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ON THE BASIS NUMBER OF SOME COMPLETE BIPARTITE GRAPHS

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This note completes the determination of the basis numbers of the complete bipartite graphs.

For a connected graph G, with p vertices and q edges, which is finite and does not contain loops or multiple edges, let $e_1, e_2, \ldots, e_q$ be an ordering of the edges. Then any subset S of the edges corresponds to a q-dimensional vector $(a_1, a_2, \ldots, a_q)$ with $a_i = 1$ if $e_i \in S$ and $a_i = 0$ if $e_i \notin S$, for $i = 1, 2, \ldots, q$. These vectors form a vector space over the field Z_2. Those vectors which correspond to cycles of G generate a subspace called the *cycle space of* G, denoted $C(G)$. Strictly speaking the vectors generate $C(G)$, but we usually think of the corresponding cycles as elements of the space.

A basis for $C(G)$ is called *k-fold* if each edge of G occurs in at most k cycles of the basis. The *basis number of* G, denoted $b(G)$, is defined as the minimum integer k such that G has a k-fold basis. Mac Lane [1] has shown that G is planar if and only if $b(G) \le 2$. Schmeichel [2] has investigated the basis number of some classes of non-planar graphs and proved that $b(K_n) = 3$ for $n \ge 5$, where K_n denotes

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the complete graph with n vertices. He also established the basis number of most complete bipartite graphs $K_{m,n}$, for $m, n \geq 5$, $b(K_{m,n}) = 4$, with the possible exceptions of $K_{5,5}, K_{5,6}, K_{5,7}, K_{5,8}, K_{6,6}, K_{6,7}, K_{6,8}$ and $K_{6,10}$. The remainder of this note constructs a 3-fold basis for each of the eight graphs listed above, showing that the basis number is three in each case.

We note here that any 4-cycle in $K_{m,n}$ is completely determined by its vertices, so will be denoted by listing them. We will represent the vertices in X by numbers and those in Y by letters, where X and Y are the two complementary, independent sets of vertices of $K_{m,n}$.

The method employed to find these 3-fold bases is to modify the 4-fold bases given by Schmeichel using a sequence of transformations performed as follows. Consider a copy of $K_{2,3}$ in $K_{m,n}$. It has three cycles C_1, C_2 and C_3. At most two of these can belong to any basis of the cycle space of $K_{m,n}$. Suppose Schmeichel's basis contains two of them, say C_1 and C_2. Our transformation is to replace either of these by C_3, thereby yielding a new basis.

Now suppose B is a basis of the cycle space of $K_{m,n}$ and B contains $12ab$ and $12bc$. These cycles span a copy of $K_{2,3}$ and so $12ab$ may be replaced by $12ac$ according to our transformation. This replacement will be denoted by $12a(b \to c)$. The successive application of $12a(b \to c)$ and $12a(c \to d)$ will be denoted by $12a(b \to c \to d)$. This notation can be extended in obvious ways.

If the letter labels are ordered, $a < b < c$ and so on, and we say $a + 1 = b$, $b + 1 = c$, and so on, then let X' (respectively Y') denote the set obtained from X (respectively Y) by deleting the vertex with the highest valued label. The set of cycles of the form $x \;\; x+1 \;\; y \;\; y+1$ with $x \in X'$ and $y \in Y'$ constitute Schmeichel's basis.

The sequences of transformations listed below yield 3-fold bases for the graphs in question, when applied to the bases given by Schmeichel.

For $K_{5,5}$

$23(c \to b)d$ $23(b \to a)c$ $3(4 \to 5)bc$ $2(3 \to 4)ab$ $12(b \to a)c$ $45(b \to a)c$
$(2 \to 1)4ab$ $1(4 \to 5)ab$ $34c(d \to e)$ $12c(d \to e)$

For $K_{5,6}$

$23(c \to b)d$ $23(b \to a)c$ $45(c \to b)d$ $45(b \to a)c$ $12(b \to a)c$ $(2 \to 1)3ac$
$23(b \to a)d$ $12(a \to b)c$ $(3 \to 2)4ab$ $(2 \to 1)3ab$ $56(b \to a)c$ $4(5 \to 6)ac$
$45(b \to a)d$ $56(a \to b)c$ $2(4 \to 5)ab$ $4(5 \to 6)ab$ $23a(d \to e)$ $45a(d \to e)$

