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Rigid Curves in Complete Intersection Calabi–Yau Threefolds

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Abstract. Working over the complex numbers, we study curves lying in a complete intersection K3 surface contained in a (nodal) complete intersection Calabi–Yau threefold. Under certain generality assumptions, we show that the linear system of curves in the surface is a connected componend of the the Hilbert scheme of the threefold. In the case of genus one, we deduce the existence of infinitesimally rigid embeddings of elliptic curves of arbitrary degree in the general complete intersection Calabi–Yau threefold.

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0. Introduction

The principal objects of study in this work are embeddings of smooth complex projective curves into smooth, three-dimensional complete intersections with trivial canonical class. These varieties – and projective Calabi–Yau threefolds in general – are currently of active interest in algebraic geometry and physics, and understanding the curves lying in them is important in both contexts.

By adjunction, a complete intersection of hypersurfaces of degrees $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{r-3}$ in $\mathbf{P}^r$ has trivial canonical bundle precisely when $\sum b_i = r + 1$, so that there are only five families of such threefolds: the quintic threefolds in $\mathbf{P}^4$, the complete intersections of types $(4, 2)$ and $(3, 3)$ in $\mathbf{P}^5$, the complete intersections of type $(3, 2, 2)$ in $\mathbf{P}^6$ and those of type $(2, 2, 2, 2)$ in $\mathbf{P}^7$. We refer to any variety of this type as a complete intersection Calabi–Yau – or ciCY – threefold.

On any complete intersection one can construct continuous families of curves by taking hyperplane sections. Even if such embeddings are excluded – say by specifying that the degree d of the embedding should satisfy $d > 2g - 2$ (where g is the genus of the curves) – a particular threefold may still admit a continuous family of curves; for example, one easily constructs families of lines on the Fermat quintic. But a dimension count suggests that for a sufficiently *general* ciCY threefold, this should not occur; curves – if they exist – should be isolated, or *rigid.* Thus for a fixed d and g, a broad goal would be to answer the following: Are there degree d

embeddings of smooth genus g curves into a general K-trivial complete intersection threefold (or Calabi–Yau threefold)? Are some of them rigid? If they are all rigid, how many are there?

Rigid embeddings may be used to construct algebraic cycles not algebraically equivalent to other cycles. With this motivation, Clemens [5] constructed rigid embeddings of arbitrarily high degree from $\mathbf{P}^1$ into a general quintic threefold $Y \subset \mathbf{P}^4$. His method consists of constructing smooth rational curves C on a smooth quartic surface X, embedding X in a nodal quintic threefold Y_0, and showing that under a general deformation of Y_0 to a smooth quintic Y, C deforms to a curve rigid in Y. It was observed by S. Katz [14] that a theorem of Mori [19] guarantees the existence of a smooth quartic surface containing a smooth rational curve of any degree, and that Clemens's deformation argument therefore constructs rigid smooth rational curves of all degrees in a general quintic threefold. Katz went on to deduce that for $d \leqslant 7$, the general quintic contains a finite positive number of smooth degree d rational curves; this has been extended to degrees $d \leqslant 9$ by Nijsse [21] and independently by Johnsen and Kleiman [13]. In fact, Clemens [6] conjectured that this finiteness should hold in all degrees.

This conjecture – and the implicit goal of *counting* the smooth rational curves of a particular degree in a general quintic (or ciCY) threefold – has inspired some remarkable mathematics over the past decade, including the development of quantum cohomology and the discovery of some surprising connections with physics and the theory of mirror symmetry; see [4, 8, 11, 14, 17, 18].

As we will show, a dimension count (2.3) suggests existence, rigidity and finiteness results when considering curves of arbitrary genus in all of the complete intersections. In the case of genus one, we prove:

THEOREM 1. *Fix $d \geqslant 3$. Then the general complete intersection Calabi–Yau threefold contains rigid elliptic curves of degree d, with the exception of degree* 3 *curves in threefolds of type* (2, 2, 2, 2).

and

THEOREM 2. *For $3 \leqslant d \leqslant b_{r-3} + 3$ (or if $4 \leqslant d \leqslant 5$ if $(b_i) = (2, 2, 2, 2)$), the general ciCY threefold of type $(b_1, \dots, b_{r-3})$ contains a finite positive number of smooth curves of degree d and genus one.*

Note. Theorem 1 does not imply that a general threefold contains embeddings of all degrees; for that, one would need a *generic* complete intersection, i.e., the complement of countably many divisors in the projective space of all complete intersections.

Because smooth cubic and quartic curves of genus one are complete intersections, algebro-geometric counts for the numbers of such curves on a general quintic exist; see [8]. In general, using the theory of mirror symmetry, predictions for the numbers of elliptic curves of degree d on a general quintic threefold have been made; cf. [3]. Recent work of Getzler [10] may lead to a mathematical proof of these formulae along the lines of that given by Givental [11] for rational curve.

OVERVIEW. Section 1 consists of general material. In Section 1. we briefly recall the fundamental properties of Hilbert schemes and give a definition of rigid embedding. In Section 1.2, we discuss zero schemes of sections of a vector bundle, their deformations over $\mathbf{C}$, and their Hilbert schemes. Specifically, we show that under a certain vanishing hypothesis, their Hilbert schemes are themselves zero-schemes (Theorem 1.5). In Section 1.3, we establish some results on linear systems on K3 surfaces. The main tool is the Atiyah exact sequence, and the section opens with a discussion of its construction.

Although we are ultimately interested in smooth curves, the proof of Theorem 1 will make use of complete linear systems on K3 surfaces. To handle all curves occurring in this way, we construct $\mathcal{CH}$ – a smooth open subscheme of the Hilbert scheme of curves in $\mathbf{P}^r$ – which parameterizes all of the smooth genus g curves of degree $d > 2g - 2$ in $\mathbf{P}^r$ as well as any deformation of them on complete intersection K3 surfaces. As an application of Section 1.2, we then establish that if $Y \subset \mathbf{P}^r$ is a (global) complete intersection, $\mathrm{Hilb}^Y \cap \mathcal{CH}$ is the scheme-theoretic zero locus of a section of a locally free sheaf $\mathcal{V}$ on $\mathcal{CH}$.

Now it turns out that in case Y is a Calabi–Yau complete intersection, $\dim \mathcal{CH} = \mathrm{rk}\mathcal{V}$, which suggests that a sufficiently general section should have some isolated zero points. Thus, we would like to show that sections of $\mathcal{V}$ arising from a general Calabi–Yau complete intersections are sufficiently general in the above sense; in the genus one case, this is precisely the content of Theorem 1, and suggests the more general:

CONJECTURE 1. *Let $g \geqslant 0$, $d > 2g - 2$ and suppose there exist degree d projective embeddings of smooth connected curves of genus g. Then there exist rigid degree d embeddings of smooth, connected curves of genus g into a general complete intersection Calabi–Yau threefold.*

A more optimistic guess would be that *all* of the zeros are reduced and isolated. Theorem 2 and the theorem of Katz [14] are very special cases, and we show how to deduce them from Theorem 1 and the existence theorem of Clemens–Katz respectively.

Section 3 contains the proof of Theorem 1. In Section 3.1, following the general idea of Clemens's proof as described above, we study the inclusion of curves on complete intersection K3 surfaces into general complete intersection Calabi–Yau threefolds. The main result – Theorem 3.5 – is that the deformations of these curves in the threefold are all present as deformations in the surface. The proof relies on the lemmas established in Section 1.3. Now the main theorem of Mori [19] and its generalizations to K3 surfaces of degrees 6 and 8 guarantee the existence of a K3 surface X carrying a good linear system, and we fix a sufficiently general threefold Y_0 containing it. At this point, we give a brief alternate proof of a result of Ekedahl, Johnsen and Sommervoll [7], to the effect that a generic ciCY threefold contains rigid rational curves of any degree.

Finally, in Section 3.2, we show that a general deformation of Y_0 contains a smooth rigid curve. Here, the main tool is the characterization from Section 1.2 of Hilb^{Y_0} and its deformations. It is only in this section that the genus one hypothesis becomes necessary.

CONVENTIONS AND NOTATION. Unless otherwise indicated, all schemes are separated and of finite type over a field k; except in Section 1, we assume $k = \mathbf{C}$, the field of complex numbers. A *curve* is a purely one-dimensional scheme. A *variety* is a reduced and irreducible scheme.

Let Y be a scheme and $\mathcal{F}$ an $\mathcal{O}_Y$-module; denote by $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)$ its dual. For a locally closed subscheme $X \hookrightarrow Y$, let $\mathcal{N}_{X/Y} := \mathcal{H}om_{\mathcal{O}_U}(\mathcal{I}, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$ be the *normal sheaf of* X *in* Y, where $\mathcal{I}$ is the ideal sheaf of X in some open $U \subset Y$ in which X is closed. For f a global section of a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ of finite rank, denote by $Z(f)$ the *zero-scheme* of f (cf. [9, B.3.4]).

A *complete intersection of type* $(b_i) = (b_1, b_2, \dots, b_n)$ in $\mathbf{P}^r$ is a subscheme $Y \subset \mathbf{P}^r$ of pure codimension n whose ideal is generated by n elements of degrees $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \geqslant 1$. These elements in turn determine a section $f \in \Gamma(\mathbf{P}^r, \bigoplus \mathcal{O}(b_i))$ such that $Y = Z(f)$.

