CHAINS OF CONGRUENCES ON A COMPLETELY 0-SIMPLE SEMIGROUP [1]

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To Alexander Doniphan Wallace on his 60th birthday

Let ρ and σ be two congruences on a completely 0-simple semigroup. Suppose that there is a maximal chain of congruences from ρ to σ which is of finite length. Then, as we shall show, any maximal chain of congruences from ρ to σ is finite and of the same length.

Throughout, all congruences considered will be on a single completely 0-simple semigroup which, without loss of generality, may be taken as a Rees matrix semigroup $\mathscr{M}^0$ $(G; I, \Lambda; P)$ (see Rees [3]; the notation and terminology of Clifford and Preston [1] is being used). Here G is the structure group of $\mathscr{M}^0$, I and Λ are index sets and P is a regular $\Lambda \times I$ matrix, the sandwich matrix of $\mathscr{M}^0$. The entries $p_{\lambda i}$ of P belong to the group with zero G^0. The elements of $\mathscr{M}^0$ are the ordered triples $(a; i, \lambda)$ where $a \in G^0$, $i \in I$, $\lambda \in \Lambda$ and where $(0; i, \lambda) = (0; j, \mu) = 0$, the zero of $\mathscr{M}^0$, for all i, j, λ, μ. The product in $\mathscr{M}^0$ is defined by

$$(a; i, \lambda)(b; j, \mu) = (ap_{\lambda j}b; i, \mu).$$

The $\mathscr{H}$-classes of $\mathscr{M}^0$, other than the set $\{0\}$, are the sets $H_{i\lambda} = \{(a; i, \lambda) \colon a \in G\}$, $i \in I$, $\lambda \in \Lambda$. For each $i \in I[\lambda \in \Lambda]$ there exists $\lambda \in \Lambda$ $[i \in I]$ such that $H_{i\lambda}$ is a subgroup of $\mathscr{M}^0$ (isomorphic to G). Without loss of generality we may assume that $1 \in I \cap \Lambda$ and that H_{11} is a group.

Let Z denote the $I \times \Lambda$ rectangle of $\mathscr{H}$-classes $\{H_{i\lambda} \colon i \in I, \lambda \in \Lambda\}$. A permissible partition of Z is a partition of Z induced by partitions of I and Λ and which is such that each subrectangle of the partition either is a completely simple subrectangle, i.e. every $\mathscr{H}$-class it contains is a group, or is a zero subrectangle i.e. it contains no $\mathscr{H}$-class which is a group. If $\mathscr{P}$ and $\mathscr{Q}$ are two permissible partitions of Z then $\mathscr{P} \subseteqq \mathscr{Q}$ means that each subrectangle of $\mathscr{P}$ is contained in a subrectangle of $\mathscr{Q}$. We shall speak of the partition classes into which $\mathscr{P}$ divides Z as $\mathscr{P}$-classes.

[1] The results in this paper were presented at the National Science Foundation Summer Institute in Algebra, held at Pennsylvania State University, 1963.

[2] I have profited from a detailed discussion of these results with Dr J. M. Howie.

With these assumptions and this notation it was proved in [2] that, except for the universal congruence, the congruences on $\mathscr{M}^0$ are in one-to-many correspondence with the quadruples $[\mathscr{P}, M, \{e_i\}, \{f_\lambda\}]$, where (i) $\mathscr{P}$ is a permissible partition of Z, (ii) M is a normal subgroup of H_{11} and (iii) $e_i \in H_{i1}$ $(i \in I)$ and $f_\lambda \in H_{1\lambda}$ $(\lambda \in \Lambda)$ are elements of $\mathscr{M}^0$ such that

$$(1) \qquad Mf_\lambda e_i = Mf_\mu e_j$$

when $H_{i\lambda}$ and $H_{j\mu}$ belong to the same completely simple $\mathscr{P}$-class. The equivalence classes of the congruence (determined by) $[\mathscr{P}, M, \{e_i\}, \{f_\lambda\}]$ are the sets $\cup\{e_i M a f_\lambda\colon i \in i^*, \lambda \in \lambda^*\}$, where $a \in H_{11}$ and $i^* (\subseteqq I)$ and $\lambda^* (\subseteqq \Lambda)$ the "sides" of a $\mathscr{P}$-class. In particular, M is the intersection with H_{11} of one of the equivalence classes. Also proved in [2] is:

