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ON THE DIOPHANTINE EQUATION $(P_n^{(k)})^2 + (P_{n+1}^{(k)})^2 = P_m^{(k)}$

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Abstract

A generalisation of the well-known Pell sequence $\{P_n\}_{n\geq 0}$ given by $P_0 = 0$, $P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geq 0$ is the k-generalised Pell sequence $\{P_n^{(k)}\}_{n\geq -(k-2)}$ whose first k terms are $0, \dots, 0, 1$ and each term afterwards is given by the linear recurrence $P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}$. For the Pell sequence, the formula $P_n^2 + P_{n+1}^2 = P_{2n+1}$ holds for all $n \geq 0$. In this paper, we prove that the Diophantine equation

$$(P_n^{(k)})^2 + (P_{n+1}^{(k)})^2 = P_m^{(k)}$$

has no solution in positive integers k, m and n with $n > 1$ and $k \geq 3$.

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1. Introduction

The Pell sequence $\{P_n\}_{n\geq 0}$ is the second-order linear recurrence defined by $P_{n+2} = 2P_{n+1} + P_n$ with initial conditions $P_0 = 0$ and $P_1 = 1$. A few terms of this sequence are

$$0, 1, 2, 5, 12, 29, 70, \dots .$$

Diophantine equations related to the sums of powers of consecutive Pell numbers were studied by several authors. For example, motivated by the well-known identity

$$P_n^2 + P_{n+1}^2 = P_{2n+1}, \tag{1.1}$$

which tells us that the sum of the squares of two consecutive Pell numbers is still a Pell number, Rihane *et al.* [6] studied the Diophantine equation

$$P_n^x + P_{n+1}^x = P_m$$

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and gave all the solutions of this equation in nonnegative integers m, n and x. Ddamulira and Luca [2] considered the more general Diophantine equation

$$U_n^x + U_{n+1}^x = U_m$$

in nonnegative integers (n, m, x), where $\boldsymbol{U} = (U_n)_{n\geq 0}$ is the Lucas sequence given by $U_0 = 0$, $U_1 = 1$ and $U_{n+2} = rU_{n+1} + U_n$ for all $n \geq 0$. (Note that $\boldsymbol{U}$ coincides with the Pell sequence when $r = 2$.)

Luca *et al.* [5] found all the solutions of the Diophantine equation

$$P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x = P_m$$

in positive integers (m, n, k, x) and Faye *et al.* [3] considered the more general exponential Diophantine equation

$$P_n^x + P_{n+1}^x = P_m^y$$

in positive integers (m, n, x, y).

For an integer $k \geq 2$, we consider a generalisation of the Pell sequence called the k-generalised Pell sequence or, for simplicity, the k-Pell sequence $P^{(k)} = \{P_n^{(k)}\}_{n\geq -(k-2)}$ given by the higher order linear recurrence,

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$

with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$. Note that this generalisation is in fact a family of sequences, where each new choice of k generates a distinct sequence. In particular, the usual Pell sequence is obtained for $k = 2$, that is, $P_n^{(2)} = P_n$.

Motivated by the above results, we look at the identity (1.1) in k-generalised Pell numbers. More precisely, we prove the following theorem.

THEOREM 1.1. *The Diophantine equation*

$$(P_n^{(k)})^2 + (P_{n+1}^{(k)})^2 = P_m^{(k)} \tag{1.2}$$

has no positive integer solutions (n, k, m) *with* $n > 1$ *and* $k \geq 3$.

Usually, such a type of Diophantine equations require Baker-type estimates for lower bounds for linear forms in the logarithms of algebraic numbers as well as reduction techniques involving the theory of continued fractions (the Baker–Davenport method and the LLL algorithm). In our case, we will use only elementary properties of k-Pell numbers.

2. Auxiliary results

In this section, we shall collect some facts and tools which will be used later.

The characteristic polynomial of $P^{(k)}$ is

$$\Phi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1$$

and it is irreducible over $\mathbb{Q}[x]$ with just one root $\gamma := \gamma(k)$ outside the unit circle. The other roots are strictly inside the unit circle, that is, γ is a Pisot number of degree k. This positive real root is called the dominant root of $\Phi_k(x)$.