1. General Material

1.1. HILBERT SCHEMES AND RIGID EMBEDDINGS

Let Y be a projective variety. We write $\mathrm{Univ}^Y \to \mathrm{Hilb}^Y$ for the universal flat family over its Hilbert scheme. Recall the following result from the local study of these schemes:

PROPOSITION 1.1 (Infinitesimal properties of Hilb). *The Zariski tangent space of* Hilb^Y *at a closed embedding* $X \hookrightarrow Y$ *is* $\mathrm{H}^0(X, \mathcal{N}_{X/Y})$. *Furthermore, if* $\mathrm{H}^1(X, \mathcal{N}_{X/Y}) = 0$, *then* Hilb^Y *is smooth at this point.*

For a construction of these schemes and a proof of this theorem, see [23] or, for a more general theory, [16, Section I.1 and Section I.2].

DEFINITION 1.1. A closed embedding $i\colon X \hookrightarrow Y$ into a projective variety is *rigid* if i is represented by an isolated point of Hilb^Y, i.e., if the component of this point is zero-dimensional. If, furthermore, this point is reduced, i is said to be *infinitesimally rigid*.

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(When a particular embedding is understood, we often say that X is rigid or infinitesimally rigid in Y.) From the infinitesimal study of Hilb^Y, it is immediate that i is infinitesimally rigid if and only if $\mathrm{H}^0(C, \mathcal{N}_{C/Y}) = 0$.

1.2. ZERO SCHEMES AND THEIR HILBERT SCHEMES

In this section, unless otherwise noted, schemes are separated and of finite type over an arbitrary field k.

Let W be a scheme and $\mathcal{E}$ a locally free $\mathcal{O}_W$-module of rank e, and fix a global section $f \in \Gamma(W, \mathcal{E})$. Let $Z = Z(f) \hookrightarrow W$ be the zero-scheme of f.

LEMMA 1.2. *There is an exact sequence of $\mathcal{O}_Z$-modules*

$$0 \to \mathcal{N}_{Z/W} \xrightarrow{m} \mathcal{E} \otimes \mathcal{O}_Z \xrightarrow{\rho} \mathcal{M} \to 0. \tag{1.1}$$

Moreover, if $Z \xrightarrow{i} W$ is a regular embedding, then $\mathcal{M}$ is locally free of rank $e - \mathrm{codim}_W Z$.

Remark. In case i is a regular embedding, $\mathcal{M}$ is the *excess normal bundle* of the fiber square

$$\begin{array}{ccc} Z & \xrightarrow{i} & W \\ \downarrow & & \downarrow f \\ W & \xrightarrow{0_E} & E, \end{array}$$

where E is the geometric bundle of $\mathcal{E}$ and 0_E is the zero section; see [9, Section 6.3].

Proof. Let $\mathcal{I}$ denote the ideal sheaf of Z in W. Let $U \subset W$ be an affine open set over which $\mathcal{E}$ is free and fix an isomorphism $\mathcal{E}|_U \cong \mathcal{O}_U^e$. With respect to this trivialization, write $f = (f_1, f_2, \dots, f_e)$. Define the morphism $m\colon \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z) \to \mathcal{E} \otimes \mathcal{O}_Z$ by $\varphi \mapsto (\varphi(f_1), \varphi(f_2), \dots, \varphi(f_e))$. By definition, $\mathcal{I}|_U = \sum f_i \mathcal{O}_U$, implying that m is a monomorphism. Since the (f_i) patch together to form the global section f, this local construction globalizes to a monomorphism $m\colon \mathcal{N}_{Z/W} \to \mathcal{E} \otimes \mathcal{O}_Z$; in particular, for a global section $\alpha \in \Gamma(Z, \mathcal{N}_{Z/W})$ it makes sense to write $m(\alpha) = \alpha(f)$. One can equally well check that the dual is an epimorphism, so that when $\mathcal{N}_{Z/W}$ is locally free – e.g., when i is a regular embedding – the cokernel $\mathcal{M}$ of m must be locally free of rank $e - \mathrm{rk}\,\mathcal{N}_{Z/W} = e - \mathrm{codim}_W Z$. $\square$

We shall need a local description of m in terms of derivations. Let $\mathcal{I}$ denote the ideal sheaf of Z in W, and let $z \in Z$ be a closed point. Denote by

$\delta\colon (\mathfrak{m}_{z,W}/\mathfrak{m}_{z,W}^2)^\vee \to (\mathcal{I}_z/\mathcal{I}_z^2)^\vee$ the natural map. Then the description of m in the proof of Lemma 1.2 implies:

LEMMA 1.3. *The composition* $\mu_z = m_z \circ \delta\colon (\mathfrak{m}_{z,W}/\mathfrak{m}_{z,W}^2)^\vee \to \mathcal{E} \otimes k(z)$ *is given on a derivation* d *by* $\mu_z(d) = d(f)$.

Over the complex numbers, zero schemes of regular sections which do not have the expected codimension may often be deformed to ones that do:

PROPOSITION 1.4. *In the above situation, suppose that* $k = \mathbf{C}$, *that* W *is smooth of dimension* e, *and that* $i\colon Z \hookrightarrow W$ *is regular of codimension* $e - c$. *If* $f' \in \Gamma(W, \mathcal{E})$ *such that* $Z(\rho(i^*f'))$ *has a reduced isolated point at a smooth point* z *of* Z, *then for general* ε, *the scheme* $Z(f + \varepsilon f')$ *has a reduced isolated point in a neighborhood of* z *in* W.

Proof. Because z is a smooth point, it is possible to choose local analytic coordinates $(u_1, \ldots, u_e)$ on a neighborhood U of z in W and a local trivialization $\varphi\colon \mathcal{E}|_U \xrightarrow{\sim} \mathcal{O}_U^e$ such that $\varphi(f) = (u_1, u_2, \ldots, u_{e-c}, 0, \ldots, 0)$. This induces a trivialization $\psi\colon \mathcal{M}|_{U \cap Z} \to \mathcal{O}_{U\cap Z}^c$ such that $\psi\rho\varphi^{-1}(g_1, \ldots, g_e) = (g_{e-c+1}, \ldots, g_e)$.

Write $\varphi(f') = (f_1', \ldots, f_e')$. Then $\psi\rho(i^*f') = (f_{e-c+1}', \ldots, f_e')$, and since this has a reduced isolated zero at z, there is a neighborhood $V \subset U$ of z on which $(u_1, \ldots, u_{e-c}, f_{e-c+1}', \ldots, f_e')$ is a system of coordinates.

By the implicit function theorem, for ε in a sufficiently small neighborhood of 0, the system

$$u_1 + \varepsilon f_1' = \cdots = u_{e-c} + \varepsilon f_{e-c}' = \varepsilon f_{e-c+1}' = \cdots = \varepsilon f_e' = 0$$

has a solution $p(\varepsilon) = (p_1, \ldots, p_{e-c}, 0, \ldots, 0)$ in V, where the p_i are analytic functions of ε. At $p(\varepsilon)$, the corresponding Jacobian matrix $J(\varepsilon)$ is of the form

$$J(\varepsilon) = \begin{pmatrix} I + \varepsilon J'(\varepsilon) & \varepsilon A \\ 0 & \varepsilon I \end{pmatrix}$$

where

$$J'(\varepsilon) = \left(\frac{\partial f_i'}{\partial u_j}(p(\varepsilon))\right)_{1 \leqslant i,j \leqslant e-c}.$$

Thus, J has maximal rank if $I + \varepsilon J'$ does, and since it does at $\varepsilon = 0$, it does for $|\varepsilon|$ sufficiently small. □

In certain cases, the Hilbert scheme of a zero scheme is itself a zero scheme: Let W be a projective scheme, $\mathcal{E}$ a locally free $\mathcal{O}_W$-module and $f \in \Gamma(W, \mathcal{E})$; set $Z = Z(f)$. Let $\mathcal{H} \subset \mathrm{Hilb}^W$ be open and denote by $p\colon \mathcal{U} \to \mathcal{H}$ the restriction of the universal family and by $q\colon \mathcal{U} \to W$ the second projection.

THEOREM 1.5. *Assume that*

$$\mathrm{H}^1(\mathcal{U}_s, (q^*\mathcal{E})_s) = 0 \quad \text{for all points } s \in \mathcal{H}.$$

Then $p_*q^*\mathcal{E}$ *is locally free over* $\mathcal{H}$, *and the global section* p_*q^*f *has the property that* $\mathrm{Hilb}^Z \cap \mathcal{H} = Z(p_*q^*f)$ *as subschemes of* $\mathcal{H}$.

Proof. Let $\mathcal{F} = q^*\mathcal{E}$, which is flat over $\mathcal{H}$. Recall the following consequence of semicontinuity (e.g., [20, Theorem, p. 46]).

Fact. Let $p: X \to Y$ be a proper morphism of Noetherian schemes and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, flat over Y. Suppose that $\mathrm{H}^1(X_y, \mathcal{F}_y) = 0$ for all points $y \in Y$. Then $p_*\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module, and for all morphisms $g: Y' \to Y$, the natural morphism $g^*p_*\mathcal{F} \to p'_*(g')^*\mathcal{F}$ is an isomorphism, where p' and g' are the morphisms from $X \times_Y Y'$ to Y' and X respectively.

