in M'^-$, the graph of f is shown to extend to a neighbourhood of (p, p') as an analytic set and hence as a holomorphic mapping by [11], and this leads to a contradiction. The case when $p' \in (M' \setminus (M'^+ \cup M'^-)) \cap T_2'$ requires several intermediate steps, indeed, as noted above, such a $p' \in \hat{D}'$ and if $f \colon D \to D'$ were proper, then all points in the cluster set of p' under the correspondence f^{-1} on ∂D would belong to $\hat{D}$ (see [9, Lemma 3.1]), and hence f would extend holomorphically past p. This reasoning does not apply here for f^{-1} is not known to be well defined even as a correspondence. Therefore, we first show that there is a sequence $p^j \in U \cap M$ such that f extends holomorphically across p^j and $f(p^j) \to p'$. For each j, consider the maximal possible extension of f along Q_{p^j} (see (4.3)) which associates with each p^j a one-dimensional analytic set C_j in a natural way. Secondly, it is shown that the cluster set of $\{C_j\}$ (which is defined in the next section) contains points near which X_f is defined. In particular, there are points on Q_p over which X_f is a local ramified cover. The proof of the extendability of f is then completed by using ideas of analytic continuation of germs of holomorphic mappings along real hypersurfaces from [22]. Similar ideas apply when $p' \in (M' \setminus (M'^+ \cup M'^-)) \cap (T'_1 \cup T'_0)$ and the conclusion is that f holomorphically extends across p even when $p' \in T'$. This evidently leads to a contradiction, since there are points near p, p' at which f is locally biholomorphic but the Levi form is not preserved. Thus if $p \in M^+_s$, then p' is forced to be in M'^+ and, consequently, f extends across p. In fact by [5], p' must be a strongly pseudoconvex point on M'.

For case (ii), note that $p \in M^+, T$ or M^-. When $p \in M^+_s$, case (i) shows that f extends past p. Note that points on M^- and $(M \setminus (M^+ \cup M^-)) \cap T_2$ belong to $\hat{D}$, and f extends past p in these cases as well. What remains to consider are the cases when $p \in T^+_2$ or $(M \setminus (M^+ \cup M^-)) \cap (T_1 \cup T_0)$, *i.e.*, points on the one- and zero-dimensional strata of the border between the pseudoconvex and pseudoconcave points. Suppose that $p \in T^+_2$ and let $\mathcal{L} = \bigcup_{w \in T^+_2} Q_w$, where w is allowed to vary near p on T^+_2. Let $U' \subset \mathbb{C}^2$ be an open neighbourhood that contains $\mathrm{cl}_f(p)$ such that $U' \cap M'$ is a closed smooth real analytic hypersurface of finite type. Then $X_f \subset U^+_1 \times U'^+$ is well defined. The goal will be to show that X_f extends as an analytic set in $U_1 \times U'$, and the obstructions in doing this arise from the limit points of X_f on $U^+_1 \times (U' \cap M')$. The limit points are shown to lie on $\mathcal{L} \times (U' \cap M')$. The extendability of X_f will follow by first showing that $\overline{X}_f$ is analytic near $\mathcal{L} \times T'^-_2$, while the other limit points on $\mathcal{L} \times ((M \setminus (M^+ \cup M^-)) \cap (T_1 \cup T_0))$ (which is pluripolar) are removable by Bishop's theorem. Thus the graph of f will be contained in an analytic set defined near $\{p\} \times (U' \cap M')$ and this will imply that f holomorphically extends past p.

A set of similar ideas works when $p \in (M \setminus (M^+ \cup M^-)) \cap (T_1 \cup T_0)$ and can be applied to prove case (iii). An essential ingredient in this is Lemma 6.1, according to which the cluster set of such a p cannot contain strongly pseudoconvex points on M' — and this is shown to hold even without f being proper. This exhausts all possibilities for p, indeed, as noted above, if p is on a two-dimensional stratum on the border, $p \in \hat{D}$ and hence f extends holomorphically across p.

3 The Fibres of f near p Are Discrete

Diederich and Pinchuk [10, §6.1] posed a conjecture on the cluster set of a sequence of pure dimensional analytic sets all of which are defined in a fixed neighbourhood of a real analytic CR manifold in $\mathbb{C}^n$ of finite type in the sense of D'Angelo. To recall the setup, let $W \subset \mathbb{C}^n$ be open and $N \subset W$ a relatively closed, real analytic CR manifold of finite type. Suppose that $A_j \subset W$ is a sequence of closed analytic sets of pure fixed dimension $k \geq 1$. Define the cluster set of $\{A_j\}$ as

$$\mathrm{cl}(A_j) = \{z \in W : \text{ there is a sequence } z_j \in A_j \text{ such that } z \text{ is a limit point of } (z_j)\}.$$

Their conjecture is that $\mathrm{cl}(A_j)$ is not entirely contained in N. Although open in general, proofs of the validity of this conjecture were given by them in [10] when $n = 2$ and N is a smooth real analytic hypersurface of finite type and in other instances (see [10, Propositions 8.2 and 8.3]) when more information is *a priori* assumed either about k or about the structure of $\mathrm{cl}(A_j)$. Their proof of this conjecture in the case $n = 2$ depends on the existence of suitable plurisubharmonic peak functions at the pseudoconvex points and along the smooth totally real strata of the Levi degenerate points on N. A different argument for the case of the totally real strata, based on the well-known fact that an analytic set of positive dimension cannot approach such a submanifold tangentially can be given as follows.

Proposition 3.1 *Let $M \subset U \subset \mathbb{C}^2$ be a closed, smooth real analytic hypersurface of finite type and suppose that $A_j \subset U$ is a sequence of closed analytic sets of pure dimension one. Then $\mathrm{cl}(A_j)$ cannot be entirely contained in M.*

Proof Suppose that $\mathrm{cl}(A_j) \subset M$. If possible, pick $a \in \mathrm{cl}(A_j) \cap M_s^+$. Let $\tilde{U}$ be an open neighbourhood of a chosen so small that $\tilde{U} \cap M \subset M_s^+$ and such that there is a continuous plurisubharmonic function ϕ on $\tilde{U}$ with $\phi(z) < \phi(a)$ for all $z \in (\tilde{U} \cap M) \setminus \{a\}$. Since $a \in \mathrm{cl}(A_j)$, it follows that $A_j \cap \tilde{U} \neq \varnothing$ for infinitely many j, but note however that $A_j \cap \tilde{U}$ may have many components for each such j. For brevity this subsequence will still be denoted by j. Choose $a_j \in A_j \cap \tilde{U}$ such that $a_j \to a$. Let $\tilde{A}_j$ be that component of $A_j \cap \tilde{U}$ which contains a_j. As $A_j \subset U$ are closed, it follows that $\partial\tilde{A}_j \subset \partial\tilde{U}$, and hence $\mathrm{cl}(A_j) \cap \partial\tilde{U} \neq \varnothing$. Now

$$\sup\{\phi(z) : z \in \mathrm{cl}(A_j) \cap \partial\tilde{U}\} < \phi(a),$$

and hence by continuity of ϕ there is an open neighbourhood V of $\mathrm{cl}(A_j) \cap \partial\tilde{U}$ such that

$$\sup\{\phi(z) : z \in V\} = c < \phi(a).$$

Having fixed V, choose an open neighbourhood $\tilde{V}$ of a such that $c < \phi(z)$ for all $z \in \tilde{V}$. It follows that ϕ restricted to $\tilde{A}_j$ attains its maximum in $\tilde{V}$, which is a contradiction. Exactly the same arguments can be applied to points in $\mathrm{cl}(A_j) \cap M_s^-$, if any. This shows that $\mathrm{cl}(A_j) \subset M \setminus (M_s^+ \cup M_s^-)$.

Now pick $a \in \mathrm{cl}(A_j) \cap T_2^+$, if possible, and choose coordinates around $a = 0$ so that T_2^+ coincides with the 2-plane spanned by $\Re z_1, \Re z_2$ near the origin and fix a polydisc $\tilde{U} = \{|z_1| < \eta, |z_2| < \eta\}$ around the origin with $\eta > 0$ small enough so that $\tilde{U} \cap T_2^+ = \{\Im z_1 = \Im z_2 = 0\}$. Then $\tau(z) = 2(\Im z_1)^2 + 2(\Im z_2)^2$ is a non-negative strongly plurisubharmonic function in $\tilde{U}$ whose zero locus is exactly $\tilde{U} \cap T_2^+$. Also note that $i\partial\overline{\partial}\tau = i\partial\overline{\partial}|z|^2$. For each $r > 0$ the domain $V_r = \{z \in \tilde{U} : \tau(z) < r\}$ is a strongly pseudoconvex tubular neighbourhood of $\tilde{U} \cap T_2^+$. As before, choose $a_j \in A_j$ converging to a, and by shifting it if necessary, we may assume that $a_j \in A_j \setminus (\tilde{U} \cap T_2^+)$.

Let $\tilde{A}_j$ be that component of $A_j \cap \tilde{U}$ which contains a_j. Fix $0 < r_0 \ll \eta$ whose precise value will be determined later. Then only finitely many $\tilde{A}_j$ can be contained in V_{r_0}. Indeed, if this does not hold, pass to a subsequence, still retaining the same index for brevity, if necessary for which $\tilde{A}_j \subset V_{r_0}$. Define

$$\rho_j(z) = \tau(z) - |z - a_j|^2/2,$$

and note that $\rho(a_j) = \tau(a_j) > 0$ for each j. Moreover,

$$i\partial\overline{\partial}\rho_j = i\partial\overline{\partial}\tau - i\partial\overline{\partial}|z - a_j|^2/2 = i\partial\overline{\partial}|z|^2 - i\partial\overline{\partial}|z|^2/2 = i\partial\overline{\partial}|z|^2 > 0$$

shows that the restriction of ρ_j to $\tilde{A}_j$ is subharmonic for all j. Now fix j and let $w \in \partial\tilde{A}_j \subset \partial\tilde{U}$. Then

$$\rho_j(w) = \tau(w) - |w - a_j|^2/2 \leq r_0 - |w - a_j|^2/2 < 0,$$

the last inequality holding whenever $r_0 > 0$ is chosen to satisfy

$$2r_0 < \eta^2 \approx (\eta - |a_j|)^2 \leq (|z| - |a_j|)^2 \leq |z - a_j|^2$$

for all $z \in \partial\tilde{U}$. This contradicts the maximum principle and hence all but finitely many $\tilde{A}_j$ must intersect $\tilde{U} \cap \{\tau(z) \geq r_0\}$. Note, however, that this set contains strongly pseudoconvex points, and hence it follows that $\mathrm{cl}(A_j) \cap M_s^+ \neq \varnothing$, which is a contradiction to the previous step. The same reasoning can be applied to show that $\mathrm{cl}(A_j)$ does not lie entirely in $T_2^- \cup T_1^{\pm} \cup T_0^{\pm}$. What remains is the border $M \setminus (M^+ \cup M^-)$, which, as discussed earlier, admits a semi-analytic stratification into real analytic submanifolds of dimension $2, 1, 0$. The top dimensional strata can be made maximally totally real after a possible refinement. The same arguments can be repeated for these strata to get a contradiction. ■