In [1], the Binet-like formula for the k-Pell numbers is given by

$$P_n^{(k)} = \sum_{i=1}^{k} \frac{\gamma_i - 1}{(k+1)\gamma_i^2 - 3k\gamma_i + k - 1}\gamma_i^n, \tag{2.1}$$

where $\gamma = \gamma_1, \ldots, \gamma_k$ are the roots of the characteristic polynomial $\Phi_k(x)$. The contribution of the roots inside the unit circle in (2.1) is almost trivial. More precisely, it was proved that the approximation

$$|P_n^{(k)} - g_k(\gamma)\gamma^n| < \tfrac{1}{2} \tag{2.2}$$

holds for all $n \geq 2 - k$, where

$$g_k(z) = \frac{z - 1}{(k+1)z^2 - 3kz + k - 1}. \tag{2.3}$$

From (2.2), we can write

$$P_n^{(k)} = g_k(\gamma)\gamma^n + \mathscr{E}_k(n) \quad \text{where } |\mathscr{E}_k(n)| < \tfrac{1}{2}.$$

Also, we have the inequality

$$\gamma^{n-2} \leq P_n^{(k)} \leq \gamma^{n-1} \quad \text{for all } n \geq 1. \tag{2.4}$$

Furthermore, Kılıç and Taşci [4] showed that the terms of the k-Pell sequence with indices $n \in \{1, 2, \ldots, k+1\}$ coincide with the first $k+1$ terms of the Fibonacci sequence with positive odd indices, that is,

$$P_n^{(k)} = F_{2n-1} \quad \text{for } 1 \leq n \leq k+1.$$

In the next lemma, we gather some technical results that will be used later.

Lemma 2.1 [1, Lemma 3.2]. *Let $k, l \geq 2$ be integers.*

(a) *If $k > l$, then $\gamma(k) > \gamma(l)$.*
(b) *$\phi^2(1 - \phi^{-k}) < \gamma(k) < \phi^2$, where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the golden section.*
(c) *If $k \geq 6$, then $\phi^2 - k^{-1} < \phi^2(1 - \phi^{-k}) < \gamma(k) < \phi^2$.*
(d) *$g_k(x)$ defined in (2.3) as a function of a real variable is positive, continuous and decreasing in the interval (c_k, ∞), where $c_k = (3k + \sqrt{5k^2 + 4})/2(k+1)$. Moreover, $g_k(\phi^2) = 1/(\phi + 2)$.*
(e) *$0.276 < g_k(\gamma(k)) < 0.5$.*

Note that for $k \geq 10$, the value of $g_k(\gamma)$ is not greater than 0.31, as can be seen from $g_k(\gamma) < g_k(\phi^2(1 - \phi^{-k}))$.

We will also require a lemma that lists all cases when the sum of squares of any two Fibonacci numbers becomes a Fibonacci number.

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Lemma 2.2 [7, Theorem 1.1]. *Let n, m, r be positive integers such that $m \leq n$ and let (n, m, r) be a solution of the Diophantine equation $F_n^2 + F_m^2 = F_r$. Then*

$$(n, m, r) \in \{(2, 2, 3), (3, 1, 5), (3, 3, 6), (n, n-1, 2n-1)\}.$$

3. Proof

Proof of Theorem 1.1. When $n = 2$, one has $(P_2^{(k)})^2 + (P_3^{(k)})^2 = 2^2 + 5^2 = 29$; however, $P_m^{(k)}$ belongs to the increasing sequence $13, 33, 34, 84, 88, \ldots$ for $k \geq 3$ and $m \geq 4$. Thus, there is no solution for $n = 2$. So we may suppose that $n \geq 3$.

From the estimates in (2.4), we obtain

$$\gamma^{m-2} \leq P_m^{(k)} = (P_n^{(k)})^2 + (P_{n+1}^{(k)})^2 \leq \gamma^{2(n-1)} + \gamma^{2n} = \gamma^{2n}(1 + \gamma^{-2}) < \gamma^{2n+1}$$

and

$$\gamma^{2(n-1)} \leq (P_{n+1}^{(k)})^2 < (P_n^{(k)})^2 + (P_{n+1}^{(k)})^2 = P_m^{(k)} \leq \gamma^{m-1},$$

where we used $1 + 1/\gamma^2(k) < 1 + 1/\gamma^2(3) < 2 < \gamma(k)$. Thus, if (m, n, k) is a solution of (1.2), then $m \in \{2n, 2n+1, 2n+2\}$.

Next, if $3 \leq n \leq 5$ and $3 \leq k \leq 12$, then finite computations in *Mathematica* yield no solutions for (1.2), whereas if $3 \leq n \leq 5$ and $k > 12$, then all three numbers $P_n^{(k)}$, $P_{n+1}^{(k)}$ and $P_m^{(k)}$ are Fibonacci numbers with odd indices:

$$P_n^{(k)} = F_{2n-1}, \quad P_{n+1}^{(k)} = F_{2n+1} \quad \text{and} \quad P_m^{(k)} \in \{F_{4n-1}, F_{4n+1}, F_{4n+3}\}.$$

By Lemma 2.2, the sum of the first two squares cannot be the third value. So, from now on, $n > 5$ independently of k.