Thus, $p_*\mathcal{F}$ is locally free. Consider the embedding $g: \mathrm{Hilb}^Z \cap \mathcal{H} \to \mathcal{H}$. Since $qg': \mathcal{U} \times_{\mathcal{H}} (\mathrm{Hilb}^Z \cap \mathcal{H}) \to W$ factors through $Z \hookrightarrow W$, the section $p'_*(g')^*q^*(f)$ is zero; using the above fact, so is $g^*p_*q^*(f)$. By the universal property of zero schemes, this implies that g factors through $h: Z(p_*q^*f) \hookrightarrow \mathcal{H}$. Conversely, the universal property of Hilb^Z implies that h factors through g. Thus, h and g represent the same closed subscheme of $\mathcal{H}$. □

With the same notation, assume the hypothesis of Theorem 1.5 holds, and set $\mathcal{H}^Z = \mathrm{Hilb}^Z \cap \mathcal{H} = Z(p_*q^*f)$. Let $z \in \mathcal{H}^Z$ correspond to a closed embedding $X \hookrightarrow Z$. Denote by $T_z\mathcal{H}^Z$ and $T_z\mathcal{H}$ the Zariski tangent spaces at z of $\mathcal{H}^Z$ and $\mathcal{H}$ respectively, and by m' the composition

$$\mathcal{N}_{X/W} \to \mathcal{N}_{Z/W} \otimes \mathcal{O}_X \xrightarrow{m|_X} \mathcal{E} \otimes \mathcal{O}_X$$

(where $m: \mathcal{N}_{Z/W} \to \mathcal{E} \otimes \mathcal{O}_Z$ is the morphism of Lemma 1.2).

PROPOSITION 1.6. *There is a commutative diagram of exact sequences of* $k(z)$*-vector spaces*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^0(X, \mathcal{N}_{X/Z}) & \longrightarrow & \mathrm{H}^0(X, \mathcal{N}_{X/W}) & \xrightarrow{\mathrm{H}^0(m')} & \mathrm{H}^0(X, \mathcal{E} \otimes_{\mathcal{O}_W} \mathcal{O}_X) \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & T_z\mathcal{H}^Z & \longrightarrow & T_z\mathcal{H} & \xrightarrow{\mu_z} & (p_*q^*\mathcal{E}) \otimes_{\mathcal{O}_{\mathcal{H}}} k(z), \end{array}$$

where the first two vertical arrows are the standard isomorphisms and the third follows from semicontinuity, and μ_z *is the morphism of Lemma 1.3 applied to the locally free sheaf* $p_*q^*\mathcal{E}$.

Proof. Let $k' = k(z)$ (which is the field of definition of X), and denote by $k'[\varepsilon]$ the ring of dual numbers over k'. To prove commutativity of the right-hand square, start with $\alpha \in H^0(X, \mathcal{N}_{X/W})$. There is a corresponding scheme $\mathcal{X} \subset W \times_k \operatorname{Spec} k'[\varepsilon]$, flat over $\operatorname{Spec} k'[\varepsilon]$, which lifts X. Over an affine open set $V = \operatorname{Spec} A \subset W$, the ideal of $\mathcal{X}$ in $\operatorname{Spec}(A \otimes_k k'[\varepsilon]) \subset W \times \operatorname{Spec} k'[\varepsilon]$ is $\bigcup_{a \in I} a + \varepsilon\alpha(a)$, where $I = I(X \cap V)$ and we think of $\alpha|_V \in \operatorname{Hom}_A(I, A/I)$. By the universal property of the Hilbert scheme, this in turn gives rise to a morphism $v = v(\alpha)\colon \operatorname{Spec} k'[\varepsilon] \to \operatorname{Hilb}^W$ whose image is the point z and such that $\mathcal{U} \times_{\operatorname{Hilb}^W} \operatorname{Spec} k'[\varepsilon] \cong \mathcal{X}$. The map $\alpha \mapsto v(\alpha)$ is precisely the middle isomorphism of our diagram (for details, see e.g. [23, Prop. 4.4]). Let $r = r(\alpha)\colon \mathcal{X} \to \mathcal{U}$ be the induced morphism. Then by the construction of $\mathcal{X}$, the section $r^*q^*f \in \Gamma(\mathcal{X}, r^*q^*\mathcal{E})$ is given by $r^*q^*f = f + \varepsilon\alpha(f)$. (As noted in the proof of Lemma 1.2, this notation makes sense.) The desired commutativity is now immediate from the descriptions of m and μ in Lemmas 1.2 and 1.3.

That the left-hand square commutes is a consequence of the above description of the vertical isomorphism. Finally, exactness at the left of either sequence is standard, and exactness at the middle follows because m is a monomorphism (Lemma 1.2). □

1.3. THE ATIYAH EXACT SEQUENCE AND CURVES ON K3 SURFACES

Recall that for $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a scheme X, there is a ∂-functorial pairing

$$H^{r-i}(X, \mathcal{F}) \times \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow H^r(X, \mathcal{G})$$

called the *Yoneda Pairing*. (See e.g. [1, Theorem IV.1.1].) As a consequence of ∂-functoriality, the pairing has a simple description in case $i = 1$: for $\xi \in \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, choose an extension $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$ representing ξ. Then the map $\times\xi\colon H^{r-1}(X, \mathcal{F}) \to H^r(X, \mathcal{G})$ is just the connecting homomorphism in the long exact sequence of cohomology arising from the above extension. Recall also ([1, Corollary IV.2.6]) that in case $\mathcal{F}$ is locally free of finite rank, there is a natural isomorphism

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong H^i(X, \mathcal{G} \otimes \mathcal{F}^\vee). \tag{1.2}$$

Now suppose that X is a compact Kähler variety and that $\mathcal{L}$ is a line bundle on X. Atiyah [2] showed that the class $2\pi i c_1(\mathcal{L}) \in H^1(X, \Omega^1_X) \cong \operatorname{Ext}^1(\mathcal{T}_X, \mathcal{O}_X)$ may be represented by the *Atiyah exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}_{\mathcal{L}} \longrightarrow \mathcal{T}_X \longrightarrow 0, \tag{1.3}$$

constructed as follows: Let $\pi\colon P \to X$ be the geometric principal bundle associated to $\mathcal{L}$. Then $\mathbf{C}^*$ acts on the exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{T}_P \longrightarrow \pi^*\mathcal{T}_X \longrightarrow 0,$$

where $\mathcal{V}$ is the sheaf of vertical vector fields. Taking $\mathbf{C}^*$-invariants, one obtains an exact sequence which descends to X as $0 \to \mathcal{A}d(\mathcal{L}) \to \mathcal{A}_{\mathcal{L}} \to \mathcal{T}_X \to 0$, where $\mathcal{A}d(\mathcal{L}) \cong \mathcal{O}_X$ is the adjoint bundle of $\mathcal{L}$; this is (1.3). One way to think of $\mathcal{A}_{\mathcal{L}}$ is

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as the sheaf of first-order deformations of the pair $(X, \mathcal{L})$, i.e., as first-order deformations of the geometric realization of $\mathcal{L}$ preserving the vector bundle structure.

EXAMPLE 1.1 [2, Section 6]. When $X = \mathbf{P} = \mathbf{P}(V)$ and $\mathcal{L} = \mathcal{O}_{\mathbf{P}}(1)$, the Atiyah sequence is the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}}(1) \longrightarrow \mathcal{T}_{\mathbf{P}} \longrightarrow 0.$$

The Atiyah sequence is functorial [2, Section 2 Remark (2)]: if $f: X' \to X$ is a holomorphic map, $\mathcal{A}_{f^*\mathcal{L}} = f^*\mathcal{A}_{\mathcal{L}} \times_{f^*\mathcal{T}_X} \mathcal{T}_{X'}$. Consequently, we have:

PROPOSITION 1.7. *Suppose X is projective:* $X \hookrightarrow \mathbf{P}^r$. *Then for all k*

$$\mathrm{H}^{k-2}(X, \mathcal{N}_{X/\mathbf{P}^r}) \xrightarrow{\delta} \mathrm{H}^{k-1}(X, \mathcal{T}_X) \xrightarrow{c} \mathrm{H}^k(X, \mathcal{O}_X)$$

is a complex, exact if $\mathrm{H}^{k-1}(X, \mathcal{O}_X(1)) = 0$; *here* δ *arises from* $0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbf{P}} \otimes \mathcal{O}_X \to \mathcal{N}_{X/\mathbf{P}} \to 0$ *and c is the Yoneda pairing* $\times c_1(\mathcal{O}_X(1))$.

