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THE DISTINCT ZEROS OF THE PRODUCT OF A POLYNOMIAL AND ITS SUCCESSIVE DERIVATIVES

BY
Q. I. RAHMAN

It has been conjectured that *if $p(z)$ is a polynomial of degree n then the product $P(z)=p(z)p'(z)p''(z)\ldots p^{(n-1)}(z)$ has at least $n+1$ distinct zeros unless $p(z)=c(z-a)^n$*. Professor P. Erdös who mentioned this problem in a lecture at the University of Montreal attributed it to Tiberiu Popoviciu.

In the present paper we consider the special case where $p(z)$ has only real zeros.

Let the statement "$f(z)$ is a constant multiple of $g(z)$" be abbreviated "$f(z) \approx g(z)$".

We prove the following

THEOREM. *If $p(z)$ is a polynomial of degree n with real zeros then the product $P(z)=p(z)p'(z)\ldots p^{(n-1)}(z)$ has*

(i) 1 *distinct zero if* $p(z)\approx(z-a)^n$;

(ii) $n+1$ *distinct zeros if* $p(z)\approx(z-a)(z-b)^{n-1}$ *or* $p(z)\approx(z-a)^2(z-b)^2$ *or* $p(z)\approx(z-a)^3(z-b)^3$;

(iii) *at least* $n+2$ *distinct zeros, in any other case.*

If $n=3$ then $p(z)=z(z^2-1)$ is a polynomial with only real zeros for which $P(z)$ has exactly 5 $(=n+2)$ distinct zeros. Again, if $n=5$ then $z(z^2-1)^2$ may be taken as a polynomial $p(z)$ with only real zeros for which $P(z)$ has exactly 7 $(=n+2)$ distinct zeros. However, we do not assert that for each n there is a polynomial $p(z)$ of degree n with only real zeros such that $P(z)$ has exactly $n+2$ distinct zeros.

If $n=1$ the product $P(z)$ has only one zero.

If $n=2$ the product $P(z)$ has 1 distinct zero or 3 $(=n+1)$ distinct zeros according as the two zeros of $p(z)$ are coincident or distinct.

Now let $n\geq 3$ and suppose that $p(z)$ has at least 3 distinct zeros. If $[a, b]$ is the smallest interval containing all the zeros then both a and b are zeros of $p(z)$. Let the multiplicity of the zero at a be κ and that of the zero at b be l. Then $\kappa+l\leq n-1$, $\max(\kappa, l)\leq n-2$. Without loss of generality we may suppose that $\kappa\leq l$. With the aid of Rolle's theorem it can be reasoned that for $\kappa\leq j\leq n-2$ the smallest zero $a^{(j)}$ of $p^{(j)}(z)$ is simple and

$$a < a^{(\kappa)} < a^{(\kappa+1)} < \cdots < a^{(n-2)}.$$

Besides, for $l\leq j\leq n-2$ the largest zero $b^{(j)}$ of $p^{(j)}(z)$ is simple and

$$b > b^{(l)} > b^{(l+1)} > \cdots > b^{(n-2)} > a^{(n-2)}.$$

Thus the product $P(z)=p(z)p'(z)\ldots p^{(n-2)}(z)$ has at least $2n-\kappa-l$ distinct zeros, namely, a, $a^{(\kappa)}$, $a^{(\kappa+1)},\ldots,a^{(n-2)}$, $b^{(n-2)}$, $b^{(n-3)},\ldots,b^{(l)}$, b. Including the zero $\frac{1}{2}(a^{(n-2)}+b^{(n-2)})$ of $p^{(n-1)}(z)$ the product $P(z)$ has at least $n+2$ distinct zeros.

Finally, let $n\geq 3$ and suppose that $p(z)$ has 2 distinct zeros, i.e. $p(z)\approx(z-\alpha)^{\kappa}(z-\beta)^{l}$ where $\kappa+l=n$. There is no loss of generality in supposing that $\alpha=-1$, $\beta=1$, and $\kappa\leq l$.

(i) If $p(z)\approx(z+1)(z-1)^{n-1}$ then for $j=1,2,\ldots,n-2$ the jth derivative $p^{(j)}(z)$ has a zero of multiplicity $n-1-j$ at 1 and a simple zero at $-1+2j/n$. Hence along with the zero $(n-2)/n$ of $p^{(n-1)}(z)$ the product $p(z)p'(z)\ldots p^{(n-1)}(z)$ has precisely $n+1$ distinct zeros.

(ii) If $p(z)\approx(z+1)^2(z-1)^2$ or $p(z)\approx(z+1)^3(z-1)^3$ then elementary direct calculation shows that $P(z)$ has 5 $(=n+1)$ and 7 $(=n+1)$ distinct zeros respectively.

We prove that in every other case the product $P(z)$ has at least $n+2$ distinct zeros.