Let $m = 2k + i$, $i \in \{0, 1, 2\}$. Then (1.2) can be written as

$$(g_k(\gamma)\gamma^n + \mathscr{E}_k(n))^2 + (g_k(\gamma)\gamma^{n+1} + \mathscr{E}_k(n+1))^2 = g_k(\gamma)\gamma^{2n+i} + \mathscr{E}_k(2n+i).$$

Divide both sides by γ^{2n} to get

$$(g_k(\gamma) + \mathscr{E}_k(n)/\gamma^n)^2 + (g_k(\gamma)\gamma + \mathscr{E}_k(n+1)/\gamma^n)^2 = g_k(\gamma)\gamma^i + \mathscr{E}_k(2n+i)/\gamma^{2n}. \tag{3.1}$$

We write

$$(g_k(\gamma) + \mathscr{E}_k(n)/\gamma^n)^2 = g_k^2(\gamma) + \mathscr{C}_1,$$

where

$$|\mathscr{C}_1| = |2g_k(\gamma)\mathscr{E}_k(n)/\gamma^n + (\mathscr{E}_k(n)/\gamma^n)^2| \leq 2 \cdot (1/2) \cdot (1/2) \cdot \gamma^{-n} + (1/4) \cdot \gamma^{-2n} < \gamma^{-n}. \tag{3.2}$$

Similarly,

$$(g_k(\gamma)\gamma + \mathscr{E}_k(n+1)/\gamma^n)^2 = g_k^2(\gamma)\gamma^2 + \mathscr{C}_2,$$

where

$$\begin{aligned} |\mathscr{C}_2| &= |2g_k(\gamma)\gamma\mathscr{E}_k(n+1)/\gamma^n + (\mathscr{E}_k(n+1)/\gamma^n)^2| \\ &\leq 2 \cdot (1/2) \cdot 3 \cdot (1/2) \cdot \gamma^{-n} + (1/4) \cdot \gamma^{-2n} < 2\gamma^{-n}. \end{aligned} \tag{3.3}$$

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Since $|\mathscr{C}_3| = |\mathscr{E}_k(2n+i)/\gamma^{2n}| < \gamma^{-n}$, we obtain from (3.1), (3.2) and (3.3) that

$$|g_k(\gamma) + g_k(\gamma)\gamma^2 - \gamma^i| = \frac{1}{g}|\mathscr{C}_3 - \mathscr{C}_1 - \mathscr{C}_2| < 4 \cdot \left(\frac{1}{\gamma^n} + \frac{2}{\gamma^n} + \frac{1}{\gamma^n}\right) = \frac{16}{\gamma^n}. \tag{3.4}$$

Computing the left-hand side of (3.4) for $k = 3, 4, \ldots, 9$ and $i \in \{0, 1, 2\}$, we obtain

$$0.241 < |g_k(\gamma) + g_k(\gamma)\gamma^2 - \gamma^i| < \frac{16}{\gamma^n} < \frac{16}{2.54^n},$$

which contradicts the fact that $n > 5$.

Now, suppose $k \geq 10$. If $i = 0$, then in the left-hand side of (3.4),

$$g_k(\gamma)\gamma^2 + g_k(\gamma) - 1 > \frac{(\phi^2(1-\phi^{-10}))^2}{\phi+2} + \frac{1}{\phi+2} - 1 > 1.14,$$

leading to $2.54^n < 14.04$, which is false for $n > 5$. So, $i \in \{1, 2\}$.

If $i = 1$, then

$$\gamma > \phi^2(1-\phi^{-10}) > 2.59$$

and $g_k(\gamma) < 0.31$, so $\gamma - g_k(\gamma)\gamma^2 - g_k(\gamma) > 2.59 - 0.31\phi^4 - 0.31 > 0.15$ and we get $\gamma^n < 106.7$, which is false for $\gamma > 2.59$ and $n > 5$.

Similarly, if $i = 2$, we also have $g_k(\gamma) < 0.31$ and $\gamma > 2.59$. It follows that $\gamma^2 - g_k(\gamma)\gamma^2 - g_k(\gamma) > 2.59^2 - 0.31\phi^4 - 0.31 > 4.27$, which is even larger than the previous estimate, giving us again a contradiction. This completes the proof. □

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