# A model of a hyperboloid of one sheet and its asymptotic cone

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In this article is described the construction of a thread model of a hyperboloid of one sheet $(H)$ and its asymptotic cone $(C)$. It is simple to make, requiring only cardboard and thread, and can be made collapsible and of pocket size if desired. The model consists of two hinged pieces of cardboard (intersecting planes $\pi$ and $\bar{\pi}$) on which are drawn circles $S_H$, $\bar{S}_H$ respectively in which the planes meet $H$, and the concentric circles $S_C$, $\bar{S}_C$ respectively in which the planes meet $C$. A number of generators of the same system on $H$ are now represented by threads joining $S_H$ and $\bar{S}_H$, and the corresponding parallel generators of $C$ are represented by threads joining $S_C$ and $\bar{S}_C$. In order to ensure that these generators are well spaced, those of $C$ are taken at equal eccentric angles apart in a principal elliptic section. The main theorem used in the design is that if $l$ is a generator of $C$, then the tangent plane to $C$ at points of $l$ meets $H$ in two generators both of which are parallel to $l$.

Fig. 1 shows two similar rectangular cards $ABCD$, $\bar{A}\bar{B}\bar{C}\bar{D}$, hinged with $AB$ to $\bar{B}\bar{A}$ so that the cards can be folded together. When in position, these cards make a chosen angle, say $2\theta$, which could well be $\pi/2$. The four circles in fig. 1 are the circles mentioned above, and the diagrams on the two cards, giving the points at which holes must be made for the threads to pass through, are exactly the same. It is therefore convenient to construct a separate diagram for one plane only (fig. 2) and to prick through this at the required points on to each of the cards. The cards are then held in position with struts (*e.g.* of wood or meccano) while the threading is done, and when finished the struts can be removed and the model closed. The model can then be held open when required at any time.

To determine the dimensions of the model, let the radii of $S_H$, $S_C$ be $b$, $c$ respectively and let the perpendicular distance of their common centre from $AB$ be $a$, the mid-point $O$ of $AB$ being the foot of this perpendicular. Then $a$, $b$, $c$, as well as $\theta$, can have any desired

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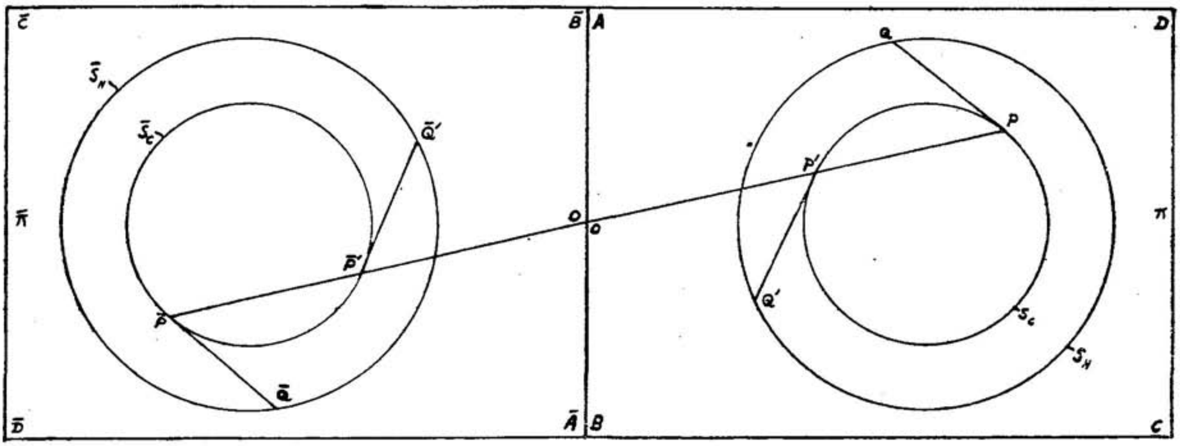


FIGURE 1.

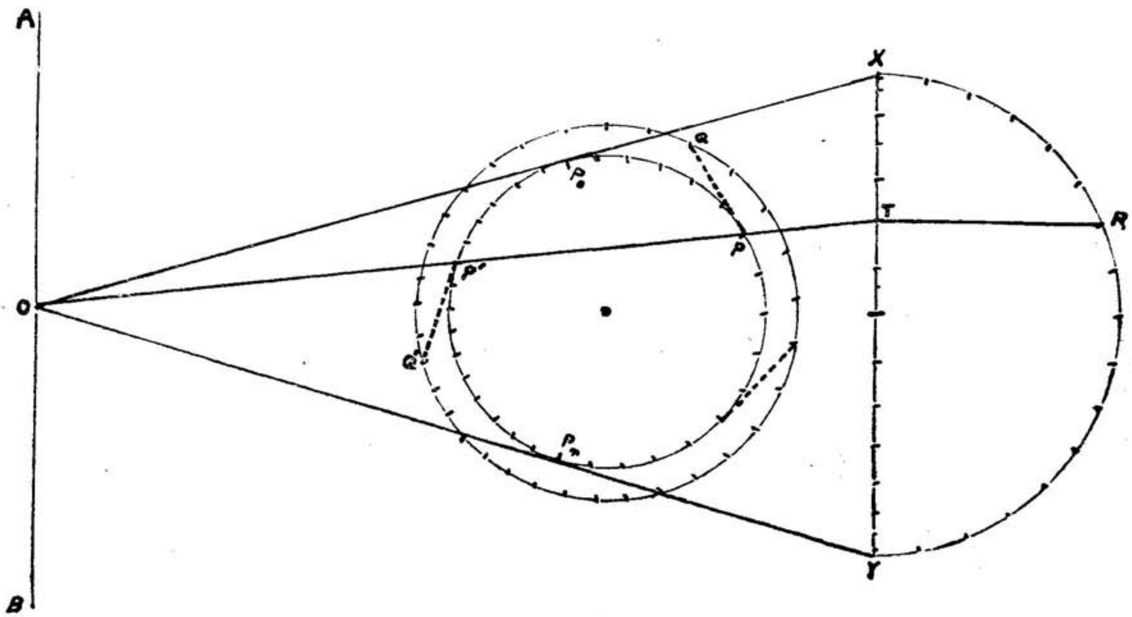


FIGURE 2.