$$(\alpha) \qquad [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}] \subseteqq [\mathscr{Q}, N, \{g_i\}, \{h_\lambda\}]$$

if and only if

(i) $\mathscr{P} \subseteqq \mathscr{Q}$,

(ii) $M \subseteqq N$,

(iii) *for each subrectangle of $\mathscr{P}$, with sides i^* and λ^*, say, there exist a_{i^*} and b_{λ^*} in H_{11} such that, for $i \in i^*$ and $\lambda \in \lambda^*$,*

$$e_i f_1 = g_i n_i a_{i^*} h_1,$$
$$e_1 f_\lambda = g_1 n_\lambda b_{\lambda^*} h_\lambda,$$

where n_i and n_λ belong to N.

As a corollary to (α) (see [2]) we have:

(β) *Let*
$$\rho = [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}],$$
$$\sigma = [\mathscr{P}, M, \{g_i\}, \{h_\lambda\}].$$

Then $\rho = \sigma$ if and only if $\rho \subseteqq \sigma$.

We shall prove our result by means of a series of lemmas. First, a comment on notation. The inclusion sign $\subset$ and the containment sign $\supset$ will be taken to mean proper inclusion and proper containment, respectively. In a partially ordered set L, ordered by $\leqq$, say, an element b will be said to *cover* an element a if (i) $a \leqq b$, (ii) $a \neq b$ and (iii) $a \leqq c \leqq b$, for c in L, implies that either $a = c$ or $c = b$. We shall write $b \succ a$, or $a \prec b$, to denote that b covers a. The set L involved will either be specified or be clear from the context.

LEMMA 1. *Let $\rho = [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}]$ and $\sigma = [\mathscr{P}, N, \{g_i\}, \{f_\lambda\}]$ be congruences on $\mathscr{M}^0$. Suppose that $\rho \subseteqq \sigma$. Then $\sigma = [\mathscr{P}, N, \{e_i\}, \{f_\lambda\}]$.*

PROOF. Since, by (α), $M \subseteqq N$, $Mf_\lambda e_i = Mf_\mu e_j$ implies $Nf_\lambda e_i = Nf_\mu e_j$. Consequently, $[\mathscr{P}, N, \{e_i\}, \{f_\lambda\}]$ is a congruence on $\mathscr{M}^0$. Using (i), (ii) and

(iii) of (α) $\rho \subseteqq \sigma$ implies that $[\mathscr{P}, N, \{e_i\}, \{f_\lambda\}] \subseteqq \sigma$. Whence, by ($\beta$), $\sigma = [\mathscr{P}, N, \{e_i\}, \{f_\lambda\}]$.

Lemma 2. *Let* $\rho = [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}]$ *and* $\sigma = [\mathscr{Q}, M, \{g_i\}, \{h_\lambda\}]$ *be congruences on* $\mathscr{M}^0$. *Suppose that* $\rho \subseteqq \sigma$. *Then* $\rho = [\mathscr{P}, M, \{g_i\}, \{h_\lambda\}]$.

Proof. $Mh_\lambda g_i = Mh_\mu g_j$ for any i, j, λ, μ such that $H_{i\lambda}$ and $H_{j\mu}$ belong to the same completely simple subrectangle of $\mathscr{Q}$. Since, by (α), $\mathscr{P} \subseteqq \mathscr{Q}$, it follows that $Mh_\lambda g_i = Mh_\mu g_j$ for any i, j, λ, μ such that $H_{i\lambda}$ and $H_{j\mu}$ belong to the same completely simple subrectangle of $\mathscr{P}$. Hence $[\mathscr{P}, M, \{g_i\}, \{h_\lambda\}]$ is a congruence on $\mathscr{M}^0$. As in the proof of lemma 1, it now follows from (α) that $\rho \subseteqq [\mathscr{P}, M, \{g_i\}, \{h_\lambda\}]$, whence, by ($\beta$), $\rho = [\mathscr{P}, M, \{g_i\}, \{h_\lambda\}]$.

The result asserted in the next lemma may be easily verified.

Lemma 3. (i) *Let* ρ *and* σ *be congruences on* $\mathscr{M}^0$ *such that* $\rho = [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}] \subseteqq [\mathscr{P}, N, \{e_i\}, \{f_\lambda\}] = \sigma$. *Let* K *be a normal subgroup of* H_{11} *such that* $M \subseteqq K \subseteqq N$. *Then* $\tau = [\mathscr{P}, K, \{e_i\}, \{f_\lambda\}]$ *is a congruence on* $\mathscr{M}^0$ *and* $\rho \subseteqq \tau \subseteqq \sigma$.