Proposition 3.2 *Let D, D' be domains in $\mathbb{C}^2$, both possibly unbounded and $f\colon D \to D'$ a non-constant holomorphic mapping. Suppose that $M \subset \partial D$ is an open, smooth real analytic hypersurface of finite type, and let $p \in M$. Let U be a neighbourhood of p in $\mathbb{C}^2$ such that the cluster set of no point on $U \cap M$ intersects D'. Then there exists a possibly smaller neighbourhood V of p such that $f\colon V^- \to D'$ has discrete fibres.*

Proof First observe that for each $z \in D$, the analytic set $f^{-1}(f(z))$ is at most one-dimensional, as otherwise the uniqueness theorem will imply that f is a constant mapping. Now if the assertion does not hold, then there is a sequence $p_j \to p$ for which $f^{-1}(f(p_j))$ is one-dimensional at p_j. Let A_j be a pure one-dimensional component of $f^{-1}(f(p_j)) \cap U^-$ that contains p_j. Since the cluster set of no point on $U \cap M$ intersects D', it follows that $A_j \subset U$ is a closed analytic set and hence Proposition 3.1 implies that $\mathrm{cl}(A_j)$ is not entirely contained in M. Pick $\zeta_0 \in \mathrm{cl}(A_j) \cap U^-$ and choose $\zeta_j \in A_j$ converging to ζ_0. Then $f(p_j) = f(\zeta_j) \to f(\zeta_0) \in D'$. On the other hand, note that since $\mathrm{cl}_f(p) \cap D' = \varnothing$, it follows that either $|f(p_j)| \to +\infty$ or $f(p_j)$ clusters only at a finite boundary point of D'. This is a contradiction. ■

Now suppose that $p \in M^+$. Then there exist (see, for example, [14, 20]) constants $\alpha, \beta > 0$ and an open neighbourhood V of p such that for every $\zeta \in V \cap \partial D$ there exists a plurisubharmonic function ϕ_ζ on V^- that is continuous on $V \cap \overline{D}$ satisfying

$$-|z-\zeta|^\alpha \lesssim \phi_\zeta(z) \lesssim -|z-\zeta|^\beta \tag{3.1}$$

for any $z \in V \cap \overline{D}$. Here $(\alpha, \beta) = (1, 2)$ or $(1, 2m)$, depending on whether p is strongly pseudoconvex or just weakly so, and in the latter case $2m$ is the type of ∂D at p. Moreover, the constants involved in these estimates are independent of $\zeta \in V \cap \partial D$. Thus ϕ_ζ is a family of local plurisubharmonic barriers at $\zeta \in V \cap \partial D$ all of which are defined in a fixed neighbourhood of p. It follows from (3.1) that there are small neighbourhoods $V_2 \subset V_1$ of p with V_2 compactly contained in V_1 and $\tau > 0$ and a smooth non-decreasing convex function θ with $\theta(t) = -\tau$ for $t \le -\tau$ and $\theta(t) = t$ for $t \ge -\tau/2$ such that $\rho_p(z) = \tau^{-1}\theta(\phi_p(z)) : D \to [-1, 0)$ is a negative continuous plurisubharmonic function on D with $\rho_p(z) = -1$ on $D \setminus V_1$ and $\rho_p(z) = \tau^{-1}\phi_p(z)$ on V_2^-. Since f has discrete fibres near p by Proposition 3.2, we may define

$$\psi_p(z') = \begin{cases} \sup\{\rho_p(z) : z \in f^{-1}(z')\} & \text{for } z \in f(V_1^-), \\ -1 & \text{for } z' \in D' \setminus f(V_1^-). \end{cases}$$

Arguments similar to those in [13, 21] show that $\psi_p(z')$ is a negative, continuous plurisubharmonic function on D'. Furthermore there is an open neighbourhood U' of p' small enough so that $U' \cap M'$ is smooth real analytic such that

$$\operatorname{dist}(f(z), U' \cap M') \lesssim \operatorname{dist}(z, U \cap M), \tag{3.2}$$

whenever $z \in U^-$ and $f(z) \in U'^-$. Now since $p' \in \mathrm{cl}_f(p)$, there is a sequence $p_j \to p$ such that $f(p_j) \to p'$. While not much can be said at this stage about the continuity of $\psi_p(z')$ at p', it does however follow from the definition of $\psi_p(z')$ that $\psi_p(f(p_j)) \to 0$. This observation will be used in the sequel.

4 Proof of Theorem 1.1: Case (i)

In this section p will be a strongly pseudoconvex point, *i.e.*, $p \in M_s^+$, and separate cases will be considered depending on whether $p' \in M'^+$, T', or M'^- for $p' \in \mathrm{cl}_f(p)$.

4.1 The Case When $\mathbf{p}' \in \mathbf{M}'^+ \cup \mathbf{M}'^-$

If $p' \in M'^+$, then by a theorem of Sukhov [21], the map f admits a Hölder continuous extension to a neighbourhood of p on M. Furthermore, by the result of Pinchuk–Tsyganov [18], f extends holomorphically, in fact, locally biholomorphically, across p.

Next, suppose that $p' \in M_s'^-$. Let p_j be a sequence of points in D such that $p_j \to p$, and $p_j' = f(p_j) \to p'$. Fix a small ball $V' \subset \mathbb{C}^2$ around p' in which M' is strictly pseudoconcave. For each p_j', let $\zeta_j' \in M' \cap V'$ be the unique point such that

$$\operatorname{dist}(p_j', M' \cap V') = |\zeta_j' - p_j'|.$$

Let $L'_j \subset V'$ be the complex line through p'_j which is obtained by translating the complex tangent space to M' at ζ'_j. Then $L'_j \to L' \subset V'$, which is the complex tangent space to M' at p'. Since M' is strictly pseudoconcave at p', it follows that $L' \cap \partial V' \Subset D'$ and hence that $L'_j \cap \partial V'$ is uniformly compactly contained in D' for all large j. Let $\phi_j \colon \overline{\Delta} \to L_j$ be a holomorphic parametrization, which is continuous on $\overline{\Delta}$ and satisfies $\phi_j(0) = p'_j$ and $\phi_j(\partial\Delta) = L'_j \cap \partial V'$. The sub-mean value property shows that

$$\psi_p(p'_j) = \psi_p \circ \phi_j(0) \lesssim \int_{\partial\Delta} \psi_p \circ \phi_j$$

for each j. Note that $\psi_p(p'_j) \to 0$, while the right side is bounded above by a uniform negative constant since $\{\phi_j(\partial\Delta)\}$ are uniformly compactly contained in D' for all large j. This is a contradiction.

Suppose now that $p' \in T'^{-} = T' \cap M'^{-}$. Let $T'^{-} = T_2'^{-} \cup T_1'^{-} \cup T_0'^{-}$ be a stratification of T'^{-} into totally real, real analytic manifolds of dimensions $2, 1$, and 0, respectively. Suppose $p' \in T_2'^{-}$. Let V and V' be small neighbourhoods of p and p'. Consider the set $A = \Gamma_f \cap (V \times V')$ where Γ_f is the graph of the map f. Then $(p, p') \in \overline{A}$. Since the cluster set of M under f does not contain points in D' and for any point $q \in M$ near p the set $\mathrm{cl}_f(q)$ cannot contain strictly pseudoconcave points by the argument above, it follows that the limit points of the set A in $V \times V'$ are contained in $(M \cap V) \times (T_2'^{-} \cap V')$. The latter is a real analytic CR manifold of CR dimension one. By a theorem of Chirka [3] (see also [4]), the set $(M \cap V) \times (T_2'^{-} \cap V')$ is a removable singularity for A, *i.e.*, A admits analytic continuation as an analytic set in $V \times V'$ after shrinking these neighbourhoods if needed. Therefore, by [11] the map f extends holomorphically to a neighbourhood of p. Arguing by induction, we may assume that $\mathrm{cl}_f(p)$ does not contain points in $T_2'^{-}$, and we then repeat the argument for $T_1'^{-}$, and for $T_0'^{-}$. This shows that in each case f admits holomorphic extension to a neighbourhood p. This again leads to a contradiction, because the extension will be locally biholomorphic away from a complex analytic set of dimension one, and biholomorphic maps preserve the Levi form.

4.2 The Set X_f

The remaining possibility is that $\mathrm{cl}_f(p) \cap U' \subset T' \setminus (M'^{+} \cup M'^{-})$. Note that under these conditions, there exists a sequence of points $p^j \to p$, $\{p^j\} \subset M$ such that f extends holomorphically to a neighbourhood of each p^j. This follows by the previously used argument. Indeed, if the cluster set of a small neighbourhood of p in M contains strictly pseudoconvex points of M', then at those points we have extension by [18,21], which gives us points of extendability arbitrarily close to p. So suppose that the limit points of the set $A = \Gamma_f \cap (V \times V')$ are contained in $(M \cap V) \times (T' \cap V')$, which is locally (after stratification and inductive argument on dimension) a CR manifold of dimension at most one. By [4], A admits analytic continuation as an analytic set in $V \times V'$, and by [11], the map f extends holomorphically to a neighbourhood of p. Then there are strongly pseudoconvex points on M near p that are mapped locally biholomorphically to strongly pseudoconcave points near p', and this is a contradiction.

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Furthermore, it is clear that for any point $p' \in \mathrm{cl}_f(p)$, the sequence $\{p^j\}$ above can be chosen in such a way that $f(p^j) = p'^j \to p'$ as $j \to \infty$.

Lemma 4.1 *Let $p \in M_s^+$ and suppose that $p' \in T' \setminus (M'^+ \cup M'^-)$. Then $X_f \subset U_1^+ \times U_1'^+$ defined by* (2.1) *is a nonempty, pure two-dimensional, closed analytic set.*

Proof First, X_f is non-empty because f extends holomorphically to a neighbourhood of p_j, with the extension sending M to M', and (2.1) simply manifests the invariance property of the Segre varieties for the extension near p_j.