Proof. Functoriality and the example give commutative diagrams with exact rows and columns:

$$\begin{array}{ccccc}
 & & \mathrm{H}^{k-1}(X, \mathcal{A}_{\mathcal{O}_X(1)}) & \longrightarrow & \mathrm{H}^{k-1}(X, \mathcal{O}_X(1)^{r+1}) \\
 & & \downarrow & & \downarrow \\
\mathrm{H}^{k-2}(X, \mathcal{N}_{X/\mathbf{P}^r}) & \xrightarrow{\delta} & \mathrm{H}^{k-1}(X, \mathcal{T}_X) & \longrightarrow & \mathrm{H}^{k-1}(X, \mathcal{T}_{\mathbf{P}^r} \otimes \mathcal{O}_X) \\
 & & \downarrow{\scriptstyle \times 2\pi i c_1(\mathcal{O}(1))} & & \downarrow \\
 & & \mathrm{H}^k(X, \mathcal{O}_X) & = & \mathrm{H}^k(X, \mathcal{O}_X).
\end{array}$$

A diagram chase establishes the proposition. □

Recall that $\pi: P \to X$ is the geometric principal bundle associated to $\mathcal{L}$. Now $\pi^*\mathcal{L}$ is canonically trivial, and for $U \subset X$ open, there are isomorphisms

$$\psi_U: \Gamma(U, \mathcal{L}) \xrightarrow{\sim} \Gamma(\pi^{-1}(U), \pi^*\mathcal{L})^{\mathbf{C}^*}.$$

Given $s \in \Gamma(U, \mathcal{L})$ and $\xi \in \Gamma(U, \mathcal{A}_{\mathcal{L}})$, we may differentiate $\psi_U(s)$ by ξ, obtaining a $\mathbf{C}^*$-invariant function on $\pi^{-1}(U)$, i.e.,

$$\xi(\psi_U(s)) \in \Gamma(\pi^{-1}(U), \pi^*\mathcal{L})^{\mathbf{C}^*} \cong \Gamma(U, \mathcal{L}).$$

Thus, given a global section $s \in \Gamma(X, \mathcal{L})$, we may define a morphism of sheaves $v_s: \mathcal{A}_{\mathcal{L}} \to \mathcal{L}$ by $v_s(\xi) = \psi_U^{-1}(\xi(\psi_U(s|_U)))$ for $\xi \in \Gamma(U, \mathcal{A}_{\mathcal{L}})$.

Suppose $s \in \Gamma(X, \mathcal{L})$, and let $D = Z(s)$ be the associated locally principal subscheme of X. One way to describe the usual morphism

$$\mu: \mathcal{T}_X \to \mathcal{T}_X \otimes \mathcal{O}_D \to \mathcal{N}_{D/X} \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{O}_D$$

is by $\mu(\xi) = \xi(s)|_D$. Thus, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A}_{\mathcal{L}} & \longrightarrow & \mathcal{T}_X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \nu_s} & & \downarrow{\scriptstyle \mu} & & \\ 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{s} & \mathcal{L} & \longrightarrow & \mathcal{L} \otimes \mathcal{O}_D & \longrightarrow & 0. \end{array}$$

This shows

PROPOSITION 1.8. *If X is a smooth projective variety over* $\mathbf{C}$ *and D an effective divisor on X, the square*

$$\begin{array}{ccc} \mathrm{H}^{k-1}(X, \mathcal{T}_X) & \xrightarrow{\times 2\pi i c_1(\mathcal{L})} & \mathrm{H}^k(X, \mathcal{O}_X) \\ \downarrow{\scriptstyle H^{k-1}(\mu)} & & \| \\ \mathrm{H}^{k-1}(X, \mathcal{L} \otimes \mathcal{O}_D) & \xrightarrow{\delta} & \mathrm{H}^k(X, \mathcal{O}_X) \end{array}$$

commutes for all k.

We apply the above results to the case $X \hookrightarrow \mathbf{P}^r$ a smooth projective K3 surface, C_0 a smooth connected curve of genus g on X, and $\mathcal{L} = \mathcal{O}_X(C_0)$. As usual, denote $|\mathcal{L}| = \mathbf{P}(\Gamma(X, \mathcal{L}))$. We shall need a preliminary result:

LEMMA 1.9. *For all $C \in |\mathcal{L}|$ and $m > 0$:*

(1) $\mathrm{h}^0(\mathcal{O}_C) = 1$ *and* $\mathrm{h}^1(\mathcal{O}_C) = g$,
(2) $\mathrm{h}^0(\mathcal{N}_{C/X}) = g$ *and* $\mathrm{h}^1(\mathcal{N}_{C/X}) = 1$, *and*
(3) *If $\mathcal{O}_{C_0}(m)$ is non-special, then $\mathcal{O}_C(m)$ is nonspecial.*

Remark. Since the linear system parameterizes a flat family of curves, (3) implies that if $\mathcal{O}_{C_0}(m)$ is nonspecial, $\mathrm{h}^0(\mathcal{O}_C(m)) = \mathrm{h}^0(\mathcal{O}_{C_0}(m))$.

Proof. Obviously, results 1 and 3 hold for $C = C_0$. For any curve $D \in |\mathcal{L}|$, there is a standard exact sequence

$$0 \longrightarrow \mathcal{L}^\vee \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0. \tag{1.4}$$

By the definition of K3 surface, $\mathrm{h}^1(\mathcal{O}_X) = 0$. Therefore, $\mathrm{h}^0(\mathcal{O}_C) = \mathrm{h}^0(\mathcal{O}_{C_0}) = 1$, implying the first assertion.

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Since $\omega_X \cong \mathcal{O}_X$, Serre duality allows us to deduce from (1.4) that

$$\mathrm{h}^0(\mathcal{L}) = g+1 \quad \text{while} \quad \mathrm{h}^1(\mathcal{L}) = \mathrm{h}^2(\mathcal{L}) = 0.$$

The second assertion then follows from

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_C \longrightarrow 0$$

and the isomorphism $\mathcal{N}_{C/X} \cong \mathcal{L} \otimes \mathcal{O}_C$.

Finally, tensor (1.4) with $\mathcal{O}_X(m)$. When $D = C_0$,

$$\mathrm{h}^1(\mathcal{O}_D(m)) = \mathrm{h}^2(\mathcal{O}_X(m)) = 0,$$

which implies $\mathrm{h}^2(\mathcal{L}^\vee(m)) = 0$. The third assertion is immediate. □

For each $C \in |\mathcal{L}|$, there is an exact sequence

$$0 \longrightarrow \mathcal{N}_{C/X} \longrightarrow \mathcal{N}_{C/\mathbf{P}^r} \longrightarrow \mathcal{N}_{X/\mathbf{P}^r} \otimes \mathcal{O}_C \longrightarrow 0. \tag{1.5}$$

LEMMA 1.10. *Suppose that $\mathcal{O}_X(1)$ and $\mathcal{L}$ are independent in* Pic X. *Then for all $C \in |\mathcal{L}|$, the composition*

$$\varphi\colon \mathrm{H}^0(X, \mathcal{N}_{X/\mathbf{P}^r}) \longrightarrow \mathrm{H}^0(C, \mathcal{N}_{X/\mathbf{P}^r} \otimes \mathcal{O}_C) \longrightarrow \mathrm{H}^1(C, \mathcal{N}_{C/X})$$

of the restriction with the connecting homomorphism arising from (1.5) *is surjective. Furthermore,* $\ker\varphi$ *is independent of C (given $\mathcal{L}$).*

Proof. From the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_X & \longrightarrow & \mathcal{T}_{\mathbf{P}^r} \otimes \mathcal{O}_X & \longrightarrow & \mathcal{N}_{X/\mathbf{P}^r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{N}_{C/X} & \longrightarrow & \mathcal{N}_{C/\mathbf{P}^r} & \longrightarrow & \mathcal{N}_{X/\mathbf{P}^r} \otimes \mathcal{O}_C & \longrightarrow & 0 \end{array}$$

of $\mathcal{O}_X$-modules we obtain a commutative square of connecting homomorphisms; in combination with the square of Proposition 1.8, it gives a commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^0(X, \mathcal{N}_{X/\mathbf{P}^r}) & \xrightarrow{\delta'} & \mathrm{H}^1(X, \mathcal{T}_X) & \xrightarrow{\times c_1(\mathcal{L})} & \mathrm{H}^2(X, \mathcal{O}_X) \\ \downarrow & & \downarrow \mathrm{H}^1(\mu) & & \| \\ \mathrm{H}^0(C, \mathcal{N}_{X/\mathbf{P}^r} \otimes \mathcal{O}_C) & \longrightarrow & \mathrm{H}^1(C, \mathcal{N}_{C/X}) & \xrightarrow{\delta} & \mathrm{H}^2(X, \mathcal{O}_X). \end{array}$$

Because of Lemma 1.9, δ is an isomorphism and since the top row is independent of C, so is $\ker\varphi$. For surjectivity it suffices to prove that the composition $(\times c_1(\mathcal{L})) \circ \delta'$ is surjective or, equivalently,

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Claim. image$(\delta') \not\subset \ker(\times c_1(\mathcal{L}))$.