For $K_{5,7}$

$23(c \to b)d$ $23(b \to a)c$ $12(b \to a)c$ $(2 \to 1)3ac$ $23(b \to a)d$ $12(a \to b)c$
$(3 \to 2)4ab$ $(2 \to 1)3ab$ $3(4 \to 5 \to 6 \to 7)bc$ $56(b \to a)c$ $56a(c \to d \to e)$
$67c(d \to e)$ $4(5 \to 6 \to 7)de$ $34c(d \to e)$ $12c(d \to e)$ $5(6 \to 7)de$
$6(7 \to 4 \to 3 \to 2 \to 1)de$

For $K_{5,8}$

$23(c \to b \to a)d$ $23a(d \to e)$ $45(c \to b \to a)d$ $45a(d \to e)$ $67(c \to b \to a)d$
$67a(d \to e)$ $(2 \to 1)3ab$ $(4 \to 3 \to 2 \to 1)5bc$ $12(b \to a)c$ $(3 \to 2)4bc$
$12a(c \to d)$ $(2 \to 1)3de$ $12c(d \to e)$ $6(7 \to 8)ab$ $1(5 \to 6 \to 7 \to 8)bc$
$78(b \to a)c$ $5(6 \to 7)bc$ $78a(c \to d)$ $6(7 \to 8)de$ $78c(d \to e)$

For $K_{6,6}$

$23(c \to b)d$ $23(b \to a)c$ $12(b \to a)c$ $(2 \to 1)3ac$ $23(b \to a)d$ $12(a \to b)c$
$(3 \to 2)4ab$ $(2 \to 1)3ab$ $45(c \to b)d$ $45(b \to a)c$ $56(b \to a)c$ $4(5 \to 6)ac$
$45(b \to a)d$ $56(a \to b)c$ $2(4 \to 5)ab$ $4(5 \to 6)ab$ $23a(d \to e)$ $23d(e \to f)$
$12d(e \to f)$ $(2 \to 1)3df$ $23a(e \to f)$ $12d(f \to e)$ $(3 \to 2)4ef$ $(2 \to 1)3ef$
$45a(d \to e)$ $45d(e \to f)$ $56d(e \to f)$ $4(5 \to 6)df$ $45a(e \to f)$ $56d(f \to e)$
$2(4 \to 5)ef$ $4(5 \to 6)ef$

For $K_{6,7}$

$3(4 \to 5 \to 6 \to 7)ab$ $(4 \to 3 \to 2 \to 1)5ef$ $23(e \to d \to c \to b \to a)f$
$56a(b \to c \to d \to e \to f)$ $(2 \to 1)3cd$ $5(6 \to 7)cd$ $(3 \to 2 \to 1)4bc$
$4(5 \to 6 \to 7)de$ $34c(d \to e \to f)$ $45(c \to b \to a)d$ $12b(c \to d \to e \to f)$
$67(d \to c \to b \to a)e$ $12(b \to a)f$ $(2 \to 1)3ab$ $67a(e \to f)$ $5(6 \to 7)ef$

For $K_{6,8}$

$(4 \to 3 \to 2 \to 1)5cd$ $1(5 \to 6 \to 7 \to 8)cd$ $23(c \to b \to a)d$ $23a(d \to e \to f)$
$67(c \to b \to a)d$ $67a(d \to e \to f)$ $45(b \to a)c$ $45d(e \to f)$ $(3 \to 2 \to 1)4ab$
$(2 \to 1)3ab$ $23d(e \to f \to a)$ $12a(b \to c \to d \to e \to f)$ $(2 \to 1)3ef$
$5(6 \to 7 \to 8)ab$ $6(7 \to 8)ab$ $67d(e \to f \to a)$ $78a(b \to c \to d \to e \to f)$
$6(7 \to 8)ef$