Secondly, X_f is a locally complex analytic set. Indeed, suppose that $(x, x') \in X_f$, so $f(Q_x \cap U^-)$ contains ${}_{^s x'}Q'_{x'}$. Let $z_0 \in Q_x \cap U^-$ be such that ${}^s x' = f(z_0)$. Let $V = V_1 \times V_2 \subset U_1^-$ be a small polydisc centred at z_0 such that $f(Q_x \cap V)$ is contained in the canonical component of $Q'_{x'}$. Such V exists because $f(Q_x \cap U^-)$ contains the germ ${}_{^s x'}Q'_{x'}$ and both sets have the same dimension. Then there exists a neighbourhood U_x of x such that for any $w \in U_x$, we have $Q_w \cap V \neq \varnothing$, and moreover, we may assume that for any $z_1 \in V_1$, the point $(z_1, h(z_1, \overline{w}))$ is in $Q_w \cap V$. Then the condition $f(z) \in Q'_{w'}$ for all $z \in Q_w \cap V$ is equivalent to $r'(f(z), \overline{w}) = 0$, or

$$r'(f(z_1, h(z_1, \overline{w})), \overline{w}') = 0, \quad z_1 \in V_1.$$

This is an infinite system of holomorphic equations (after conjugation) that describes the property that $f(Q_w \cap V) \subset Q'_{w'}$ for all $w \in U_x$ and $w' \in U_1'^+$. By analyticity, the latter inclusion implies that $f(Q_w \cap U_1^-)$ contains ${}_{^s w'}Q'_{w'}$ provided that U_x is sufficiently small. This shows that in $U_x \times U_1'^+$, the set X_f is described by a system of holomorphic equations, and so X_f is a locally complex analytic set.

Thirdly, X_f is closed. Indeed, let

$$E = \{z \in Q_w \cap U_2^- : f(z) = {}^s w', f({}_z Q_w) \supset {}_{^s w'}Q'_{w'} \text{ and } (w, w') \in X_f\}.$$

Since $p \in M_s^+$, it follows that $Q_p \cap \overline{D} = \{p\}$ and hence [9, Lemma 8.4] shows that E is relatively compactly contained in U_2. Now if $(w^j, w'^j) \in X_f$ converges to $(w^0, w'^0) \in U_1^+ \times U_1'^+$, then we need to show that $(w^0, w'^0) \in X_f$, *i.e.*,

$$f(Q_{w^0} \cap U_1^-) \supset {}_{^s w'^0}Q'_{w'^0}. \tag{4.1}$$

For this, choose $\zeta^j \in Q_{w^j} \cap U_2^-$ such that $f(\zeta^j) = {}^s w'^j$ and $f({}_{\zeta^j}Q_{w^j}) \supset {}_{^s w'^j}Q'_{w'^j}$. Since E is compactly contained in U_2, it follows that $\zeta^j \to \zeta^0 \in U_2^-$ so that $f(\zeta^0) = {}^s w'^0$, and the analytic dependence of Q_w on w shows that $f({}_{\zeta^0}Q_{w^0}) \supset {}_{^s w'^0}Q'_{w'^0}$, which evidently implies (4.1). ■

Suppose that p^j is a sequence of points on M converging to p such that f extends holomorphically to a neighbourhood U_j of each p_j, and the sequence $p'^j = f(p^j)$ converges to $p' \in M'$. Let $V_j \subset U_1$ be a neighbourhood of Q_{p^j}. We may choose U_j and V_j in such a way that U_j is a bidisc, and for any point $w \in V_j$, $Q_w \cap U_j$ is a non-empty connected graph over the z_1-axis. For each j consider the following set:

$$X_j = \{(w, w') \in V_j \times U_1' : f(Q_w \cap U_j) \subset Q'_{w'}\}, \tag{4.2}$$

where by f we mean the extension of f to U_j.

Lemma 4.2 *X_j is a closed complex analytic subset of $V_j \times U_1'$.*

Proof The proof of this lemma is similar to that of Lemma 4.1. ■

Let X_j be given by (4.2). We may consider only the irreducible component of X_j of dimension two that coincides near (p^j, p'^j) with the graph of f. For simplicity denote this component again by X_j. Define

$$C_j = X_j \cap ((Q_{p^j} \setminus \{p_j\}) \times Q'_{p'^j}). \tag{4.3}$$

Since M is strictly pseudoconvex, $(Q_{p^j} \setminus \{p^j\}) \subset U_1^+$ (provided that U_1 is sufficiently small), and therefore, C_j is a closed complex analytic subset of $U_1^+ \times U_1'$.

4.3 The Case When $\mathbf{p}' \in (\mathbf{T}' \setminus (\mathbf{M}'^+ \cup \mathbf{M}'^-)) \cap \mathbf{T}_2'$

In this subsection we will concentrate on the case when $p' \in \mathrm{cl}_f(p)$ is a point on the totally real two-dimensional stratum of the border separating the pseudoconvex and pseudoconcave points of M'. Our first goal is to prove the inclusion $C_j \subset X_f$. This will be done after a careful choice of the neighbourhoods in the target space. Choose a neighbourhood U of p such that all the previous conclusions are valid and let $\tilde{U}' \Subset U'$ be a pair of neighbourhoods of the point p' such that the following hold:

(i) $\tilde{U}'$ and U' are bidiscs, and for any point w' in $\tilde{U}'^+$ the symmetric point w'^s is contained in U'^-,

(ii) for any $w' \in \tilde{U}'$ the set $Q'_{w'} \cap U'$ is a holomorphic graph $z_2' = h'(w', z_1')$,

(iii) $Q'_{w'} \cap M'$ intersects $\partial U'$ transversally for $w' \in \tilde{U}'$. In particular, this means that $Q'_{w'} \cap M'$ intersects $\partial U'$ at points near which both $Q'_{w'} \cap M'$ and $\partial U'$ are smooth submanifolds.

Note that the above conditions are possible to meet because $Q'_{p'} \cap M'$ is a finite union of isolated points and real analytic curves with isolated singularities, and so U' can be chosen to satisfy (iii) for $w' = p'$. Since the singularities of $Q'_{w'} \cap M'$ vary analytically with w' this is also possible to achieve for all w' in a small neighbourhood of p'.

Lemma 4.3 *If $(p^j, p'^j) \in U \times \tilde{U}'$, then $C_j \cap (U \times \tilde{U}') \subset X_f \cap (U \times \tilde{U}')$.*

Proof Note that the inclusion holds near (p^j, p'^j) because X_f and X_j both agree with the graph of the extension of f to U_j by the invariance property of Segre varieties. If C_j is a closed analytic subset of $U^+ \times U'^+$ (where X_f is defined), then the inclusion $C_j \subset X_f$ holds by the uniqueness property for complex analytic sets. Thus, to prove the lemma we only need to show that C_j is contained in $(U^+ \times \tilde{U}'^+)$. Arguing by contradiction, suppose that $(a, a') \in C_j \cap (U^+ \times M')$ for some fixed $j > 0$, and that (a, a') is a limit point of X_f. Let $\gamma(t) \subset Q_{p_j}$ be a smooth curve that connects $p_j = \gamma(0)$ and $a = \gamma(1)$, and consider a continuous family of points a_t as t varies from 0 to 1. Suppose that for all points on this curve except the terminal point a, we have $(a_t, a_t') \in X_f$ for some point $a_t' \in U'$, *i.e.*, the curve γ is contained in the projection of X_f to the first component. We will need the following lemma.

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Lemma 4.4 *For any* t, $0 \le t < 1$ *the set* $f(Q_{a_t} \cap U^-) \cap U'^-$ *contains a connected component, which we denote by* Z_t, *such that* $p'_j \in \overline{Z}_t$, *and* ${}^s a'_t \in Z_t$. *In particular, there is a path* $\tau_t \subset Z_t \cup \{p'_j\}$ *starting at* p'_j *and terminating at* ${}^s a'_t$.

Proof of Lemma 4.4 Note here that the point p'_j and the path γ' (consisting of points a'_t) is contained in $\tilde{U}'$. Let R be the subset of $[0,1)$ for which the lemma holds. Then $R \neq \varnothing$, because it holds for sufficiently small t.

Claim 1. R is an open set. Indeed, if $t_0 \in R$, then since all the data is analytic, and the path τ_{t_0} is compactly contained in U', a small perturbation of t near t_0 will preserve the property described by the lemma.

Claim 2. If $[0, t_0) \subset R$, then $t_0 \in R$. Indeed, let $U' = U'_1 \times U'_2 \subset \mathbb{C}_{z'_1} \times \mathbb{C}_{z'_2}$, and let $P\colon U' \to U'_1$ be the coordinate projection. Consider the sets $P(Z_t)$ for $t < t_0$. Then $P(Z_t)$ is a connected open set. Its boundary consists of points from the boundary of U'_1 and the interior points of U'_1. The latter are the projections of the points in the closure of $f(Q_{a_t} \cap U^-) \cap U'^-$ that are contained in M'. To see this, observe that $f(Q_{a_t} \cap U^-) \subset Q'_{a'_t}$, and for any w' the restriction $P|_{Q'_{w'}}$ is a biholomorphic map because $Q'_{w'}$ is a graph over U'_1. By assumption, the point $P(p'_j)$ is on the boundary of $P(Z_t)$, and $P({}^s a'_t)$ is an interior point of $P(Z_t)$ for $t < t_0$. We denote by $P(Z_{t_0})$ the limit set of the sequence $P(Z_t)$ as $t \to t_0$, which by construction is a subset of $P(f(Q_{a_{t_0}} \cap U^-) \cap U'^-)$. We will show that $P(Z_{t_0})$ has the same property, which will complete the proof of Claim 2.

The claim may fail only if the set $P(Z_{t_0})$ is disconnected and the point $P(p'_j)$ is not on the boundary of the connected component of $P(Z_{t_0})$ containing $P({}^s a'_{t_0})$. Suppose this is the case. Then, since at any time $t < t_0$, the set $P(Z_t)$ is connected, it follows that the connected components of $P(Z_{t_0})$ containing $P(p'_j)$ (on the boundary) and $P({}^s a'_{t_0})$ must have at least one common boundary point. Denote by S_1 the component of $P(Z_{t_0})$ that contains $P(p'_j)$ on the boundary, and by S_2 the component that contains $P({}^s a'_{t_0})$. Two cases are possible.

(i) S_1 and S_2 have common boundary points only in $\partial U'_1$.
(ii) S_1 and S_2 have at least one common boundary point in U'_1.

Suppose (i) holds. Then $Q'_{a'_{t_0}} \cap M'$ intersects $\partial U'$ either tangentially or at a singular point, which is not allowed by the choice of $\tilde{U}'$.

Now suppose (ii) holds and $\zeta'_1 \in \overline{S}_1 \cap \overline{S}_2 \cap U'_1$. Then as discussed above, ζ'_1 is the projection of a point $\zeta' = (\zeta'_1, \zeta'_2) \in M'$ that is in the closure of

$$f(Q_{a_{t_0}} \cap U^-) \cap U'^-.$$

Because only points from the boundary of D can be mapped by f into the boundary of D', we conclude that for any sequence of points in $P^{-1}(S_1) \cap Z_{t_0}$ converging to ζ', the sequence of preimages under f converges to $Q_{a_{t_0}} \cap M$. By passing to a subsequence, we may assume that the latter sequence converges to a point $\zeta \in Q_{a_{t_0}} \cap M$. We show that f extends holomorphically to a neighbourhood of ζ. This can be proved as follows.