Since $\omega_X \cong \mathcal{O}_X$, Serre duality on X states that the Yoneda pairing

$$\mathrm{H}^1(X, \mathcal{T}_X) \times \mathrm{H}^1(X, \Omega^1_X) \longrightarrow \mathrm{H}^2(X, \mathcal{O}_X)$$

is non-degenerate. In particular, the maps $\times c_1(\mathcal{L})$ and $\times c_1(\mathcal{O}_X(1))$ are surjective and because $\mathcal{L}$ and $\mathcal{O}(1)$ are independent in Pic$X$,

$$\ker(\times c_1 \mathcal{O}(1)) \neq \ker(\times c_1 \mathcal{L}).$$

By Kodaira vanishing $\mathrm{h}^1(\mathcal{O}_X(1)) = 0$, so by Proposition 1.7, image$(\delta') = \ker(\times c_1 \mathcal{O}(1))$, proving the claim. □

Now suppose that X is a complete intersection of type $(a_1, a_2, \ldots, a_{r-2})$ in $\mathbf{P}^r$ and that $\mathcal{O}_{C_0}(\min\{a_i\})$ is nonspecial; let $C \in |\mathcal{L}|$. By Lemma 1.9, $\mathcal{O}_C(\min\{a_i\})$ is nonspecial, and since $\mathcal{N}_{X/\mathbf{P}^r} \otimes \mathcal{O}_C \cong \bigoplus \mathcal{O}_C(a_i)$, this implies that $\mathrm{h}^1(\mathcal{N}_{X/\mathbf{P}^r} \otimes \mathcal{O}_C) = 0$. Then via (1.5), Lemma 1.10 implies

$$\mathrm{H}^1(C, \mathcal{N}_{C/\mathbf{P}^r}) = 0.$$

In particular, we have

COROLLARY 1.11. *Let X be a complete intersection K3 surface in $\mathbf{P}^r$ and C_0 a smooth genus g curve on X. If $\mathcal{O}_X(1)$ and $\mathcal{L} = \mathcal{O}_X(C_0)$ are independent in* Pic X *and $\mathcal{O}_{C_0}(1)$ is nonspecial,* Hilb^r *is smooth at all points representing curves $C \in |\mathcal{L}|$.*

2. A Framework for Studying Curves on Complete Intersection Threefolds

Fix integers $d \geqslant 1$, $g \geqslant 0$ and $r \geqslant 4$. For the remainder of this work, unless the contrary is specifically stated, we assume that

$$d > 2g - 2 \text{ and } (\mathrm{d,g}) \notin \{(1,1), (2,1), (3,2), (4,2), (5,3)\}. \tag{2.1}$$

For a projective variety Y, let CUniv$^Y \xrightarrow{p}$ CHilb^Y be the universal family over the Hilbert scheme of curves of degree d and genus g in Y, and q: CUniv$^Y \to Y$ the second projection. Then CHilb^Y is a finite union of connected components of Hilb^Y, and p is the corresponding restriction of the universal Univ$^Y \to$ Hilb^Y. Let CHilb$^Y_{\mathrm{sm}}$ be the open subscheme of CHilb^Y parameterizing smooth connected curves. In case $Y = \mathbf{P}^r$, we abbreviate to CHilb^r etc.

PROPOSITION 2.1. *With the hypotheses (2.1),* CHilb$^r_{\mathrm{sm}}$ *is smooth, connected and of dimension $(r+1)d - (r-3)g + r - 3$.*

Proof. Apply Riemann-Roch and the infinitesimal properties of the Hilbert scheme. □

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Let $\overline{\mathrm{CHilb}^r_{\mathrm{sm}}}$ be the closure in CHilb^r. Now $q^*\mathcal{O}(1)$ is flat over CHilb^r, so by the semi-continuity theorem, the set

$$S = \{s \in \overline{\mathrm{CHilb}^r_{\mathrm{sm}}} : \dim_{k(s)} \mathrm{H}^0(\mathrm{CUniv}^r_s, q^*(\mathcal{O}(1))_s) = d - g + 1\}$$
$$\text{and } \mathrm{CHilb}^r \text{ is irreducible at } s\}$$

is open in CHilb^r.

DEFINITION 2.1. Define $\mathcal{CH}^r \supset \mathrm{CHilb}^r_{\mathrm{sm}}$ to be the smooth locus of the open subscheme corresponding to S. If $Y \subset \mathbf{P}^r$ is a closed subvariety, let $\mathcal{CH}^Y$ be the scheme-theoretic intersection

$$\mathcal{CH}^Y := \mathcal{CH}^r \cap \mathrm{CHilb}^Y.$$

By abuse of notation, we denote by $\mathcal{CU}^Y \xrightarrow{p} \mathcal{CH}^Y$ the restriction of the universal family, and $q\colon \mathcal{CU}^Y \to Y$ the second projection.

Two results illustrate the utility of the definition of $\mathcal{CH}^r$. First, as a direct consequence of Corollary 1.11:

PROPOSITION 2.2. *With the notation of Corollary 1.11, if* $d = \mathcal{L} \cdot \mathcal{O}(1) > 2g - 2$, *all curves* $C \in |\mathcal{L}|$ *are parameterized by points of* $\mathcal{CH}^r$

PROPOSITION 2.3. *Let* $Y \subset \mathbf{P}^r$ *be a complete intersection of type* $(b_1, \dots, b_{r-c})$, *corresponding to a section* $f \in \Gamma(\mathbf{P}^r, \bigoplus \mathcal{O}(b_i))$. *Then* $p_*q^*(\bigoplus \mathcal{O}(b_i))$ *is locally free of rank* $(\sum b_i)d + (r-c)(1-g)$ *on* $\mathcal{CH}^r$. *Furthermore,* $\mathcal{CH}^Y = Z(p_*q^*(f))$ *as schemes.*

Proof. Riemann–Roch and the invariance of Euler characteristics in flat families imply that for all $n > 0$, $\mathrm{H}^1(C, \mathcal{O}(n)) = 0$ for all curves $C \hookrightarrow \mathbf{P}^r$ parameterized by points of $\mathcal{CH}^r$. The proposition is then a special case of Theorem 1.5. □

Henceforth, we work with a fixed $(b_1, \dots, b_{r-3})$ satisfying

$$\sum b_i = r + 1,$$

so that complete intersections of this type are Calabi–Yau threefolds. For a scheme $X \subset \mathbf{P}^r$, set

$$\mathcal{B}_X := \mathcal{O}_X(b_1) \oplus \cdots \oplus \mathcal{O}_X(b_{r-3})$$

and let

$$\mathcal{V} := p_*q^*\mathcal{B}_{\mathbf{P}^r}. \tag{2.2}$$

Combining Propositions 2.1 and 2.3, we have

$$\mathrm{rk}\,\mathcal{V} = \dim \mathcal{CH}^r. \tag{2.3}$$

Let $Y = Z(f)$ be a complete intersection of type (b_i). A smooth curve of degree d and genus g which is infinitesimally rigid in Y corresponds to a reduced isolated point of $\mathrm{CHilb}^Y_{\mathrm{sm}}$, i.e., to a point of $\mathrm{CHilb}^r_{\mathrm{sm}}$ where $p_*q^*(f)$ meets the zero section transversely. Therefore, Theorem 1 follows from

THEOREM 2.4. *If $g = 1$ and $d \geqslant 3$ (or $d \geqslant 4$ if $(b_i) = (2, 2, 2, 2)$), then for general $f \in \Gamma(\mathbf{P}^r, \mathcal{B})$, $Z(p_*q^*(f))$ has reduced isolated points in* $\mathrm{CHilb}^r_{\mathrm{sm}}$.

Given the dimension count (2.3), such a theorem seems reasonable enough. More optimistically, one hopes that for general $f \in \Gamma(\mathcal{B})$, the corresponding $p_*q^*(f)$ have finite, reduced zero schemes.

Let us examine how one proves finiteness in low degrees. A standard consequence of the algebraic version of Sard's theorem is

PROPOSITION 2.5. *Let $\mathcal{E}$ be a locally free sheaf of rank r on a smooth complex variety X of dimension n. If $\mathcal{E}$ is globally generated by a linear subspace $\Lambda \subset \Gamma(X, \mathcal{E})$, then for general $s \in \Lambda$, $Z(s)$ is either empty or smooth of dimension $n - r$.*

Remark. In case $r \geqslant n$, the smoothness hypothesis may be weakened: if X has a dense, smooth open subscheme X', the proposition still holds as can be seen by applying the smooth case on X' and on the smooth strata of the reduced singular scheme of X.

Now $\mathrm{h}^0(\mathcal{B}_{\mathbf{P}^r})$ is independent of d, whereas $\mathrm{rk}(\mathcal{V})$ increases with d, so for d sufficiently large, $\mathcal{V}$ cannot possibly be generated by sections of the form $p_*q^*(f)$. On the other hand, if one continues to assume that $d > 2g - 2$, an immediate consequence of a theorem of Gruson, Lazarsfeld and Peskine [12, Theorem p. 492] is

PROPOSITION 2.6. *The bundle $\mathcal{V}$ is generated over* $\mathrm{CHilb}^r_{\mathrm{sm}}$ *by sections of the form $p_*q^*(f)$ whenever $b_{r-3} \geqslant d - 2$ or when $g \geqslant 1$ and $b_{r-3} \geqslant d - 3$.*

Note. Actually, the theorem of [12] is considerably stronger: it gives the bound $n \geqslant d - r$ for nondegenerate nonrational curves in $\mathbf{P}^r$. Since we must consider degenerate curves, however, this is the best we can do.

Combining the two propositions in our language we have:

COROLLARY 2.7. *For a general complex complete intersection threefold Y of type $(b_1, \ldots, b_{r-3})$ (with $\sum b_i = r + 1$), if $g = 0$ and $d \leqslant b_{r-3} + 2$ or if $g \geqslant 1$ and $2g - 2 \leqslant d \leqslant b_{r-3} + 3$, then* $\mathrm{CHilb}^Y_{\mathrm{sm}}$ *is finite and reduced.*

Theorem 2 is a combination of this Corollary with Theorem 1.

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Note. Since the theorem of [12] and hence Proposition 2.6 are sharp, the proof of finiteness for rational curves in degrees $d = 8$, 9 on quintics by Nijsse [21] and by Johnsen and Kleiman [13] requires careful analysis of those curves C for which $\Gamma(\mathbf{P}^4, \mathcal{O}(5)) \to \Gamma(C, \mathcal{O}(5))$ fails to be surjective.