In fact, the polynomial $p'(z)$ has a zero of multiplicity $\kappa-1$ at -1, a zero of multiplicity $l-1$ at $+1$ and a zero $\gamma_1=-(l-\kappa)/(\kappa+l)$ in the open interval $(-1,1)$. Thus the product $p(z)p'(z)$ has 3 distinct zeros. The second derivative $p''(z)$ has a zero of multiplicity $\kappa-2$ at -1, a zero of multiplicity $l-2$ at 1, a simple zero $\gamma_{2,1}$ in the open interval $(-1,\gamma_1)$ and a simple zero $\gamma_{2,2}$ in the open interval $(\gamma_1, 1)$. The product $p(z)p'(z)p''(z)$ has therefore 5 distinct zeros. As long as $p^{(j)}(z)$ has a zero at -1 the smallest zero $\gamma_{j+1,1}$ of $p^{(j+1)}(z)$ lying in the open interval $(-1,1)$ cannot be a zero of any of the polynomials $p(z)$, $p'(z),\ldots,p^{(j)}(z)$. Similarly, as long as $p^{(j)}(z)$ has a zero at 1 the largest zero $\gamma_{j+1,2}$ of $p^{(j+1)}(z)$ lying in the open interval $(-1,1)$ cannot be a zero of any of the polynomials $p(z)$, $p'(z),\ldots,p^{(j)}(z)$. Thus the product $p(z)p'(z)\ldots p^{(n-1)}(z)$ vanishes at least at the $n+1$ points -1, $+1$, $\gamma_1,\gamma_{2,1},\ldots,\gamma_{\kappa,1},\gamma_{2,2},\ldots,\gamma_{l,2}$ where

$$-1<\gamma_{\kappa,1}<\gamma_{\kappa-1,1}<\cdots<\gamma_{2,1}<\gamma_1<\gamma_{2,2}<\cdots<\gamma_{l,2}<1.$$

According to Rolle's theorem the polynomial $p'''(z)$ has a zero $\gamma_{3,0}$ between $\gamma_{2,1}$ and $\gamma_{2,2}$ which we have not counted since it may possibly coincide with γ_1. But according as $\kappa=2$ or $\kappa>2$, $p'''(z)$ is proportional to

$$(z-1)^{n-5}\{n(n-1)z^2+2(n-1)(n-6)z+(n-4)(n-9)\}$$

or to

$$(z+1)^{\kappa-3}(z-1)^{l-3}\{\kappa(\kappa-1)(\kappa-2)(z-1)^3+3\kappa l(\kappa-1)(z-1)^2(z+1)$$
$$+3\kappa l(l-1)(z-1)(z+1)^2+l(l-1)(l-2)(z+1)^3\}.$$

Hence $p'''(\gamma_1)=p'''[-(l-\kappa)/(\kappa+l)]\neq 0$ unless $k=l$. It follows that if $\kappa\neq l$ then $\gamma_{3,0}\neq\gamma_1$ and the product $P(z)$ has at least $n+2$ distinct zeros.

Since the case $\kappa=l=2$ and $\kappa=l=3$ have already been considered let $\kappa=l$ and $n\geq 8$. In our preceding discussion of the case $p(z)\approx(z+1)^{\kappa}(z-1)^{l}$ where $\kappa+l=n$ we have ignored the fact that the polynomial $p^{(\mathrm{iv})}(z)$ has a simple zero $\gamma_{4,-0}$ in the open interval $(\gamma_{3,1}, \gamma_{3,0})$ and another simple zero $\gamma_{4,+0}$ in the open interval $(\gamma_{3,0}, \gamma_{3,2})$. We did so in order to allow the possibility that $\gamma_{4,-0}, \gamma_{4,+0}$ may respectively be equal to $\gamma_{2,1}, \gamma_{2,2}$. However, if $p(z) \approx (1-z^2)^\kappa$ then

$$\gamma_{2,1} = -\frac{1}{\sqrt{2\kappa-1}}, \qquad \gamma_{2,2} = \frac{1}{\sqrt{2\kappa-1}},$$

and

$$p^{(\text{iv})}(z) \approx (1-z^2)^{\kappa-4}\{(2\kappa-1)(2\kappa-3)z^4 - 6(2\kappa-3)z^2 + 3\}.$$

Hence neither $p^{(\text{iv})}(\gamma_{2,1}) = 0$ nor $p^{(\text{iv})}(\gamma_{2,2}) = 0$, i.e. $\gamma_{4,-0} \neq \gamma_{2,1}$ and $\gamma_{4,+0} \neq \gamma_{2,2}$. It follows that the product $P(z)$ has at least $n+3$ distinct zeros.

UNIVERSITÉ DE MONTRÉAL,
MONTRÉAL, QUÉBEC

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