There is a path $\sigma' \subset f(Q_{a_{t_0}} \cap U^-) \cap U'^-$ that connects p'_j and ζ'. It can be obtained, for example, by taking a path in the closure of S_1 connecting $P(p'_j)$ and ζ'_1, and

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lifting it to the closure of $f(Q_{a_{t_0}} \cap U^-) \cap U'^-$. Take $\sigma = f^{-1}(\sigma')$; more precisely, let σ be the component of $f^{-1}(\sigma')$ that contains p_j. We claim that the germ of the map f at p_j extends analytically to a neighbourhood of the closure of σ. For the proof, we choose a small neighbourhood $U_{a_{t_0}}$ of the point a_{t_0} and a thin neighbourhood V_{t_0} of $Q_{a_{t_0}} \cap U$. If $U_{a_{t_0}}$ is small enough, then over $U_{a_{t_0}}$ the set X_j is a ramified covering; in particular, it defines a holomorphic correspondence $F_{t_0} \colon U_{a_{t_0}} \to U'$. Now choose V_{t_0} such that for any $w \in V_{t_0}$, the set $Q_w \cap U_{a_{t_0}}$ is nonempty and connected, and consider the set

$$Y_{t_0} = \{(w, w') \in V_{t_0} \times U' : F_{t_0}(Q_w \cap U_{a_{t_0}}) \subset Q'_{w'}\}.$$

An argument similar to that of Lemma 4.1 (see [22]) shows that Y_{t_0} is a nonempty complex analytic set. Furthermore, there exists an irreducible component of Y_{t_0} that agrees with the graph of f near (p_j, p'_j). Indeed, let z be any point in $U_j \cap V_j \cap V_{t_0}$, where U_j and V_j are the neighbourhoods from (4.2). Let $w \in Q_z \cap U_{a_{t_0}}$ be an arbitrary point. It follows that $z \in Q_w$. Let $w' \in F_{t_0}(w)$. This means by definition of F_{t_0} that $f(Q_w \cap U_j) \subset Q'_{w'}$, in particular, $f(z) \in Q'_{w'}$ or $w' \in Q'_{f(z)}$. From this we conclude that $F_{t_0}(w) \subset Q'_{f(z)}$. Since w was an arbitrary point of $Q_z \cap U_{t_0}$, it follows that $F_{t_0}(Q_z \cap U_{a_{t_0}}) \subset Q'_{f(z)}$, which means that $(z, f(z)) \in Y$. We will consider only this irreducible component of Y_{t_0}, which for simplicity we denote again by Y_{t_0}.

It follows that the same inclusion holds in a neighbourhood of the curve σ. By continuity and using the fact that $f(\sigma) = \sigma'$, we have that $Q_\zeta \cap U_{a_{t_0}}$ is mapped by F_{t_0} into $Q'_{\zeta'}$, and therefore, we actually have the set Y_{t_0} defined in a neighbourhood of ζ, which gives the extension of f to a neighbourhood of the point ζ. Let $\tilde{f}$ be the extension of f near ζ. Then $\tilde{f}(\zeta) = \zeta'$. Since $\zeta \in M_s^+$, by the invariance of the Levi form ζ' can only be a strictly pseudoconvex point of M', and therefore, $\tilde{f}$ is locally biholomorphic in some neighbourhood V of ζ. By the invariance of the Segre varieties, we have $\tilde{f}(V \cap Q_{a_{t_0}}) = \tilde{f}(V) \cap Q'_{a'_{t_0}}$. But now we reach a contradiction: near ζ the set $Q_{a_{t_0}} \cap U^-$ is connected, while near ζ' the set $Q'_{a'_{t_0}} \cap U'^-$ has at least two components. Thus, case (ii) is also not possible, and this completes the proof of Lemma 4.4. ∎

We continue with the proof of Lemma 4.3. For a fixed j we are interested in understanding the points on C_j on $(U^+ \times M')$ that are limit points for X_f. Let (a, a') and $\gamma(t)$ be as at the beginning of the proof of the lemma. Observe that in this case $a' \in T'_2$. Indeed, if $a' \in M_s'^+$, then for t close to 1, the set $Q'_{a'_t} \cap U'^-$ has a small connected component near a', which contains ${}^s a'_t$. These components shrink to the point a' as t approaches 1. But this contradicts Lemma 4.4, which states that there exists a component $Z_t \subset f(Q_{a_t} \cap U^-) \cap U'^-$ containing p'_j in its closure and ${}^s a'_t$ for all $0 \le t < 1$. Suppose now that $a' \in M_s'^-$. Then for all $t < 1$, the set $Q_{a_t} \cap U^-$ contains a point which is mapped by f to ${}^s a'_t$. Since ${}^s a'_t$ approaches a' as $t \to 1$, we conclude that there exists a point on $Q_a \cap M$ whose cluster set under the map f contains a' , a strictly pseudoconcave point. However, by the previous considerations we know that this is not possible.

Let $(a, a') \in C_j \cap (U^+ \times T'_2)$ be a limit point for X_f. By Lemma 4.4 there exists a curve $t \mapsto (a_t, a'_t) \in (Q_{p_j} \times Q'_{p'_j}) \cap (U^+ \times U'^+)$ parametrized by $[0, 1]$ that converges to (a, a'), and a component $Z_t \subset f(Q_{a_t} \cap U^-) \cap U'^-$ containing p'_j in its closure and ${}^s a'_t$ for all $0 \le t < 1$. For each $0 \le t < 1$ let σ'_t be a path in $Z_t \cup \{p'_j\}$ that joins p'_j and ${}^s a'_t$. Let Ω' be a tubular neighbourhood of T'_2 in $\mathbb{C}^2$ chosen so small that it does not contain p'_j. As $t \to 1$, $a'_t \to a' \in T'_2$, and hence the symmetric points ${}^s a'_t \to a'$ which implies that ${}^s a'_t \in \Omega'$ for t close to 1. Since σ'_t join p'_j and ${}^s a'_t$, it follows that these paths must leave Ω' for each $0 \le t < 1$. Let γ'_t be the component of $\sigma'_t \cap \Omega'$ that contains ${}^s a'_t$. Then the other end point of γ'_t lies on $\partial\Omega'$. It is possible to choose a subsequence of γ'_t that converges in the Hausdorff metric to a continuum, say γ'_1, which is contained in $\overline{\Omega}'$ and which contains a' and points on $\partial\Omega'$. Furthermore, each γ'_t is contained in $f(Q_{a_t} \cap U^-) \subset Q'_{a'_t}$ by the invariance property and hence the limiting continuum γ'_1 must lie in $Q'_{a'}$. Recall that $Q'_{a'} \cap M'$ is a finite union of smooth real analytic arcs and singular points by the choice of $\tilde{U}'$ and U'. Moreover the various components of $Q'_{a'} \cap U'^-$ have their boundaries contained in the union of these real analytic arcs and points and each component contains points on T'_2 in its closure. Two cases arise:

(i) Suppose that γ'_1 contains a point $z'_0 \in \partial\Omega' \cap D'$. Let C' be the component of $Q'_{a'} \cap D'$ which contains z'_0. Since $f\colon Q_a \cap D \to D'$ is proper, it follows that the image $f(Q_a \cap D)$ is a closed subvariety in D'. Now $f(Q_a \cap D)$ contains z'_0 (in fact, points on γ'_1 near z'_0 are also contained in $f(Q_a \cap D)$) and this implies that $C' \subset f(Q_a \cap D)$. Let Γ' be a boundary component of C'; evidently $\Gamma' \subset Q'_{a'} \cap M'$. Pick a point $z' \in \Gamma' \cap T'_2$ and join it to z'_0 by a path $r(t)$ that lies in $C' \cup \{z'\}$. Then extend f^{-1} analytically along $r(t)$. This process gives rise to an extension of f^{-1} at z' and it means that points on T'_2 are mapped to strongly pseudoconvex points, which is a contradiction.

(ii) Suppose that γ'_1 is contained entirely in $Q'_{a'} \cap M'$. It follows that γ'_1 must be contained in the connected union of one or more of the arcs that make up $Q'_{a'} \cap M'$. Let Γ' be one such smooth real analytic arc that is contained in γ'_1 and which contains points on $\partial\Omega' \cap M'$ as well as those on T'_2. Repeating the same argument as in case (i) it is possible to get a contradiction for exactly the same reason.

This completes the proof of Lemma 4.3. ■

Since $X_f \subset U^+ \times U'^+$, Lemma 4.3 immediately implies the following.

Corollary 4.5 *If* $(p^j, p'^j) \in U \times \tilde{U}'$, *then* $C_j \cap (U^+ \times U'^-) = \varnothing$.

Lemma 4.6 *Let* X_f *and* C_j *be defined as above. Then at least one of the following statements holds:*

(i) $\overline{X}_f \cap (U^+ \times (U' \cap M')) = \varnothing$.

(ii) *The cluster set of* $\{C_j\}$ *cannot be* $\{p\} \times Q'_{p'}$.

Proof Suppose (i) does not hold, $(\zeta, \zeta') \in \overline{X}_f \cap (U \times (U' \cap M'))$, and $(\zeta^j, \zeta'^j) \to (\zeta, \zeta')$ as $j \to \infty$, $(\zeta^j, \zeta'^j) \in X_f$. Then ζ' cannot be a strictly pseudoconvex point, as otherwise $f\colon D \to D'$ would have values outside D'. Also ζ' cannot be a strictly pseudoconcave point by previous considerations. So we conclude that $\zeta' \in T'_2$. We may as well assume that $\zeta' = p'$. By analyticity, the set $f(Q_\zeta \cap U^-)$ is the limit of $f(Q_{\zeta^j} \cap U^-)$. Therefore, we conclude that $f(Q_\zeta \cap U^-) \cap U'^-$ must be contained in $Q'_{p'} \cap U'^-$. In particular, this means that $Q'_{p'} \cap U'^-$ is not empty.