3. Finding Rigid Curves

3.1. CONSTRUCTING Y_0 AS A PENCIL OF K3 SURFACES

In order to prove the existence results, we need a source of curves, such as the main result of [19]:

THEOREM 3.1 (Mori). *Let k be an algebraically closed field of characteristic* 0 *and* $d > 0$ *and* $g \geqslant 0$ *be integers. Then there is a nonsingular curve* C_0 *of degree d and genus g on a nonsingular quartic surface X in* $\mathbf{P}^3$ *if and only if (1)* $g = d^2/8 + 1$, *or (2)* $g < d^2/8$ *and* $(d, g) \neq (5, 3)$.

Note. The cases $g = d^2/8 + 1$ occur when C_0 is a complete intersection of X and another surface. In all other cases, $\mathcal{O}(1)$ and $\mathcal{O}(C_0)$ are independent in Pic$X$.

The existence statement of Mori's theorem may be generalized to K3 surfaces of higher degree. Oguiso [22, Theorem 3] established the existence of $K3$ surfaces of arbitrary degree containing a smooth rational curve of given degree. For the other complete intersections, one has:

THEOREM 3.2. *Let k be an algebraically closed field of characteristic* 0 *and* $d > 0$ *and* $g \geqslant 0$ *be integers.*

(1) *If* $(d, g) = (3, 1)$*, or* $g < d^2/12$ *and* $(d, g) \neq (7, 4)$*, there exist a nonsingular complete intersection surface X of type* $(3, 2)$ *in* $\mathbf{P}^4$ *and a nonsingular curve* $C_0 \subset X$ *of degree d and genus g.*
(2) *If* $(d, g) = (4, 1)$*, or* $g < d^2/16$ *and* $(d, g) \neq (9, 5)$*, there exist a nonsingular complete intersection surface X of type* $(2, 2, 2)$ *in* $\mathbf{P}^5$ *and a nonsingular curve* $C_0 \subset X$ *of degree d and genus g.*

Furthermore, $\mathcal{O}(C_0)$ *and* $\mathcal{O}(1)$ *are independent in* Pic X.

Proof. The proof follows Mori's almost verbatim. In Proposition 3 [19] one makes the obvious numerical modification to obtain a polarization of degree 6 (respectively 8.) A cohomology calculation shows that every K3 surface with a degree 6 (respectively 8) polarization may be realized as a complete intersection of type (3, 2) (respectively (2, 2, 2)), so Mori's Remark 4 remains valid. In the final inductive step of the proof, one again makes obvious numerical modifications. In the degree six case, one needs to explicitly construct curves with $(d, g) = (1, 0)$, $(2, 0)$, $(3, 0)$, $(4, 1)$ and $(5, 2)$ while in the degree eight case, one needs curves with $(d, g) = (1, 0)$, $(2, 0)$, $(3, 0)$, $(4, 0)$, $(5, 1)$, $(6, 2)$ and $(7, 3)$ to start the induction. All of these may be constructed explicitly. □

Note. While this theorem is adequate for our purposes, A. L. Knutsen [15] has recently given a complete characterization à la Mori of smooth curves on complete intersection K3s, and in fact, on K3s with polarization of arbitrary degrees.

We now fix d and g subject to the continuing conditions (2.1). The idea of using K3 surfaces to construct rigid curves on Calabi–Yau threefolds first appears in [5], where Clemens uses a quartic surface X containing rational curves, realizes it as $\{q = \ell = 0\}$ in $\mathbf{P}^4$ for $q \in \Gamma(\mathbf{P}^4, \mathcal{O}(4))$ and $\ell \in \Gamma(\mathbf{P}^4, \mathcal{O}(1))$, and considers the (nodal) quintic $Y_0 = \{f_0 = 0\}$, where

$$f_0 = \alpha q + \beta \ell$$

for general $\alpha \in \Gamma(\mathbf{P}^4, \mathcal{O}(1))$ and $\beta \in \Gamma(\mathbf{P}^4, \mathcal{O}(4))$.

In general, to construct a nodal complete intersection Calabi–Yau threefold of type $(b_1, \ldots, b_{r-3})$ in $\mathbf{P}^r$, start with a smooth complete intersection K3 surface X of type $(a_1, \ldots, a_{r-2})$ in $\mathbf{P}^r$, with

$$1 \leqslant a_i \leqslant b_i \text{ for all } i \text{ and } \sum a_i = r + 1.$$

(Note that several of the a_i may well be 1.) Choose generators g_i of degrees a_i for the ideal of X. By Theorems 3.1 or 3.2, we may assume that X contains a smooth curve C_0 of genus g and degree d, and that Pic X is spanned by $\mathcal{O}(1)$ and $\mathcal{L} = \mathcal{O}(C_0)$. Set

$$\mathcal{A} := \bigoplus \mathcal{O}_{\mathbf{P}^r}(a_i).$$

For general $\alpha_{ij} \in \Gamma(\mathbf{P}^r, \mathcal{O}(b_i - a_j))$, set

$$f_i := \sum \alpha_{ij} g_j$$

and let

$$Y_0 := V(f_1, f_2, \ldots, f_{r-3})$$

be the corresponding complete intersection threefold. We will occasionally refer to the fourfold

$$Z_0 := (f_1 = \cdots = f_{r-4} = 0).$$

PROPOSITION 3.3. *If the* (α_{ij}) *are sufficiently general,* Z_0 *is smooth, and* Y_0 *has* $\ell > 2$ *ordinary double points. The precise value of* ℓ *and some of the corresponding equations for* Y_0^{sing} *in* X *are shown at the top of the next page.*

Proof. The proposition must be established case-by-case, where it follows from generality of the α_{ij} via repeated applications of Bertini's theorem and the Jacobian criterion. □

(b_i)	(a_j)	ℓ	Y_0^{sing}
(5)	(4, 1)	16	$X \cap (\alpha_{11} = \alpha_{12} = 0)$
(5)	(3, 2)	36	$X \cap (\alpha_{11} = \alpha_{12} = 0)$
(4, 2)	(4, 1, 1)	4	$X \cap (\alpha_{22} = \alpha_{23} = 0)$
(4, 2)	(3, 2, 1)	18	$X \cap (\alpha_{11} = \alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22} = 0)$
(4, 2)	(2, 2, 2)	32	$X \cap (\alpha_{21}\alpha_{12} - \alpha_{22}\alpha_{11} = \alpha_{21}\alpha_{13} - \alpha_{23}\alpha_{11} = 0)$
(3, 3)	(3, 2, 1)	12	$X \cap (\alpha_{21}\alpha_{12} - \alpha_{22}\alpha_{11} = \alpha_{21}\alpha_{13} - \alpha_{23}\alpha_{11} = 0)$
(3, 3)	(2, 2, 2)	32	$X \cap (\alpha_{21}\alpha_{12} - \alpha_{22}\alpha_{11} = \alpha_{21}\alpha_{13} - \alpha_{23}\alpha_{11} = 0)$
(3, 2, 2)	(3, 2, 1, 1)	6	$X \cap (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} = \alpha_{22}\alpha_{34} - \alpha_{24}\alpha_{32} = 0)$
(3, 2, 2)	(2, 2, 2, 1)	16	$X \cap$ *linear* $\cap$ *quadratic*
(2, 2, 2, 2)	(2, 2, 2, 1, 1)	8	$X \cap$ *linear* $\cap$ *linear*

LEMMA 3.4. *There are isomorphisms*

$$\mathcal{N}_{X/\mathbf{P}^r} \cong \mathcal{A}_X \quad \text{and} \quad \mathcal{N}_{Y_0/\mathbf{P}^r} \otimes \mathcal{O}_X \cong \mathcal{B}_X, \tag{3.1}$$

under which the usual map $\mathrm{H}^0(X, \mathcal{N}_{X/\mathbf{P}^r}) \to \mathrm{H}^0(X, \mathcal{N}_{Y_0/\mathbf{P}^r} \otimes \mathcal{O}_X)$ *is given by the matrix* $(\overline{\alpha}_{ij})$*, where* $\overline{\alpha}_{ij}$ *is the restriction of* α_{ij} *to* X*. Furthermore.*

$$\mathcal{N}_{X/Y_0} \cong \mathcal{O}_X. \tag{3.2}$$

Proof. The existence of the isomorphisms 3.1 is a special case of Lemma 1.2; the description of the map on global sections is standard and follows from the same description at the level of sheaves.

Finally, the isomorphism $\mathcal{N}_{X/Y_0} \cong \mathcal{O}_X$ holds away from $Y_0^{\text{sing.}}$ by the adjunction formula. Locally, at the nodes, $X \subset Y_0$ can be transformed by an analytic change of coordinates to the inclusion

$$(x = y = 0) \subset (xz - yw = 0)$$

in affine 4-space, where one checks directly that $\mathcal{N}_{X/Y_0}$ is locally free. □

THEOREM 3.5. *Suppose that the* α_{ij} *are general. Then for all* $C \in |\mathcal{L}|$, $\mathrm{h}^0(\mathcal{N}_{C/Y_0}) = g$

We shall interpret this to mean that most of the curves in $|\mathcal{L}|$ acquire no additional deformations when they are considered as curves in Y_0.