(ii) *Let* ρ *and* σ *be congruences on* $\mathscr{M}^0$ *such that* $\rho = [\mathscr{P}, M, \{g_i\}, \{h_\lambda\}] \subseteqq [\mathscr{Q}, M, \{g_i\}, \{h_\lambda\}] = \sigma$. *Let* $\mathscr{S}$ *be a permissible partition of* Z *such that* $\mathscr{P} \subseteqq \mathscr{S} \subseteqq \mathscr{Q}$. *Then* $\tau = [\mathscr{S}, M, \{g_i\}, \{h_\lambda\}]$ *is a congruence on* $\mathscr{M}^0$ *and* $\rho \subseteqq \tau \subseteqq \sigma$.

Lemma 4. *Let* $\rho = [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}]$ *and* $\sigma = [\mathscr{Q}, N, \{g_i\}, \{h_\lambda\}]$ *be two congruences on* $\mathscr{M}^0$ *and suppose that* $\sigma \succ \rho$ *in the lattice of congruences on* $\mathscr{M}^0$. *Then, either* (i) $\mathscr{P} = \mathscr{Q}$ *and* $N \succ M$ *in the lattice of normal subgroups of* H_{11}, *or* (ii) $M = N$ *and* $\mathscr{Q} \succ \mathscr{P}$ *in the lattice of permissible partitions of* Z.

Proof. Since $\rho \subseteqq \sigma$ we know by (α) that $\mathscr{P} \subseteqq \mathscr{Q}$ and $M \subseteqq N$. Since $\mathscr{P} \subseteqq \mathscr{Q}$ and σ is a congruence, it follows that $[\mathscr{P}, N, \{g_i\}, \{h_\lambda\}], = \tau$, say, is a congruence on $\mathscr{M}^0$. Again using (α) it follows from $\rho \subseteqq \sigma$ that $\rho \subseteqq \tau \subseteqq \sigma$. Whence, if $\mathscr{P} \subset \mathscr{Q}$ and $M \subset N$, $\rho \subset \tau \subset \sigma$. This is contrary to hypothesis, for $\sigma \succ \rho$. Hence either (i) $\mathscr{P} = \mathscr{Q}$ or (ii) $M = N$. The remaining assertions of the lemma under (i) and (ii) follow immediately from lemma 3 part (i) and part (ii), respectively.

We now come to the key interchange lemma which enables us to prove our theorem. If $\sigma \supset \rho$ and σ and ρ have the same permissible partition of Z, then σ and ρ will be said to be obtained from one another by a *group change*. We shall write this as $\sigma \supset_g \rho$; and, when also $\sigma \succ \rho$, we shall similarly write $\sigma \succ_g \rho$. If $\sigma \supset \rho$ and σ and ρ determine the same normal subgroup of H_{11}, then σ and ρ will be said to be obtained from one another by a *partition change*. We shall write this as $\sigma \supset_p \rho$; and, when also $\sigma \succ \rho$, we shall similarly write $\sigma \succ_p \rho$.

Lemma 5. *Let* ρ, σ *and* τ *be congruences on* $\mathscr{M}^0$ *such that* $\sigma \supset_g \tau \supset_p \rho$.

Then there exists a congruence τ' on $\mathscr{M}^0$ such that $\sigma \supset_p \tau' \supset_g \rho$.

Moreover, if $\sigma \succ_g \tau \succ_p \rho$, then also $\sigma \succ_p \tau' \succ_g \rho$.

PROOF. Suppose that

$$\rho = [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}],$$
$$\tau = [\mathscr{Q}, M, \{g_i\}, \{h_\lambda\}],$$
$$\sigma = [\mathscr{Q}, N, \{p_i\}, \{q_\lambda\}];$$

the assumptions here being justified because τ is obtained from ρ by a partition change and σ from τ by a group change. By lemma 1, $\sigma = [\mathscr{Q}, N, \{g_i\}, \{h_\lambda\}]$ whence $Nh_\lambda g_i = Nh_\mu g_j$ provided i, j, λ, μ are such that $H_{i\lambda}$ and $H_{j\mu}$ belong to the same completely simple subrectangle of $\mathscr{Q}$. *A fortiori*, since $\mathscr{P} \subseteq \mathscr{Q}$, it follows that $[\mathscr{P}, N, \{g_i\}, \{h_\lambda\}] = \tau'$, say, is a congruence on $\mathscr{M}^0$. Since $\mathscr{P} \subset \mathscr{Q}$, it follows that $\tau' \subset_p \sigma$. By lemma 2, $\rho = [\mathscr{P}, M, \{g_i\}, \{h_\lambda\}]$; whence, since $M \subset N$, $\rho \subset_g \tau'$.