Suppose that $\{p\} \times Q'_{p'}$ is the cluster set of C_j as $j \to \infty$. Since $Q'_{p'} \cap U'^- \neq \varnothing$, we conclude that $C_j \cap (U^+ \times U'^-) \neq \varnothing$ for sufficiently large j. But this contradicts Corollary 4.5. ■

We now show that f extends holomorphically across p in either of the possibilities listed in Lemma 4.6. Indeed, suppose that Lemma 4.6(ii) holds. It follows that there exists a point $(a, a') \in X_f$ such that $a \in Q_p \setminus p$, and $\pi(X_f)$ contains some open neighbourhood U_a of a in U. We show that f extends holomorphically to a neighbourhood of $p \in M$. Our argument is similar to the argument used in Lemma 4.3. We choose neighbourhoods U_a and $U'_{a'}$ of a and a', respectively, in such a way that the projection $\pi\colon X_f \cap (U_a \times U'_{a'}) \to U_a$ is a ramified covering, which gives rise to a holomorphic correspondence $F_a\colon U_a \to U'_{a'}$. Let V_a be a neighbourhood of Q_a such that for any $w \in V_a$, $Q_w \cap U_a$ is a non-empty connected set. Note that $p \in V_a$ because $a \in Q_p$. Define the set $Y = \{(w, w') \in V_a \times U' : F_a(Q_w \cap U_a) \subset Q'_{w'}\}$. As before, Y is a closed complex analytic subset of $V_a \times U'$. We now show that $Y \neq \varnothing$ by showing that Y contains a piece of the graph of f. Let j be sufficiently large such that $U_j \cap V_a \neq \varnothing$. We may fix such j and assume further that $U_j \subset V_a$, and $V_j \cap U_a \neq \varnothing$ (this is possible because $Q_{p^j} \to Q_p$). Let $z \in U_j$, and let $w \in Q_z \cap U_a \cap V_j$ be arbitrary. Let $w' \in F_a(w)$. Then by definition of X_j, we have $f(Q_w \cap U_j) \subset Q'_{w'}$, in particular, $f(z) \in Q'_{w'}$. This implies that $w' \in Q'_{f(z)}$. Since w was an arbitrary point in $Q_z \cap U_a \cap V_j$, and w' was any point in $F_a(w)$, it follows that

$$F_a(Q_z \cap U_a \cap V_j) \subset Q'_{f(z)}.$$

Since $Q_z \cap U_a$ is a connected set of which $Q_z \cap U_a \cap V_j$ is a non-empty open subset, we conclude that $F_a(Q_z \cap U_a) \subset Q'_{f(z)}$, which means that $(z, f(z)) \in Y$. This proves the claim. Using standard arguments, one can conclude that the set Y gives a holomorphic extension of f to a neighbourhood of any point in $Q_a \cap M$, in particular, to a neighbourhood of p.

Thus, it remains to consider Lemma 4.6(i), *i.e.*, when $\overline{X}_f \cap (U \times M') = \varnothing$. For this, first observe the following lemma.

Lemma 4.7 *X_f has no limit points on $(U \cap M) \times \tilde{U}'^+$.*

Proof Suppose that $(w^0, w'^0) \in \overline{X}_f \cap ((U \cap M) \times \tilde{U}'^+)$ and let $(w^j, w'^j) \in X_f \subset U^+ \times \tilde{U}'^+$ be such that $(w^j, w'^j) \to (w^0, w'^0)$. Then $f(Q_{w^j} \cap D) \supset {}_{^s w'^j}Q'_{w'^j}$ holds for all j. Choose $\zeta^j \in Q_{w^j} \cap D$ such that $f(\zeta^j) = {}^s w'^j$. By continuity ${}^s w'^j \to {}^s w'^0 \in D'$. The strict pseudoconvexity of $U \cap M$ implies that $Q_{w^j} \cap D$ shrinks to w^0 as $j \to \infty$ and therefore $\zeta^j \to w^0$. This contradicts (3.2). ■

Lemma 4.8 *$\overline{C}_j \subset U \times \tilde{U}'$ is a closed analytic set of pure dimension one for each j.*

Proof By Lemma 4.3 $C_j \subset X_f$, and hence C_j has no limit points on $(U \cap M) \times \tilde{U}'^+$. Moreover, there are no limit points for C_j on $U^+ \times (U' \cap M')$ either, by the assumption that X_f has no limit points there. Therefore, $\overline{C}_j \setminus C_j \subset \{p^j\} \times (Q'_{f(p^j)} \cap M')$. Suppose that $(p^j, q) \in \overline{C}_j \setminus C_j$ and choose $(w^k, w'^k) \in C_j$ converging to (p^j, q) as $k \to \infty$. Then

$$f(Q_{w^k} \cap D) \supset {}_{^s w'^k}Q'_{w'^k}$$

for all k, and let $\zeta^k \in Q_{w^k} \cap D$ be such that $f(\zeta^k) = {}^s w'^k$. By continuity, ${}^s w'^k \to {}^s q = q$, and since $U \cap M$ is strictly pseudoconvex, it follows that $\zeta^k \to p^j$ as $k \to \infty$. However, f extends across p^j and hence $f(\zeta^k) \to q$. It follows that $q = f(p^j) = p'^j$. This shows that (p^j, p'^j) is the only limit point for C_j and by the Remmert–Stein theorem, $\overline{C}_j \subset U \times \tilde{U}'$ is a closed complex analytic set of pure dimension one for each j. ■

Lemma 4.9 *The cluster set of* $\{C_j\}$ *is non-empty in* $U^+ \times \tilde{U}'^+$.

Proof For $\epsilon > 0$ small, let $U_\epsilon \Subset U$ and $U'_\epsilon \Subset \tilde{U}'$ be bidiscs of size ϵ around p, p' whose sides are parallel to those of U and $\tilde{U}'$, respectively. Consider the non-empty analytic sets $\overline{C}_j \cap (U_\epsilon \times U'_\epsilon)$ and examine the coordinate projection

$$\pi' \colon \overline{C}_j \cap (U_\epsilon \times U'_\epsilon) \to U'_\epsilon.$$

There are two cases to be considered.

Case 1: If π' is proper for all large j, then $\pi'(\overline{C}_j \cap (U_\epsilon \times U'_\epsilon)) = Q'_{p'_j} \cap U'_\epsilon$. In this case, it follows that $Q'_{p'} \cap U'_\epsilon$ cannot intersect D'. Indeed, if possible, choose $\tau'^0 \in Q'_{p'} \cap U'^-_\epsilon$. Choose $\tau'^j \to Q'_{p'^j} \cap U'_\epsilon$ such that $\tau'^j \to \tau'^0$. Then $\tau'^j \in U'^-_\epsilon$ for j large which contradicts Lemma 4.3, according to which $\overline{C}_j$ does not contain points over $\tilde{U}'$. For $\eta < \epsilon$, let U_η, U'_η be bidiscs around p, p', respectively, of size η. Pick $w'^0 \in Q'_{p'} \cap \partial U'^+_{\epsilon/2}$ and let $w'^j \in Q'_{p'_j} \cap \partial U'^+_{\epsilon/2}$ be such that $w'^j \to w'^0$. Since π' is proper, choose $w^j \in Q_{p^j}$ such that $(w^j, w'^j) \in \overline{C}_j \cap (U_\epsilon \times U'_\epsilon)$. After passing to a subsequence, we may assume that $(w^j, w'^j) \to (w^0, w'^0)$, where $w^0 \in Q_p$. Now Lemma 4.7 shows that $w^0 \notin U \cap M$ since $w'^0 \in U'^+_{\epsilon/2}$. Hence $w^0 \in U^+_{\epsilon/2}$, and therefore $(w^0, w'^0) \in \mathrm{cl}(C_j) \cap (U^+ \times \tilde{U}'^+)$.

Case 2: If for some subsequence still indexed by j the projection π' is not proper, then it is possible to choose $(w^j, w'^j) \in \overline{C}_j \cap (U_\epsilon \times U'_\epsilon)$ with $w^j \in Q_{p^j} \cap \partial U_\epsilon$. If for some fixed $\eta > 0$, $Q'_{p'^j} \cap U'_\eta \subset \pi'(\overline{C}_j \cap (U_\epsilon \times U'_\epsilon))$ for all large j, then this is exactly the situation addressed in Case (i) above and hence we may assume without loss of generality that $w'^j \to p'$. The strict pseudoconvexity of $U \cap M$ implies that $Q_{p^j} \cap \partial U_\epsilon \Subset U^+$ uniformly and hence $w^j \to w^0 \in U^+$ after passing to a subsequence. Thus $(w^0, p') \in U^+ \times (U' \cap M')$ is a limit point for X_f, which is a contradiction. ■

By [10, Theorem 7.4] it is known that the volumes of $\{C_j\} \subset U^+ \times \tilde{U}'^+$ are uniformly bounded on each compact subset of $U^+ \times \tilde{U}'^+$ after perhaps passing to a subsequence. By Bishop's theorem, C_j converges to a pure one-dimensional analytic set, say $C_p \subset U^+ \times \tilde{U}'^+$. Since $C_j \subset Q_{p^j} \times Q'_{p'^j}$, it follows that $C_p \subset Q_p \times Q'_{p'}$. In particular there are points on $Q_p \setminus \{p\}$ over which X_f is defined. This is exactly the situation considered in Case (ii) of Lemma 4.6 and the arguments presented there show that f extends holomorphically across p. Consequently, when $p \in M^+_s$ and $p' \in (M' \setminus (M'^+ \cup M'^-)) \cap T'_2$, f extends holomorphically across p and $f(p) = p'$. The invariance property of Segre varieties shows that ([9]) f^{-1} extends across p' as a holomorphic correspondence and thus there are strictly pseudoconcave points near p' that are mapped locally biholomorphically by some branch of f^{-1} to strictly pseudoconvex points near p and this is a contradiction. This completes the discussion in case p' is on a two-dimensional totally real stratum of the border between the pseudoconvex and pseudoconcave points.

4.4 The Case When $\mathbf{p}' \in (\mathbf{T}' \setminus (\mathbf{M}'^{+} \cup \mathbf{M}'^{-})) \cap (\mathbf{T}'_1 \cup \mathbf{T}'_0)$

Exactly the same arguments can be applied when p' is on a one-dimensional stratum of the border. The proof uses the additional fact that we know from the above reasoning, *i.e.*, the cluster set of a strongly pseudoconvex point cannot intersect a totally real two-dimensional stratum of the border. The case when p' is a point on the zero-dimensional stratum of the border goes as follows: observe that the cluster set of a strongly pseudoconvex point cannot intersect either a two- or one-dimensional stratum of the border. So if $p' \in \mathrm{cl}_f(p)$ is on the zero-dimensional stratum, it must be isolated in the cluster set of p and hence f is continuous up to M near p. Therefore, by [10] f admits a holomorphic extension across p with $f(p) = p'$. This is a contradiction, as explained before.

5 Proof of Theorem 1.1: Case (ii)

The cases to be considered are $p \in T_2^+ \cup T_1^+ \cup T_0^+$ for the other possibility that $p \in M_s^+$ is covered by the previous section.