For $K_{6,10}$

$(5 \to 4 \to 3 \to 2)6cd$ $2(6 \to 7 \to 8 \to 9)cd$ $23c(d \to e)$ $23(c \to b)e$ $(2 \to 1)3bc$
$(2 \to 1)3de$ $(2 \to 1)3ab$ $12(b \to a)c$ $(2 \to 1)3ef$ $12d(e \to f)$ $(4 \to 3)5cd$
$34(b \to a)c$ $34(c \to a)d$ $34d(e \to f)$ $34a(d \to f)$ $(3 \to 1 \to 2)4ab$
$(3 \to 1 \to 2)4ef$ $2(4 \to 5)ab$ $2(4 \to 5)ef$ $56a(b \to c)$ $56(e \to d)f$ $(5 \to 4)6bc$
$(5 \to 4)6de$ $89(c \to b)d$ $89b(d \to e)$ $8(9 \to 0)bc$ $8(9 \to 0)de$ $90(b \to a)c$
$8(9 \to 0)ab$ $90d(e \to f)$ $8(9 \to 0)ef$ $6(7 \to 8)cd$ $78(b \to a)c$ $78(c \to a)d$
$78d(e \to f)$ $78a(d \to f)$ $7(8 \to 0 \to 9)ab$ $7(8 \to 0 \to 9)ef$ $(7 \to 6)9ab$
$(7 \to 6)9ef$ $4(6 \to 7)bc$ $4(6 \to 7)de$

These transformations yield the following 3-fold basis. The bases are listed in the order which results from Schmeichel's basis, that is when the cycle C replaces cycle C' in the basis, the cycle C is listed where C' would have been.

For $K_{5,5}$

12*ab* 12*ac* 12*ce* 12*de* 15*ab* 23*ac* 23*bd* 23*de* 34*ab* 35*bc* 34*ce* 34*de*
45*ab* 45*ac* 45*cd* 45*de*

For $K_{5,6}$

12*ab* 12*bc* 12*cd* 12*de* 13*ab* 13*ac* 23*ae* 23*de* 25*ab* 34*bc* 34*cd* 34*de*
46*ab* 46*ac* 45*ae* 45*de* 56*ab* 56*bc* 56*cd* 56*de*

For $K_{5,7}$

12*ab* 12*bc* 12*ce* 12*de* 13*ab* 13*ac* 23*ad* 23*de* 24*ab* 37*bd* 34*ce* 34*de*
45*ab* 45*bc* 45*cd* 47*de* 56*ab* 56*ac* 56*cd* 57*de* 67*ab* 67*bc* 67*ce* 16*de*

For $K_{5,8}$

12ab 12ad 12ce 12de 13ab 23bc 23ae 13de 34ab 24bc 34cd 34de
45ab 18bc 45ae 45de 56ab 57bc 56cd 56de 68ab 67bc 67ae 68de
78ab 78ad 78ce 78de

For $K_{6,6}$

12ab 12bc 12cd 12de 12ef 13ab 13ac 23af 13df 13ed 25ab 34bc
34cd 34de 25ef 46ab 46ac 45af 46df 46ef 56ab 56bc 56cd 56de
56ef

For $K_{6,7}$

12ab 12af 12cd 12de 12ef 13ab 23bc 13cd 23de 23af 37ab 14bc
34cf 34de 34ef 45ab 45bc 45ad 47de 15ef 56af 56bc 57cd 56de
57ef 67ab 67bc 67cd 67af 67ef

For $K_{6,8}$

12af 12bc 12cd 12de 12ef 13ab 23bc 23af 23ad 13ef 14ab 34bc
34cd 34de 34ef 45ab 45ac 13cd 45df 45ef 58ab 56bc 56cd 56de
56ef 68ab 67bc 67af 67ad 68ef 78af 78bc 78cd 78de 78ef

For $K_{6,10}$

12ab 12ac 12cd 12df 12ef 13ab 13bc 23be 13de 13ef 25ab 34ac
34af 34df 25ef 45ab 45bc 35cd 45de 45ef 56ac 47bc 29cd 47de
56df 67ab 67bc 68cd 67de 67ef 69ab 78ac 78af 78df 69ef 80ab
89bc 89be 80de 80ef 90ab 90ac 90cd 90df 90ef

References

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[2] E.F. Schmeichel, "The basis number of a graph", *J. Combin. Theory Ser. B* **30** (1981), 123-129.

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