Proof. For *any* $C \in |\mathcal{L}|$ we have the exact sequence of normal sheaves:

$$0 \longrightarrow \mathcal{N}_{C/Y_0} \longrightarrow \mathcal{N}_{C/\mathbf{P}^r} \longrightarrow \mathcal{N}_{Y_0/\mathbf{P}^r} \otimes \mathcal{O}_C, \tag{3.3}$$

which we combine with (1.5) into a commutative diagram with exact rows and columns:

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$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{N}_{C/X} & = & \mathcal{N}_{C/X} & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_{C/Y_0} & \longrightarrow & \mathcal{N}_{C/\mathbf{P}^r} & \longrightarrow & \mathcal{N}_{Y_0/\mathbf{P}^r}\otimes\mathcal{O}_C \\
 & & \downarrow{\scriptstyle\mu} & & \downarrow & & \| \\
 & & \mathcal{N}_{X/Y_0}\otimes\mathcal{O}_C & \longrightarrow & \mathcal{N}_{X/\mathbf{P}^r}\otimes\mathcal{O}_C & \xrightarrow{\Phi_C} & \mathcal{N}_{Y_0/\mathbf{P}^r}\otimes\mathcal{O}_C \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array} \tag{3.4}$$

Given that $\mathrm{h}^0(\mathcal{N}_{C/X}) = g$ while $\mathrm{h}^0(\mathcal{N}_{X/Y_0} \otimes \mathcal{O}_C) = 1$ ((3.2) and Lemma 1.9), the theorem will be proved if we can show that for any $C \in |\mathcal{L}|$, $\mathrm{H}^0(\mu)$ is not surjective. But a diagram chase reduces this to

Claim. Let the α_{ij} be general. Then for all $C \in |\mathcal{L}|$,

$$\delta_C(\ker \mathrm{H}^0(\Phi_C)) \neq 0,$$

where $\delta_C \colon \mathrm{H}^0(C, \mathcal{N}_{X/\mathbf{P}^r} \otimes \mathcal{O}_C) \to \mathrm{H}^1(C, \mathcal{N}_{C/X})$ is the connecting homomorphism.

To prove this claim, use Gaussian elimination to find a generator $N = N(\alpha_{ij})$ of the null-space of the linear map $\mathbf{C}^{r-2} \xrightarrow{(\alpha_{ij})} \mathbf{C}^{r-3}$. Keeping track of degrees, this can be done in such a way that the ith coordinate of N is of degree a_i. For example, if $(b_i) = (4, 2)$ and $(a_j) = (3, 2, 1)$, the vector $N = (\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22}, -\alpha_{14}\alpha_{23}, \alpha_{11}\alpha_{22})$. Since N is well-defined only up to scalar, we view it as a line in $\mathrm{H}^0(\mathbf{P}^r, \mathcal{A})$ or a point of $\mathbf{P}(\mathrm{H}^0(\mathbf{P}^r, \mathcal{A}))$.

Now each term of each coordinate of N is a term of a determinant of (α_{ij}), so that as (α_{ij}) varies, N hits a multiple of each element of the form $(0, \dots, 0, \lambda_1\lambda_2 \dots \lambda_{a_i}, 0, \dots, 0)$, where the λ_k's are of degree 1. Since the image of the Segre embedding of $\mathbf{P}(\mathrm{H}^0(\mathbf{P}^r, \mathcal{O}(1)))^{\times a_i}$ in $\mathbf{P}(\mathrm{H}^0(\mathbf{P}^r, \mathcal{O}(a_i)))$ is non-degenerate, the linear span of the image of N includes the spaces

$$0 \oplus \cdots \oplus 0 \oplus \mathrm{H}^0(\mathbf{P}^r, \mathcal{O}(a_i)) \oplus 0 \oplus \cdots \oplus 0$$

and hence all of $\bigoplus_i \mathrm{H}^0(\mathbf{P}^r, \mathcal{O}(a_i))$.

Now in light of Lemma 3.4, the claim follows from Lemma 1.10 since $\mathrm{H}^0(\mathbf{P}^r, \mathcal{A}) \to \mathrm{H}^0(X, \mathcal{A}_X)$ is surjective for any complete intersection X. □

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Let

$$S := Y_0^{\text{sing}}$$

be the set of nodes of Y_0 so that $\ell = |S|$. Let $Y_0' := Y_0^{\text{ns}} = Y_0 \setminus S$ and $X' := X \setminus S$.

COROLLARY 3.6. *If $C \in |\mathcal{L}|$ there is a commutative diagram with exact rows and vertical isomorphisms:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{\mathrm{H}^0(C, \mathcal{N}_{C/\mathbf{P}^r})}{\mathrm{H}^0(C, \mathcal{N}_{C/Y_0})} & \longrightarrow & \mathrm{H}^0(C, \mathcal{B}_C) & \longrightarrow & \mathrm{H}^1(C, \mathcal{N}_{C/Y_0}) & \longrightarrow & 0 \\ & & \downarrow\wr & & \| & & \downarrow\wr & & \\ 0 & \longrightarrow & \dfrac{\mathrm{H}^0(C, \mathcal{A}_C)}{\mathrm{H}^0(C, \mathcal{O}_C)} & \longrightarrow & \mathrm{H}^0(C, \mathcal{B}_C) & \longrightarrow & \mathrm{H}^1(C, \mathcal{O}_C) & \longrightarrow & 0. \end{array}$$

Proof. For curves supported on X', all columns and rows of (3.4) are short exact. Given the further cohomology calculations of Corollary 1.11 and taking into account (3.1), the diagram of cohomology groups arising from (3.4) reduces to the given one. □

The existence of rigid rational curves in quintic threefolds due to Clemens [5] and Katz [14] has been generalized to type (4,2) threefolds by Oguiso [22] and recently to all ciCY threefolds [7]:

THEOREM 3.7 (Ekedahl, Johnsen and Sommervoll). *For any $d > 0$, a general, smooth, Calabi–Yau complete intersection threefold admits a rigid degree d embedding of $\mathbf{P}^1$.*

Our methods provide an alternate proof:

Proof. Let $g = 0$. The existence of a smooth rational curve C_0 of degree d on a relevant K3 surface follows from Theorems 3.1 and 3.2. For general α_{ij}, Theorem 3.5 implies immediately that C_0 is infinitesimally rigid in Y_0. So a general ciCY threefold must also contain an infinitesimally rigid rational curve of degree d. (For an elaboration of this last point, see the proof of Theorem 2.4 given in the following section.)

3.2. GENUS ONE: DEFORMING Y_0

In this section, we consider the case $g = 1$. Then X is elliptically fibered by

$$\pi \colon X \to \Lambda := \mathbf{P}(\mathrm{H}^0(X, \mathcal{O}_X(C_0))) \cong \mathbf{P}^1.$$

By the generality of the α_{ij}, we may assume

$$C^{\rm sing} \cap S = \emptyset \quad \text{for all fibers C of } \pi. \tag{3.5}$$

Set

$$\Delta := \Lambda \setminus \pi(S).$$

The universal property of the Hilbert scheme shows that $\Lambda \to \mathrm{CHilb}^{Y_0}$ is a closed subscheme, and by Corollary 3.6, $\Delta \to \mathrm{CHilb}^{Y_0}$ is an open subscheme. Using Proposition 2.2 one concludes

COROLLARY 3.8. *Δ is an open subscheme of $\mathcal{CH}^{Y_0} = Z(p_* q^*(f_1, \ldots, f_{r-3}))$.*

To deform Y_0 to a smooth threefold containing a rigid curve, we shall apply Proposition 1.4. In our case, the sequence (1.1) is

$$0 \longrightarrow \mathcal{N}_{\Delta/\mathcal{CH}^r} \stackrel{m}{\to} \mathcal{V}|_\Delta \stackrel{\rho}{\to} \mathcal{M} \longrightarrow 0 \tag{3.6}$$

and $g = \mathrm{rk}\,\mathcal{M} = 1$. To find a section of $\mathcal{M}$ with isolated zeros, we shall extend ρ to a map of locally free sheaves on Λ. (See Definition 2.1 and (2.2) for notation.)

Recall that S denotes the singular scheme of Y_0. Let $\widetilde{X} = \mathrm{Bl}_S X \stackrel{b}{\to} X$ with exceptional divisor $E = E_1 + \cdots + E_\ell$, and let $\tilde{\pi} = \pi \circ b \colon \widetilde{X} \to \Lambda$. Define

$$\overline{\mathcal{M}} = \mathrm{R}^1 \tilde{\pi}_* \mathcal{O}_{\widetilde{\mathrm{X}}}(2\mathrm{E}).$$

PROPOSITION 3.9. *The sheaf $\overline{\mathcal{M}}$ has the following properties:*

(1) *There exists a morphism $\bar{\rho} \colon \mathcal{V}|_\Lambda \to \overline{\mathcal{M}}$ extending ρ,*
(2) *$\overline{\mathcal{M}}$ is invertible,*
(3) *The composition $\Gamma(\mathbf{P}^r, \mathcal{B}) \longrightarrow \Gamma(X, \mathcal{B}_X) \longrightarrow \Gamma(\Lambda, \mathcal{V}|_\Lambda) \longrightarrow \Gamma(\Lambda, \overline{\mathcal{M}})$ is surjective, and*
(4) $\deg \overline{\mathcal{M}} = \ell - 2 > 0$.