5.1 The Case When $\mathbf{p} \in \mathbf{T}_2^+$

Recall that $\mathrm{cl}_f(p) \subset M'$ and hence a point $p' \in \mathrm{cl}_f(p)$ could belong to either M'^{+}, T', or M'^{-}. The arguments when $p' \in M'^{\pm}$ are similar to those used in Section 4, and therefore we shall be brief in these cases. Indeed, when $p' \in M'^{+}$, then by [21] the map f admits a Hölder continuous extension to a neighbourhood of p on M, and hence by [10] it follows that f extends holomorphically across p. In case $p' \in M_s'^{-}$, the same argument from Section 4 applies without any changes. Indeed, the main ingredient there is the negative, continuous, plurisubharmonic function $\psi_p(z')$ on D', which can be constructed in this case as well because $p \in T_2^+$.

Now suppose that $p' \in T'^{-} = T' \cap M'^{-}$. Let $T'^{-} = T_2'^{-} \cup T_1'^{-} \cup T_0'^{-}$ be a stratification of T'^{-} into totally real, real analytic manifolds of dimensions 2, 1, and 0, respectively. Suppose that $p' \in T_2'^{-}$ and let V, V' be small neighbourhoods of p, p' respectively. Evidently $A = \Gamma_f \cap (V \times V')$, where Γ_f is the graph of f, contains (p, p') in its closure. Then $(\overline{A} \setminus A) \cap (V \times V')$ cannot be contained in $(T_2'^{+} \times T_2'^{-}) \cap (V \times V')$ for if not, A will admit analytic continuation as an analytic set across the totally real manifold $T_2^+ \times T_2'^{-}$. Hence by [11], f will extend holomorphically to a neighbourhood of p. Proceeding by induction, we may assume that $\mathrm{cl}_f(p)$ does not contain points in $T_2'^{-}$ and repeat the argument for the lower dimensional strata. Thus in each case f admits holomorphic extension to a neighbourhood of p. This is a contradiction, because the extension will be locally biholomorphic away from a codimension one analytic set, and biholomorphic maps preserve the Levi form.

The remaining possibility is that $\mathrm{cl}_f(p) \subset T' \setminus (M'^{+} \cup M'^{-})$, in which case the arguments used above show that for every $p' \in \mathrm{cl}_f(p)$ there is a sequence $p^j \to p$,

$\{p^j\} \subset M$ such that f extends holomorphically to a neighbourhood of each p^j and $f(p^j) \to p' \in T' \setminus (M'^+ \cup M'^-)$. To deal with this case, let $U' \subset \mathbb{C}^2$ be an open neighbourhood that compactly contains $\mathrm{cl}_f(p) \subset M'$ and such that $U' \cap M'$ is a closed, smooth, real analytic hypersurface of finite type. We may also assume that U' is small enough to guarantee the existence of $Q'_{w'}$ as a local complex manifold for $w' \in U'^+$. Having chosen such a U', fix a standard pair of neighbourhoods $U_1 \subset U_2$ around p so that $(Q_p \setminus \{p\}) \cap M \cap U_2 \subset M_s^+$ (this is possible by [9, Lemma 12.1]) and such that $f(U_2^-)$ is compactly contained in U'. This latter condition can be fulfilled since $\mathrm{cl}_f(p) \subset M'$. By shrinking U_2 further if needed, we may additionally assume that for $w' \in \partial U' \setminus D'$ the symmetric point ${}^s w' \notin f(U_2^-)$. This is possible since

$$\mathrm{dist}(w', M') \backsimeq \mathrm{dist}({}^s w', M')$$

for all w' in a given compact set in $\mathbb{C}^2$ that intersects M'. Then

$$X_f = \{(w, w') \in U_1^+ \times U'^+ : f(Q_w \cap D) \supset {}_{^s w'}Q'_{w'}\}$$

is closed by [9, Lemma 12.2] and also complex analytic by the arguments used before in Lemma 4.1. Furthermore, X_f is non-empty because of the existence of the sequence $p^j \to p$ such that f extends across p^j as mentioned above.

Lemma 5.1 *X_f does not have limit points on $U_1^+ \times (\partial U' \setminus D')$.*

Proof Suppose that (w^0, w'^0) is a limit point for X_f on $U_1^+ \times (\partial U' \setminus D')$ and let $(w^j, w'^j) \in X_f$ converge to (w^0, w'^0). Then $f(Q_{w^j} \cap D) \supset {}_{^s w'^j}Q'_{w'^j}$ holds for all j, and choose $\zeta^j \in Q_{w^j} \cap D$ such that $f(\zeta^j) = {}^s w'^j$. After passing to a subsequence, $\zeta^j \to \zeta^0$ for some ζ^0 in the closure of $U_2 \cap D$. This is because

$$E = \{z \in Q_w \cap U_2^- : f(z) = {}^s w', f({}_z Q_w) \supset {}_{^s w'}Q'_{w'} \text{ and } (w, w') \in X_f\}$$

is closed. Evidently, ${}^s w'^j \to {}^s w'^0$. Note that ${}^s w'^0$ is contained in the cluster set of ζ^0, which is a contradiction since ${}^s w'^0 \notin f(U_2^-)$ by construction. ∎

Define $\mathcal{L} = \bigcup_{w \in T_2^+} Q_w$, where the neighbourhoods U_1, U_2 are small enough so that $(Q_p \setminus \{p\}) \cap M \cap U_2 \subset M_s^+$.

Lemma 5.2 *All limit points of X_f on $U_1^+ \times (U' \cap M')$ are contained in*

$$\mathcal{L} \times (T' \setminus (M'^+ \cup M'^-)).$$

Proof Let (w^0, w'^0) be a limit point for X_f in $U_1^+ \times (U' \cap M')$, and suppose that $(w^j, w'^j) \in X_f$ converges to (w^0, w'^0). Then

$$f(Q_{w^j} \cap D) \supset {}_{^s w'^j}Q'_{w'^j} \tag{5.1}$$

holds for all j, so let $z^j \in Q_{w^j} \cap D$ be such that $f(z^j) = {}^s w'^j$. Note that ${}^s w'^j \to {}^s w'^0 = w'^0$, and since E (as defined in the previous lemma) is compactly contained in U_2, it follows that $z^j \to z^0 \in Q_{w^0} \cap U_2 \cap M$. In particular, $z^0 \in M_s^+$ or T_2^+, and

$w'^0 \in \mathrm{cl}_f(z^0)$. If $w'^0 \in M'^+$, then f holomorphically extends to a neighbourhood Ω of z^0, and $f(z^0) = w'^0$. It is therefore possible to choose ζ^j close to z^0 such that $f(\zeta^j) = w'^j$. The invariance property of Segre varieties shows that

$$f(Q_{\zeta^j} \cap \Omega) \supset {}_{s_{w'^j}}Q'_{w'^j},$$

which when combined with (5.1) shows that $Q_{\zeta^j} = Q_{w^j}$. By passing to the limit, we get $Q_{z^0} = Q_{w^0}$. This is evidently a contradiction, since $I_{w^0} = \lambda^{-1}(\lambda(w^0)) \subset M$.

The case $w'^0 \in M'^-$ does not arise, as seen before. Hence the only possibility is that $w'^0 \in T' \setminus (M'^+ \cup M'^-)$. In this case, $z^0 \notin M_s^+$ again by the results of Section 4, and hence $z^0 \in T_2^+$. Consequently, $w^0 \in Q_{z^0} \subset \mathcal{L}$. ■

Lemma 5.3 *$\mathcal{L}$ is everywhere a finite union of smooth real analytic three-dimensional submanifolds of U_2. In particular, at all of its smooth points, the CR dimension of $\mathcal{L}$ is one.*

Proof It is possible to choose coordinates around $p = 0$ so that T_2^+ becomes the totally real plane $i\mathbb{R}^2 \subset \mathbb{C}^2$. The defining function for M near $p = 0$ can then be written as $r(z, \overline{w}) = 2x_2 + (2x_1)^{2m} a(z_1, y_2)$, where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ with $m > 1$, and $a(z_1, y_2)$ is a real analytic function that is positive near the origin. Recall that the complexification of M is given by $r(z, \overline{w})$, where $(z, w) \in U_2 \times U_2$. Let U_2^* denote the open set U_2 equipped with the conjugate holomorphic structure. Then

$$M^{\mathbb{C}} = \{(z, w) \in U_2 \times U_2^* : r(z, \overline{w}) = 0\}$$

is a smooth, closed complex manifold in $U_2 \times U_2^*$ of dimension 3. Note that

$$r(z, \overline{w}) = z_2 + \overline{w}_2 + (z_1 \overline{w}_1)^{2m} \tilde{a}(z_1, w),$$

where $\tilde{a}(0, 0) > 0$. By [9, Lemma 12.1] it follows that $(Q_0 \setminus \{0\}) \cap M \subset M_s^+$, *i.e.*, Q_0 intersects T_2^+ only at the origin. Let $w_1 = u_1 + iv_1, w_2 = u_2 + iv_2$ and define

$$\begin{aligned} \tilde{\mathcal{L}} &= \{(z, w) \in U_2 \times U_2^* : r(z, \overline{w}) = 0, u_1 = u_2 = 0\} \\ &= M^{\mathbb{C}} \cap \{(z, w) \in U_2 \times U_2^* : u_1 = u_2 = 0\}. \end{aligned}$$

Let $\pi \colon U_2 \times U_2^* \to U_2$ be the coordinate projection onto the (z_1, z_2) variables. Then it can be seen that $\pi(\tilde{\mathcal{L}}) = \mathcal{L}$. The equations that define $\tilde{\mathcal{L}} \subset U_2 \times U_2^*$ are

$$f_1 = x_2 + u_2 + \Re((z_1 + \overline{w}_1)^{2m} \tilde{a}(z_1, w)),$$

$$f_2 = y_2 - v_2 + \Im((z_1 + \overline{w}_1)^{2m} \tilde{a}(z_1, w)),$$

$$f_3 = u_1, \qquad f_4 = u_2,$$

where these are regarded as functions of $x_1, y_1, x_2, y_2, u_1, v_1, u_2$, and v_2. Define the map $F \colon U_2 \times U_2^* \to \mathbb{R}^4$ by $F = (f_1, f_2, f_3, f_4)$ and note that $\tilde{\mathcal{L}} = F^{-1}(0)$. Then the derivative of F at the origin is

$$DF(0) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where the rows are the gradients of f_1, f_2, f_3, f_4 with respect to the variables mentioned above in that order. This matrix has full rank, for the minor formed by the partial derivatives with respect to x_2, y_2, u_1, u_2 is non-zero. The implicit function theorem therefore shows that $\tilde{\mathcal{L}}$ is a smooth real four-dimensional manifold near the origin and the local coordinates on $\tilde{\mathcal{L}}$ are given by x_1, y_1, v_1, v_2. Moreover, there are real analytic functions h_1, h_2, h_3, h_4 defined in a neighbourhood of the origin in the x_1, y_1, v_1, v_2 variables such that $\tilde{\mathcal{L}}$ is described by

$$x_2 = h_1(x_1, y_1, v_1, v_2), \qquad y_2 = h_2(x_1, y_1, v_1, v_2),$$
$$u_1 = h_3(x_1, y_1, v_1, v_2), \qquad u_2 = h_4(x_1, y_1, v_1, v_2).$$

Working with the f_i's, it can be seen that the real tangent space to $\tilde{\mathcal{L}}$ at the origin is the direct sum of the complex line spanned by $z_1 = x_1 + iy_1$ and a totally real plane spanned by v_1, v_2. This description of the tangent space to $\tilde{\mathcal{L}}$ persists in a neighbourhood of the origin, and therefore the CR dimension of $\tilde{\mathcal{L}}$ is one near the origin.