Suppose now that $\sigma \succ_g \tau \succ_p \rho$ and that τ' is a congruence on $\mathscr{M}^0$ such that $\sigma \supset_p \tau' \supset_g \rho$. Now $\sigma \succ_g \tau$ implies that $N \succ M$; from which it follows that $\tau' \succ_g \rho$. Similarly, $\sigma \succ_p \tau'$.

We will also need

LEMMA 6. *Let $\rho \subset \sigma$ be neither a partition change nor a group change. Then there exists a congruence τ on $\mathscr{M}^0$ such that $\rho \subset_g \tau \subset_p \sigma$.*

PROOF. Let $\rho = [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}]$ and $\sigma = [\mathscr{Q}, N, \{g_i\}, \{h_\lambda\}]$. Since ρ to σ is neither a partition change nor a group change, $\mathscr{P} \neq \mathscr{Q}$ and $M \neq N$. It is easily verified that $[\mathscr{P}, N, \{g_i\}, \{h_\lambda\}], = \tau$, say, is a congruence on $\mathscr{M}^0$, since $\mathscr{P} \subsetneqq \mathscr{Q}$ and $M \subsetneqq N$. Moreover, $\rho \subsetneqq \sigma$ implies that $\rho \subsetneqq \tau$ and $\tau \subsetneqq \sigma$. Whence we have $\rho \subset_g \tau \subset_p \sigma$, as required.

Before proceeding let us examine more closely the lattice L, say, of permissible partitions of Z. Define the equivalence relations $\omega(I)$ on the set I and $\omega(\Lambda)$ on the set Λ thus:

$\omega(I) = \{(i, j) \in I \times I\colon \quad H_{i\lambda}$ is a group iff $H_{j\lambda}$ is a group$\}$,

$\omega(\Lambda) = \{(\lambda, \mu) \in \Lambda \times \Lambda\colon H_{i\lambda}$ is a group iff $H_{i\mu}$ is a group$\}$.

Let $\eta(I)$ be the partition of I into the equivalence classes of $\omega(I)$ and $\eta(\Lambda)$ be, similarly, the partition of Λ into the equivalence classes of $\omega(\Lambda)$. Let L_1 be the set of all partitions of I contained in $\eta(I)$ and let L_2 be the set of all partitions of Λ which are contained in $\eta(\Lambda)$.

Then we have

LEMMA 7. *L is isomorphic to the direct product $L_1 \times L_2$.*

PROOF. Let $\mathscr{P}$ be a permissible partition of Z. Suppose that partitions π_1 and π_2 of I and Λ, respectively, induce $\mathscr{P}$. Thus $H_{i\lambda}$ and $H_{j\mu}$ belong to the same $\mathscr{P}$-class iff i and j belong to the same π_1-class and also λ and

μ belong to the same π_2-class. Then $\pi_1 \subseteqq \eta(I)$ and $\pi_2 \subseteqq \eta(\Lambda)$. For let i, j belong to a π_1-class. Then $H_{i\lambda}$ is a group iff $H_{j\lambda}$ is a group, for $H_{i\lambda}$ and $H_{j\lambda}$ belong to the same $\mathscr{P}$-class. Hence $(i, j) \in \omega(I)$, i.e. i, j belong to the same $\eta(I)$-class. This shows that $\pi_1 \subseteqq \eta(I)$. Similarly, $\pi_2 \subseteqq \eta(\Lambda)$.

Conversely, let $\pi_1 \subseteqq \eta(I)$ and $\pi_2 \subseteqq \eta(\Lambda)$ be partitions of I and of Λ, respectively. Let $\mathscr{P}$ be the partition of Z induced by π_1 and π_2. If $H_{i\lambda}$, $H_{j\mu}$ belong to a $\mathscr{P}$-class, then i, j belong to a π_1-class and so $H_{i\lambda}$ is a group iff $H_{j\lambda}$ is a group; and λ, μ belong to a π_2-class and so $H_{j\lambda}$ is a group iff $H_{j\mu}$ is a group. And this shows that $\mathscr{P}$ is a permissible partition of Z.