Proof of Theorem 2.4. A line bundle of degree $n > 0$ on $\Lambda \cong \mathbf{P}^1$ has a section with n isolated reduced zeros at specified points, so by Proposition 3.9, there exists $f \in \Gamma(\mathbf{P}^r, \mathcal{B})$ such that $\rho\big(p_* q^*(f)|_\Delta\big)$ has $\ell - 2 >$ isolated reduced zeros at points in Δ over which π has smooth fibers. By Proposition 1.4, for general ε, $Z(p_* q^*((f_i) + \varepsilon f))$ will have $\ell - 2$ reduced isolated points representing smooth curves. The scheme $\mathrm{CHilb}^r_{\rm sm}$ parameterizing smooth curves is an open subscheme of $\mathcal{CH}^r$, and having a reduced isolated zero is an open condition on sections of a locally free sheaf. Thus, if $Y = V(F)$ for general $F \in \Gamma(\mathbf{P}^r, \mathcal{B})$, then $\mathcal{CH}^Y = Z(p_* q^*(F))$ has $\ell - 2$ reduced isolated points representing smooth curves. □

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Proof of Proposition 3.9. There is an exact sequence

$$0 \longrightarrow \mathcal{N}_{X/Y_0} \longrightarrow \mathcal{N}_{X/\mathbf{P}^r} \longrightarrow \mathcal{N}_{Y_0/\mathbf{P}^r} \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

which, in light of (3.1), becomes

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}_X \longrightarrow \mathcal{B}_X \longrightarrow \mathcal{O}_S \longrightarrow 0. \tag{3.7}$$

Recall that we had $b\colon \widetilde{X} = \mathrm{Bl}_S X \to X$ and $\tilde{\pi} = \pi \circ b$. For all n, set $\mathcal{O}_{\widetilde{X}}(n) = b^*\mathcal{O}_X(n)$, and let $\mathcal{A}_{\widetilde{X}} = b^*\mathcal{A}_X$ and $\mathcal{B}_{\widetilde{X}} = b^*\mathcal{B}_X$. Set

$$\mathcal{C}_{\widetilde{X}} := \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_{r-4}),$$

so that $\mathcal{B}_{\widetilde{X}} = \mathcal{C}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}(b_{r-3})$. Then twisting the normal bundle sequence for the regular embeddings $\widetilde{X} \hookrightarrow \mathrm{Bl}_S\, Y_0 \hookrightarrow \mathrm{Bl}_S\, \mathbf{P}^r$ by $\mathcal{O}(2E)$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(2E) \longrightarrow \mathcal{A}_{\widetilde{X}}(E) \longrightarrow \mathcal{C}_{\widetilde{X}}(E) \oplus \mathcal{O}_{\widetilde{X}}(b_{r-3}) \longrightarrow 0. \tag{3.8}$$

of $\mathcal{O}_{\widetilde{X}}$-modules.

If we define $\bar{\rho} = j \circ \delta$

$$\mathcal{V}|_\Lambda = \tilde{\pi}_*\mathcal{B} \xrightarrow{j} \tilde{\pi}_*(\mathcal{C}(E) \oplus \mathcal{O}(b_{r-3})) \xrightarrow{\delta} \mathrm{R}^1\tilde{\pi}_*\mathcal{O}(2E) = \overline{\mathcal{M}}$$

as the composition of the natural monomorphism with the morphism of derived functors arising from (3.8), then part (1) of the proposition is a consequence of Nakayama's lemma, Proposition 1.6 and Corollary 3.6.

Suppose $C \in |\mathcal{L}|$ with $C \cap S = \{p\}$. In $\widetilde{X}$, let $\overline{C}$ be the proper transform of C and let E_0 be the exceptional curve over p. Then

$$\widetilde{C} = \pi^{-1}(C) = \overline{C} + E_0,$$

and since p is a non-singular point of C (3.5), $\overline{C} \cdot E_0 = 1$; set $\overline{C} \cap E_0 = \{q\}$.

There is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{C}}(-q) \longrightarrow \mathcal{O}_{\widetilde{C}} \longrightarrow \mathcal{O}_{E_0} \longrightarrow 0$$

of $\mathcal{O}_{\widetilde{X}}$-modules, which becomes

$$0 \longrightarrow \mathcal{O}_{\overline{C}}(q) \longrightarrow \mathcal{O}_{\widetilde{X}}(2E) \otimes \mathcal{O}_{\widetilde{C}} \longrightarrow \mathcal{O}_{E_0}(-2) \longrightarrow 0$$

upon tensoring with $\mathcal{O}_{\widetilde{X}}(2E)$.

Since $p \notin C^{\mathrm{sing}}$, $\overline{C} \cong C$, implying $\mathrm{h}^0(\mathcal{O}_{\overline{C}}(q)) = 1$ and $\mathrm{h}^1(\mathcal{O}_{\overline{C}}(q)) = 0$. We conclude that

$$\mathrm{h}^i(\mathcal{O}_{\widetilde{X}}(2E) \otimes \mathcal{O}_{\widetilde{C}}) = 1 = \mathrm{h}^i(\mathcal{O}_C) \text{ for i=0, 1.}$$

The proof shows by induction on the order of $C \cap S$ that these equalities hold for all C. Since $\tilde{\pi}$ is flat and Λ reduced, semi-continuity implies that $\overline{\mathcal{M}}$ is invertible, which is assertion (2) of the proposition.

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LEMMA 3.10. *For all* n, i, *there are isomorphisms*

$$\mathrm{H}^i(X, \mathcal{O}_X(n)) \xrightarrow{\sim} \mathrm{H}^i(\widetilde{X}, \mathcal{O}(n)) \xrightarrow{\sim} \mathrm{H}^i(\widetilde{X}, \mathcal{O}(n) \otimes \mathcal{O}(E)).$$

Proof. $E = E_1 + \cdots + E_\ell$. The E_i are pairwise disjoint (-1)-curves, so there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(n) \longrightarrow \mathcal{O}_{\widetilde{X}}(E) \otimes \mathcal{O}_{\widetilde{X}}(n) \longrightarrow \bigoplus \mathcal{O}_{E_i}(-1) \longrightarrow 0$$

of $\mathcal{O}_{\widetilde{X}}$-modules. whence the second isomorphism. The first comes from the projection formula and the degeneration of the Leray spectral sequence for the map $b: \widetilde{X} \to X$. □

COROLLARY 3.11. $\mathrm{h}^0(\mathcal{O}_{\widetilde{X}}(2E)) = 1$, $\mathrm{h}^1(\mathcal{O}_{\widetilde{X}}(2E)) = \ell - 1$ *and* $\mathrm{h}^2(\mathcal{O}_{\widetilde{X}}(2E)) = 0$.

Proof. E is a non-zero effective divisor on $\widetilde{X}$ so $\mathrm{h}^0(\mathcal{O}(-E)) = 0$. Since $\omega_{\widetilde{X}} = b^*\omega_X \otimes \mathcal{O}(E)$, Serre duality shows that $\mathrm{h}^2(\mathcal{O}(2E)) = 0$. The remaining assertions are then immediate from the sequence

$$0 \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{O}(2E) \longrightarrow \bigoplus \mathcal{O}_{E_i}(-2) \longrightarrow 0$$

and the lemma. □

We return to the proof of Proposition 3.9. Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccc}
\mathrm{H}^0(\widetilde{X}, \mathcal{C}(E) \oplus \mathcal{O}(b_{r-3})) & \longrightarrow & \mathrm{H}^1(\widetilde{X}, \mathcal{O}(2E)) & \longrightarrow & \mathrm{H}^1(\widetilde{X}, (\mathcal{A}(E))) \\
\| & & \downarrow & & \\
\mathrm{H}^0(\Lambda, \tilde{\pi}_*(\mathcal{C}(E) \oplus \mathcal{O}(b_{r-3}))) & \longrightarrow & \mathrm{H}^0(\Lambda, \overline{\mathcal{M}}) & & \\
& & \downarrow & & \\
& & 0 & &
\end{array}$$

where vertical exactness is a consequence of the degeneration of the Leray spectral sequence for $\tilde{\pi}$. By Lemma 3.10, the group at the right is 0. Since X is a complete intersection, the restriction $\mathrm{H}^0(\mathbf{P}^r, \mathcal{B}) \longrightarrow \mathrm{H}^0(X, \mathcal{B})$ is a surjection, so the third assertion of Proposition 3.9 follows from another application of Lemma 3.10.

By Hartogs's theorem, $b_*\mathcal{O}_{\widetilde{X}}(2E) \cong \mathcal{O}_X$. From the degeneration of the Leray spectral sequence of π, one easily shows

$$\pi_*\mathcal{O}_X \cong \mathcal{O}_\Lambda,$$

so that

$$\tilde{\pi}_*\mathcal{O}_{\widetilde{X}}(2E) \cong \mathcal{O}_\Lambda.$$

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Finally, by Leray again, there is an extension

$$0 \longrightarrow \mathrm{H}^1(\Lambda, \mathcal{O}) \longrightarrow \mathrm{H}^0(\widetilde{X}, \mathcal{O}(2E)) \longrightarrow \mathrm{H}^0(\Lambda, \overline{\mathcal{M}}) \longrightarrow 0,$$

and $\deg \overline{\mathcal{M}} = \ell - 2$ follows from Corollary 3.11. □

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