Now if $(0, w^0) \in \pi^{-1} \cap \tilde{\mathcal{L}}$, then $0 \in Q_{w^0}$ and hence $w^0 \in Q_0$. But it is known that Q_0 intersects T_2^+ only at the origin if U_2 is small enough, and therefore with this choice of U_2 it follows that $w^0 = 0$. This shows that $\pi\colon \tilde{\mathcal{L}} \to U_2$ is proper. Hence $\mathcal{L}$ is subanalytic and therefore admits a subanalytic stratification by real analytic submanifolds [15]. To see what the dimension of $\mathcal{L}$ is, observe that the holomorphic projection π restricted to $\tilde{\mathcal{L}}$ is of the form $\pi(x_1, y_1, v_1, v_2) = (x_1, y_1, x_2, y_2)$ in terms of the local coordinates on $\tilde{\mathcal{L}}$. The differential

$$d\pi = \begin{pmatrix} e_1 \\ e_2 \\ \nabla h_1 \\ \nabla h_2 \end{pmatrix},$$

where $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$ and ∇h_1, ∇h_2 are the gradients with respect to x_1, y_1, v_1, v_2. To compute them, note that if $h = (h_1, h_2, h_3, h_4)$, the implicit function theorem again gives $Dh = -(\partial f_i/\partial a_j)^{-1}_{i,j} \cdot (\partial f_i/\partial b_j)_{i,j}$, where $(a_1, a_2, a_3, a_4) = (x_2, y_2, u_1, u_2)$ and $(b_1, b_2, b_3, b_4) = (x_1, y_1, v_1, v_2)$. Therefore,

$$Dh(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and hence $\nabla h_1(0) = (0, 0, 0, 0)$ and $\nabla h_2(0) = (0, 0, 0, 1)$. This implies that the rank of $d\pi$ equals 3 at the origin. Note that $d\pi$ cannot have full rank, *i.e.*, 4 at points close to the origin, for if it does have full rank at $a \in \tilde{\mathcal{L}}$, then $d\pi(a) \longrightarrow T_a\tilde{\mathcal{L}} \to \mathbb{C}^2$ is an isomorphism. But $T_a\tilde{\mathcal{L}}$ is the sum of a complex line (close to $z_1 = x_1 + iy_1$) and a totally real subspace (close to that spanned by v_1, v_2). Hence the kernel of $d\pi(a)$ is the sum of a complex line (close to $z_2 = x_2 + iy_2$) and a totally real subspace (close to that spanned by u_1, u_2). This, however, is a contradiction, since $\ker d\pi(a)$ must be a complex subspace. Thus the rank of π equals 3 everywhere on $\tilde{\mathcal{L}}$ and the rank theorem combined with the properness of π (see [15, Proposition 3.5, Lemma 3.5.1]) show that $\mathcal{L} = \pi(\tilde{\mathcal{L}})$ is locally everywhere a finite union of real analytic three-dimensional submanifolds of U_2. ■

Lemma 5.4 *$\overline{X}_f$ is complex analytic near $\mathcal{L} \times T_2'^-$.*

Proof It suffices to consider the behaviour of X_f near $(a, a') \in (\overline{X}_f \setminus X_f) \cap (\mathcal{L} \times T_2'^-)$. Fix small neighbourhoods $U_a, U'_{a'}$ around a, a', respectively. We may assume that

$$\mathcal{L} \cap U_a = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \cdots \cup \mathcal{L}_\mu,$$

where each $\mathcal{L}_j$ is a closed, real analytic three-dimensional submanifold of U_a. To start with, we will assume that $a \in \mathcal{L}_j \setminus \bigcup_{i \neq j} \mathcal{L}_i$ for some $1 \leq j \leq \mu$, so that a is a smooth point on $\mathcal{L}$. Since the CR dimension of $\mathcal{L} \times T_2'^-$ is one and X_f has pure dimension two, it follows that X_f admits analytic continuation, say $X_f^{\text{ext}} \subset U_a \times U'_{a'}$, which is a closed analytic set after shrinking these neighbourhoods if necessary. As before, let π, π' be the coordinate projections onto the factors $U_a, U'_{a'}$, respectively, and define

$$S = \{(w, w') \in X_f^{\text{ext}} : \dim_{(w,w')} (\pi')^{-1}(w') \geq 1\}.$$

The defining condition for X_f, *i.e.*, (2.1), forces $\pi' \colon X_f^{\text{ext}} \to U'_{a'}$ to be locally proper, and hence $\pi'(X_f^{\text{ext}})$ contains an open subset of $U'_{a'}$. Therefore, the Cartan–Remmert theorem (in [16], for example) implies that $\dim S \leq 1$.

Suppose that $(b, b') \in (\overline{X}_f \setminus X_f) \setminus S \subset \mathcal{L} \times T_2'^-$. It is then possible to choose neighbourhoods $W_b, W'_{b'}$ around b, b', respectively, such that the projection

$$\pi' \colon X_f^{\text{ext}} \cap (W_b \times W'_{b'}) \to W'_{b'}$$

is proper. Since $\overline{X}_f \setminus X_f \subset \mathcal{L} \times T_2'^-$, it follows that $\pi'(X_f \cap (W_b \times W'_{b'}))$ contains a one-sided neighbourhood of b', say, $\Omega' \subset U_1'^+$. Evidently, $\partial\Omega'$ contains a point from $M_s'^+$, and this contradicts the fact that the limit points of X_f in $U_1^+ \times (U' \cap M')$ are contained in $\mathcal{L} \times T_2'^-$. Thus $\overline{X}_f \setminus X_f \subset S$. But the three-dimensional Hausdorff measure of S is zero, and Shiffman's theorem implies that $\overline{X}_f$ itself is analytic in $W_b \times W'_{b'}$. Therefore, $X_f^{\text{ext}} = \overline{X}_f$. This argument works when $a \in \mathcal{L}$ is a smooth point. If $a \in \mathcal{L}_\alpha \cap \mathcal{L}_\beta$ for $\alpha \neq \beta$, $1 \leq \alpha, \beta \leq \mu$, the theorems of Cartan–Bruhat (see, for example, [17]) show that the singular locus of $\mathcal{L}_\alpha \cap \mathcal{L}_\beta$ is contained in a real analytic set of strictly lower dimension. Thus it is possible to proceed by downward induction to conclude that $\mathcal{L} \times T_2'^-$ is a removable singularity for X_f. ■

Lemma 5.5 *There exists a closed complex analytic set $\hat{X}_f \subset U_1 \times U'$ of pure dimension two such that $X_f \subset \hat{X}_f \cap (U_1^+ \times U'^+)$. In particular, f extends holomorphically across $p \in T_2^+$.*

Proof It suffices to show that X_f can be continued across $\mathcal{L} \times M'_e$. For this, note that by Lemmas 5.1 and 5.4, the projection $\pi \colon \overline{X}_f \setminus (\mathcal{L} \times M'_e) \to U_1^+$ is proper. The exceptional set M'_e, being a locally finite union of real analytic arcs and points, is locally pluripolar and hence globally so by Josefson's theorem. Let ϱ be a plurisubharmonic

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function on $\mathbb{C}^4$ such that $U_1^+ \times M'_e \subset \{\varrho = -\infty\}$. Since M is of finite type near p, it follows that $\mathcal{L} \cap M$ has real dimension at most two and hence it is possible to choose p^j, across which f holomorphically extends, to not lie on $\mathcal{L} \cap M$. Fix a small ball $B \subset U_1^+ \setminus \mathcal{L}$ on which f is well defined and note that by Lemma 5.2, $\overline{X}_f$ has no limit points on $B \times (U' \cap M')$. Therefore, the pluripolar set $\{\varrho = -\infty\}$ is a removable singularity for the non-empty analytic set $(\overline{X}_f \setminus (\mathcal{L} \times M'_e)) \setminus \{\varrho = -\infty\}$ by Bishop's theorem. Hence $\overline{X}_f \subset U_1^+ \times U'$ is analytic and the projection

$$\pi \colon \overline{X}_f \to U_1^+$$

remains proper. The coordinate functions z'_i (for $i = 1, 2$) restricted to $\overline{X}_f$ satisfy a monic polynomial whose coefficients are holomorphic functions on U_1^+. By Trepreau's theorem, each of these functions extend to a fixed, full neighbourhood of p in $\mathbb{C}^2$ and the zero locus of the resulting pair of polynomials, which are still monic in z'_i (for $i = 1, 2$), provides the continuation of X_f as an analytic set, say $\hat{X}_f \subset U_1 \times U'$. By [9, Theorem 7.4] it follows that f extends holomorphically across $p \in T_2^+$. ■

5.2 The Case When $\mathbf{p} \in \mathbf{T}_1^+ \cup \mathbf{T}_0^+$

Let $\gamma \subset T_1^+$ be a smooth real analytic arc and suppose that $p \in \gamma$. Then

$$C = \{w \in U_1 : \gamma \cap U_1 \subset Q_w\}$$

is a finite set. Indeed, $\gamma \cap U_1 \subset Q_w$ implies that Q_w is the unique complexification of $\gamma \cap U_1$. Since the Segre map has finite fibres near p, it follows that C must be finite. We may therefore assume that $Q_p \cap \gamma = \{p\}$ locally. All the previous arguments used in Subsection 5.1 can now be applied to show that f holomorphically extends across $p \in \gamma$. The remaining set is discrete in γ, and again the same arguments apply to show the extendability of f across all points on T_1^+ and hence also across T_0^+.