The correspondence between permissible partitions $\mathscr{P}$ and ordered pairs (π_1, π_2) is clearly one-to-one and is an isomorphism between L and $L_1 \times L_2$.

Corollary 8. *L possesses a unique maximal element, viz. the element of L corresponding to the element $(\eta(I), \eta(\Lambda))$ of $L_1 \times L_2$.*

We will denote this maximum element of L by $\mathscr{U}$.

We shall need the following fact about L.

Lemma 9. *Let $\mathscr{P}$, $\mathscr{Q}$ be elements of L such that there exists a finite maximal chain of elements of L*

$$\mathscr{P} = \mathscr{P}_0 \prec \mathscr{P}_1 \prec \cdots \prec \mathscr{P}_t = \mathscr{Q}$$

of length t from $\mathscr{P}$ to $\mathscr{Q}$. Then any chain from $\mathscr{P}$ to $\mathscr{Q}$ of distinct elements of L is of length not greater than t and any maximal chain is of length t.

Proof. By lemma 7 it will suffice to prove the result for the lattice $L_1 \times L_2$. Since, for $(a, b), (c, d) \in L_1 \times L_2$, $(a, b) \subseteqq (c, d)$ iff $a \subseteqq c$ and $b \subseteqq d$, the assertion of the lemma will hold for L if we prove it for each of L_1 and L_2. We prove it holds for L_1; it then follows that it holds for L_2 also.

Let $a, b \in L_1$. Then a, b are partitions of I each contained in $\eta(I)$; and $a \subseteqq b$ means that each partition class of a is contained in a partition class of b. Moreover, $a \prec b$ iff a coincides with b except that precisely one of the b-classes has been split into two a-classes. A sequence $a = a_0 \prec a_1 \prec \cdots \prec a_t = b$, from a to b therefore implies that a is obtained from b simply by t successive splittings of partition classes into two. It is now evident that all chains of distinct elements of L from a to b are finite and that all maximal such chains have length t.

This completes the proof of the lemma.

To deal with the special case of a sequence of congruences with an end-point which is the universal congruence — recall that the description we have given of congruences on $\mathscr{M}^0$ excludes the universal congruence — we shall need the following result.

Lemma 10. *Let υ denote the universal congruence on $\mathscr{M}^0$ and let ω be* *a congruence on* $\mathscr{M}^0$ *such that* $\upsilon \succ \omega$. *Then* $\omega = [\mathscr{U}, H_{11}, \{e_i\}, \{f_\lambda\}]$, *where* $\{e_i\}$ *and* $\{f_\lambda\}$ *may be chosen arbitrarily.*

Moreover, any congruence on $\mathscr{M}^0$, *other than* υ, *is contained in* ω.

PROOF. Since $f_\lambda e_i \in H_{11}$ if and only if $H_{i\lambda}$ is a group (see [2]), $[\mathscr{P}, H_{11}, \{e_i\}, \{f_\lambda\}]$ is a congruence on $\mathscr{M}^0$ for any permissible partition $\mathscr{P}$ and for any choice of $\{e_i\}$ and $\{f_\lambda\}$. Thus, in particular, $[\mathscr{U}, H_{11}, \{e_i\}, \{f_\lambda\}]$ is always a congruence on $\mathscr{M}^0$. Consider the congruences $\omega = [\mathscr{U}, H_{11}, \{e_i\}, \{f_\lambda\}]$ and $\sigma = [\mathscr{P}, M, \{g_i\},\{f_\lambda\}]$. Let $\{H_{i\lambda}: i \in i^*, \lambda \in \lambda^*\}$ be a subrectangle of $\mathscr{P}$. Then, there exists $n_i \in H_{11}$, such that $e_i f_1 = g_i n_i h_1$, for $i \in i^*$: take $n_i = x_i f_1 h_1^{-1}$, where $x_i \in H_{11}$ and $g_i x_i = e_i$. Similarly, there exists $n_\lambda \in H_{11}$, such that $e_1 f_\lambda = g_1 n_\lambda h_\lambda$, for $\lambda \in \lambda^*$. Whence, by (α), $\sigma \subseteqq \omega$.