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values ($a > b > c$), and the model is fixed when these values are known. The equation of $H$ referred to its principal axes is found to be

$$(a^2 - c^2)\,x^2 + a^2 \sec^2\theta \,.\, y^2 - c^2 \operatorname{cosec}^2\theta \,.\, z^2 = (a^2 - c^2)\,(b^2 - c^2).$$

For a well shaped model, $c$ should be nearly equal to $b$. In my own model, $a = 7{\cdot}5\ cm.$, $b = 2{\cdot}5\ cm.$, $c = 2{\cdot}1\ cm.$

Returning for a moment to fig. 1, consider any point $P$ of $S_C$, let $O\,P$ meet $S_C$ again at $P'$, and let $\bar{P}$, $\bar{P}'$ be the points similarly placed on $\bar{S}_C$. Then with the cards in position, the lines $P\,\bar{P}'$, $\bar{P}\,P'$ are generators of $C$. To find the corresponding (parallel) generators of one system on $H$, draw the tangents to $S_C$ at $P$ and $P'$ to meet $S_H$ at $Q$ and $Q'$ respectively, and construct $\bar{Q}$ and $\bar{Q}'$ similarly as shown, all tangents being drawn in the counter-clockwise direction. Then the lines $Q\,\bar{Q}'$, $\bar{Q}\,Q'$ are generators of $H$ of the same system, $Q\,\bar{Q}'$ being parallel to $P\,\bar{P}'$ and $\bar{Q}\,Q'$ to $\bar{P}\,P'$.

It only remains to construct suitable points on $S_C$ and the corresponding points on $S_H$ in fig. 2. For a well spaced set of $2\,n$ generators on $C$, as described earlier, there is the following construction. Draw equal tangents $O\,X$, $O\,Y$ to $S_C$, of any convenient length, and let the points of contact be $P_0$, $P_n$. Draw a semi-circle on $X\,Y$ as diameter and divide its arc into $n$ equal parts by points $R_r$ ($r = 1, \ldots, n-1$); let $T_r$ be the foot of the perpendicular from $R_r$ to $X\,Y$ and let $O\,T_r$ meet $S_C$ at $P_r$ and $P'_r$. In this way we get $n - 1$ pairs $P_r$, $P'_r$ which with the double-points $P_0$, $P_n$ give the required $2\,n$ points on $S_C$. The corresponding points on $S_H$ are found by drawing tangents to $S_C$ at the $P$'s; the simplest way of doing this is to find the length of $P\,Q$ for one such tangent and to describe circles with this radius and centres at the $P$'s. Fig. 2 shows the completed construction for $n = 16$. These $4\,n$ points, together with $A$, $B$ and $O$, must now be transferred to each of the cards, which are then ready for threading.

The cards need not be thick but should be fairly stiff, and the holes should be made just big enough for a needle to pass through. For a small model, linen crochet thread is very satisfactory, say red for the generators of $C$ and blue for the generators of $H$; this can be used in lengths of three or four yards. When completed, the lacing and knots can be covered by other cards or paper glued to the backs of the original cards, which are thereby stiffened.

In a large scale model, generators of the other system on $H$ can be included by drawing tangents to $S_C$ in the clockwise direction and proceeding as before. This would not be practicable in a small model.

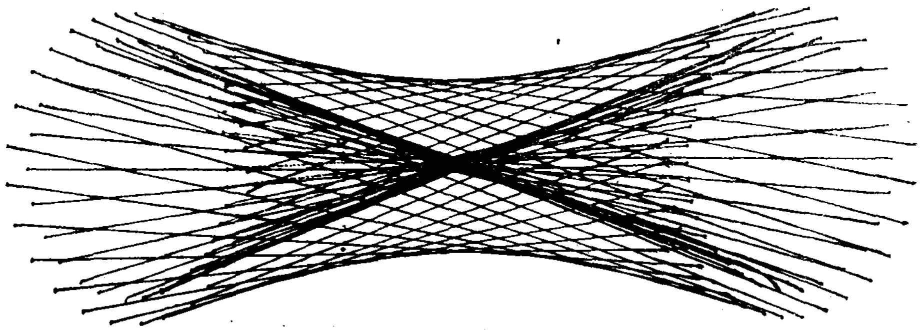

FIGURE 3. Photograph of the model[1].

[1] The author's model (measuring 5″ by 4″ when closed) was shown at a meeting of the Edinburgh Mathematical Society, 2nd June, 1945, along with some other pocket quadrics also made by the author. For descriptions of the others (folding string models and collapsible models made from circular cardboard sections) see W. H. McCrea, *Analytical geometry of three dimensions*, University Mathematical Texts, Edinburgh (Oliver & Boyd), 1942, pp. 110-124; H. W. Turnbull, "Collapsible circular sections of quadric surfaces," *Edinburgh Math. Notes*, No. 32, 1941, pp. xvi-xix.

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