6 Proof of Theorem 1.1: Case (iii)

As discussed in Section 2, it suffices to consider the case when p is on either a one- or zero-dimensional stratum of the border. Let γ be a smooth real analytic arc in the border and suppose that $p \in \gamma$. We may also assume that $Q_p \cap \gamma = \{p\}$ locally near p to start with. In particular, if U is a neighbourhood of p in $\mathbb{C}^2$, it follows that $Q_p \cap M \cap \partial U$ is contained in the union of M^+, M^- and the two-dimensional strata of the border. Therefore, the behaviour of f near points on $Q_p \cap M \cap \partial U$ is known by cases (i) and (ii) of the main theorem. Suppose that $p \in \mathrm{cl}_f(p) \cap M_s'^{+}$. Choose a standard pair of neighbourhoods $U_1 \subset U_2$ around p and $U'_1 \subset U'_2$ around p' so that X_f as in (2.1) is a non-empty closed complex analytic set of pure dimension two. Let p^j be a sequence on M converging to p, across which f holomorphically extends for each j, and such that $f(p^j) \to p'$. Such a sequence exists by the reasoning given earlier. By shrinking U'_1, U'_2 if needed, we may assume that $Q'_{w'} \cap D'$ is relatively compactly contained in U'_2 and connected for all $w' \in U_1'^+$. Consider

$$\tilde{X}_f = \{(w, w') \in U_1^+ \times U_1'^+ : f({}_{s_w}Q_w) \subset Q'_{w'} \cap D\},$$

which evidently contains X_f near points of extendability of f and is also a pure two-dimensional local analytic set, by the arguments of Lemma 4.1. It is also closed, since $Q'_{w'} \cap D'$ is connected and compactly contained in U'_2, *i.e.*, the germs $f({}_{s_w}Q_w)$ cannot escape U'_2, because they are contained in $Q'_{w'} \cap D'$. By analytic continuation, $X_f \subset \tilde{X}_f$ everywhere in $U_1^+ \times U_1'^+$. Now [23, Proposition 4.3] shows that X_f has no limit points either on $(U_1 \cap M) \times U_1'^+$ or $U_1^+ \times (U' \cap M')$. Let X_j and C_j be defined as in (4.2) and (4.3). The absence of limit points of X_f on the aforementioned sets implies that C_j is a one-dimensional analytic set in $U_1 \times U_1'^+$. Again, [23, Proposition 4.3, Lemma 5.1] (see the proof of Lemma 4.8 in Section 4 as well), show that $\overline{C}_j$ is analytic in $U_1 \times U'_1$. By [23, Lemma 5.2] (or by Lemma 4.8 in Section 4) it follows that the cluster set of $\{C_j\}$ is non-empty in $U_1^+ \times U_1'^+$. Moreover, the volumes of $\{C_j\}$ are locally uniformly bounded in $U_1^+ \times U_1'^+$, and hence the sequence converges to a pure one-dimensional analytic set, say $C_p \subset U_1^+ \times U_1'^+$, that contains (p, p') in its closure. Furthermore, by continuity it is evident that $C_p \subset Q_p \times Q'_{p'}$. Thus there are points on $Q_p \setminus \{p\}$ over which X_f is a well-defined ramified cover. By adapting the arguments in [22] as done in Lemma 4.6(ii), it follows that the graph of f extends as an analytic set near (p, p'), and hence f extends holomorphically across p with $f(p) = p'$. This is clearly a contradiction, for it is possible to find strongly pseudoconcave points near p that are mapped locally biholomorphically to strongly pseudoconvex points near p'. It may be noted that the properness of f is not required here: it suffices for f to have discrete fibres near p, and this is guaranteed by Proposition 3.2. Thus the following lemma has been proved.

Lemma 6.1 *Let D, D' be domains in $\mathbb{C}^2$, both possibly unbounded, and $f\colon D \to D'$ a holomorphic mapping. Suppose that $M \subset \partial D$ and $M' \subset \partial D'$ are open pieces which are smooth real analytic and of finite type and let $p \in \gamma$, where γ is a one-dimensional stratum in the border between the pseudoconvex and pseudoconcave points on M. Let U be an open neighbourhood of p in $\mathbb{C}^2$ such that the cluster set of $U \cap M$ is bounded and contained in M'. Then the cluster set of p does not intersect $M_s'^+$.*

Continuing with the proof of Theorem 1.1(iii), note that since f is proper, it follows that $\mathrm{cl}_f(p)$ cannot contain points on M' that lie in $\hat{D}'$. Indeed, if $p' \in \mathrm{cl}_f(p) \cap M' \cap \hat{D}'$, the cluster set of p' under the correspondence $f^{-1}\colon D' \to D$ will only contain points in $\hat{D}$ by [9, Lemma 3.1], and in particular it would follow that $p \in \hat{D}$. Hence f would extend across p. Now choose a standard pair of neighbourhoods $U_1 \subset U_2$ around p and a neighbourhood U' containing $\mathrm{cl}_f(U_2 \cap M)$ as done in Section 5, so that X_f (as in (2.1)) is a non-empty closed complex analytic set in $U_1^+ \times U'^+$. By Lemma 5.1, it follows that X_f has no limit points on $U_1^+ \times (\partial U' \setminus D')$. Define

$$\mathcal{L}_\gamma = \bigcup_{w \in \gamma \cap U_1} Q_w,$$

which is seen to be locally foliated by open pieces of Segre varieties at all its regular points and have CR dimension one by the reasoning given in Lemma 5.3. Then by Lemmas 5.4 and 6.1, $\overline{X}_f$ is analytic near $\mathcal{L}_\gamma \times T_2'^+$. Again, by Lemma 6.1 and the

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remark made above about p not belonging to $\hat{D}$, it follows that X_f possibly has limit points only on $U_1^+ \times (M_e' \cup T_1'^+ \cup T_0'^+)$, which is a pluripolar set. The arguments used in Lemma 5.5 show that there is a closed complex analytic set $\hat{X}_f \subset U_1 \times U'$ that extends the graph of f near $\{p\} \times \mathrm{cl}_f(p)$, and hence f extends holomorphically across p. The remaining set C is discrete in γ and the same arguments apply to show that f extends across each point on γ and hence across the zero-dimensional strata on the border as well. This completes the proof of Theorem 1.1(iii).

References

[1] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, *Real Submanifolds in Complex Space and Their Mappings.* Princeton Mathematical Series 47. Princeton University Press, Princeton, NJ, 1999.

[2] F. Berteloot and A. Sukhov, *On the continuous extension of holomorphic correspondences.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **24**(1997), no. 4, 747–766.

[3] E. Chirka, *Regularity of the boundaries of analytic sets.* Mat. Sb. (N.S.) **117(159)**(1982), no. 3, 291–336, 431.

[4] ______, *Complex Analytic Sets.* Mathematics and its Applications (Soviet Series) 46. Kluwer Academic Publishers Group, Dordrecht, 1989.

[5] B. Coupet, S. Pinchuk, and A. Sukhov, *On boundary rigidity and regularity of holomorphic mappings.* Internat. J. Math. **7**(1996), no. 5, 617–643. http://dx.doi.org/10.1142/S0129167X96000335

[6] K. Diederich and J. E. Fornaess, *Proper holomorphic mappings between real-analytic pseudoconvex domains in* $\mathbb{C}^n$. Math. Ann. **282**(1988), no. 4, 681–700. http://dx.doi.org/10.1007/BF01462892

[7] ______, *Biholomorphic mappings between certain real analytic domains in* $\mathbb{C}^2$. Math. Ann. **245**(1979), no. 3, 255–272. http://dx.doi.org/10.1007/BF01673510

[8] K. Diederich, J. E. Fornaess, and Z. Ye, *Biholomorphisms in dimension* 2. J. Geom. Anal. **4**(1994), no. 4, 539–552. http://dx.doi.org/10.1007/BF01896407

[9] K. Diederich and S. Pinchuk, *Proper holomorphic maps in dimension* 2 *extend*. Indiana Univ. Math. J. **44**(1995), no. 4, 1089–1126.

[10] ______, *Regularity of continuous CR maps in arbitrary dimension*. Michigan Math. J. **51**(2003), no. 1, 111–140. http://dx.doi.org/10.1307/mmj/1049832896

[11] ______, *Analytic sets extending the graphs of holomorphic mappings*. J. Geom. Anal. **14**(2004), no. 2, 231–239.

[12] K. Diederich and S. M. Webster, *A reflection principle for degenerate real hypersurfaces*. Duke Math. J. **47**(1980), no. 4, 835–843. http://dx.doi.org/10.1215/S0012-7094-80-04749-3

[13] J. E. Fornaess and E. Low, *Proper holomorphic mappings*. Math. Scand. **58**(1986), no. 2, 311–322.

[14] J. E. Fornaess and N. Sibony, *Construction of P.S.H. functions on weakly pseudoconvex domains*. Duke Math. J. **58**(1989), no. 3, 633–655. http://dx.doi.org/10.1215/S0012-7094-89-05830-4

[15] H. Hironaka, *Subanalytic sets*. In: Number Theory, Algebraic Geometry and Commutative Algebra, in Honor of Yasuo Akizuki. Kinokuniya, Tokyo, 1973, pp. 453–493.

[16] S. Łojasiewicz, *Introduction to Complex Analytic Geometry.* Birkhäuser-Verlag, Basel, 1991.

[17] R. Narasimhan, *Introduction to the Theory of Analytic Spaces.* Lecture Notes in Mathematics 25. Springer-Verlag, Berlin, 1966.

[18] S. Pinchuk and S. Tsyganov, *Smoothness of CR-mappings between strictly pseudoconvex hypersurfaces*. Izv. Akad. Nauk SSSR Ser. Mat. **53**(1989), no. 5, 1120–1129, 1136; translation in Math. USSR-Izv. **35**(1990), no. 2, 457–467.

[19] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$. Grundlehren der Mathematischen Wissenschaften 241. Springer-Verlag, New York, 1980.

[20] N. Sibony, *Some aspects of weakly pseudoconvex domains*. In: Several Complex Variables and Complex Geometry, Part 1. Proc. Sympos. Pure Math. 52. American Mathematical Society, Providence, RI, 1991, pp. 199–231.

[21] A. Sukhov, *On the boundary regularity of holomorphic mappings*. Mat. Sb. **185**(1994), no. 12, 131–142; translation in Russian Acad. Sci. Sb. Math. **83**(1995), no. 2, 541–551

[22] R. Shafikov, *Analytic continuation of germs of holomorphic mappings between real hypersurfaces in* $\mathbb{C}^n$. Michigan Math. J. **47**(2000), 133–149. http://dx.doi.org/10.1307/mmj/1030374673

[23] R. Shafikov and K. Verma, *A local extension theorem for proper holomorphic mappings in* $\mathbb{C}^2$. J. Geom. Anal. **13**(2003), no. 4, 697–714.

[24] J.-M. Trépreau, *Sur le prolongement holomorphe des fonctions C-R définés sur une hypersurface réelle de classe C^2 dans $\mathbb{C}^n$*. Invent. Math. **83**(1986), no. 3, 583–592. http://dx.doi.org/10.1007/BF01394424

[25] K. Verma, *Boundary regularity of correspondences in* $\mathbb{C}^2$. Math. Z. **231**(1999), no. 2, 253–299. http://dx.doi.org/10.1007/PL00004728

Department of Mathematics, The University of Western Ontario, London, On, N6A 5B7
e-mail: shafikov@uwo.ca

Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
e-mail: kverma@math.iisc.ernet.in