Thus every congruence, other than υ, is contained in $\omega = [\mathscr{U}, H_{11}, \{e_i\}, \{f_\lambda\}]$. It follows that ω is independent of the choice made of the sets $\{e_i\}$ and $\{f_\lambda\}$. The remaining assertions of the lemma are now evident.

We turn now to the proof of the theorem. Let ρ and σ be two congruences on $\mathscr{M}^0$ such that there is a maximal sequence of congruences

$$\rho = \rho_0 \prec \rho_1 \prec \cdots \prec \rho_m = \sigma$$

from ρ to σ which is of finite length m. We aim to show that any other sequence of distinct congruences from ρ to σ is of finite length less than or equal to m, and that any maximal such sequence is of length m. Let us dispose first of the case of $\sigma = \upsilon$, the universal congruence on $\mathscr{M}^0$. In this event, from lemma 9, we conclude that $\rho_{m-1} = \omega$, the congruence on $\mathscr{M}^0$ containing all congruences on $\mathscr{M}^0$ other than υ. If we prove our theorem for sequences of congruences from ρ to ω we can therefore conclude that it also holds for sequences from ρ to υ. Thus it will suffice to assume that σ is not the universal congruence on $\mathscr{M}^0$.

With this assumption it then follows from lemma 4 that each step $\rho_i \prec \rho_{i+1}$ is either a group change or a partition change. Successive applications of lemma 5 may therefore be made to replace the sequence ρ_i, $i = 0, 1, \cdots, m$, by a maximal sequence, of the same length, in which all the group changes occur first and are then followed by the partition changes. Without loss of generality, we can therefore assume that

$$\rho = \rho_0 \prec_g \rho_1 \prec_g \cdots \prec_g \rho_k,$$

and

$$\rho_k \prec_p \rho_{k+1} \prec_p \cdots \prec_p \rho_m = \sigma,$$

for some k with $0 \leqq k \leqq m$.

Suppose that $\rho = [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}]$. From lemma 1, it follows that $\rho_s = [\mathscr{P}, M_s, \{e_i\}, \{f_\lambda\}]$, for $0 \leqq s \leqq k$, writing, in particular, $M = M_0$.

Suppose that $\sigma = [\mathscr{Q}, N, \{g_i\}, \{h_\lambda\}]$. From lemma 2, it follows that

$\rho_t = [\mathscr{Q}_t, N, \{g_i\}, \{h_\lambda\}]$, for $k \leqq t \leqq m$, writing, in particular, $\mathscr{Q} = \mathscr{Q}_m$. Because of the double description of ρ_k, it follows that $\mathscr{P} = \mathscr{Q}_k$ and $M_k = N$.

From lemma 4, we conclude that

$$M = M_0 \prec M_1 \prec \cdots \prec M_k = N$$

in the lattice of normal subgroups of H_{11}; and that

$$\mathscr{P} = \mathscr{Q}_k \prec \mathscr{Q}_{k+1} \prec \cdots \prec \mathscr{Q}_m = \mathscr{Q}$$

in the lattice L of permissible partitions of Z.

By a well-known result from group-theory it follows that every sequence from M to N, of distinct normal subgroups of H_{11}, is finite, and that every maximal such sequence has length k. From lemma 9, it follows that any sequence from $\mathscr{P}$ to $\mathscr{Q}$, of distinct elements of L, is also finite and, when maximal is of length $m-k$.

Now consider any sequence of distinct congruences from

$$\rho = [\mathscr{P}, M, \{e_i\}, \{f_\lambda\}] \text{ to } \sigma = [\mathscr{Q}, N, \{g_i\}, \{h_\lambda\}]$$

and let

$$\rho = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_t = \sigma \tag{2}$$

be a finite portion of this sequence. If, for any i, $\sigma_i \subset \sigma_{i+1}$ is neither a group change nor a partition change, then, by lemma 6, there exists a congruence τ on $\mathscr{M}^0$ such that $\sigma_i \subset_g \tau \subset_p \sigma_{i+1}$. Consequently, without loss of generality, we can assume that each change in (2) is either a group change or a partition change. From lemma 5 it then follows that we can assume that all the group changes are performed first and then followed by the partition changes. It now follows immediately, by considerations similar to those already used, that $t \leqq m$. And this completes the proof of our theorem.

References

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