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Hermitian Yang–Mills–Higgs Metrics on Complete Kähler Manifolds

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Abstract. In this paper, first, we will investigate the Dirichlet problem for one type of vortex equation, which generalizes the well-known Hermitian Einstein equation. Secondly, we will give existence results for solutions of these vortex equations over various complete noncompact Kähler manifolds.

1 Introduction

The Hermitian Yang–Mills theory plays an important role for holomorphic vector bundles over a compact Kähler manifold M. The relation between the existence of Hermitian Einstein metrics and stable holomorphic vector bundles over closed Kähler manifolds is by now well understood, due to the work of Narasimhan–Seshadri [18], Donaldson [4], Siu [21], Uhlenbeck–Yau [22] and others. The Higgs bundle, which is introduced by Hitchin in [10] on a Riemann surface, is a holomorphic bundle E together with a given holomorphic linear map

$$\theta\colon \Omega^0(M,E) \to \Omega^{1,0}(M,E)$$

which satisfies $\theta\wedge\theta = 0$. Simpson [20] generalized the above results about Hermitian Einstein metric to Higgs bundles. Different to the Higgs bundles, Bradlow [1, 2] considered holomorphic vector bundles on which additional data in the form of a prescribed holomorphic global section is given. Bradlow investigated the following vortex equation

$$\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \lambda\frac{\sqrt{-1}}{2}\,\mathrm{Id} = 0, \tag{1.1}$$

Here F_H is the curvature of the metric connection determined by $\bar\partial_E$ and a Hermitian metric H, ϕ is a holomorphic section of E, ϕ^{*H} is the adjoint of ϕ with respect to metric H. This vortex equation looks like the Hermitian Einstein (or Hermitian Yang–Mills) equation with an extra zeroth order term, and we will call a Hermitian metric satisfying (1.1) a Hermitian Yang–Mills–Higgs metric. In [2], Bradlow proved the equivalence between the existence of Hermitian Yang–Mills–Higgs metric and the ϕ-stability of holomorphic bundle by minimizing the so called Donaldson's

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functional. More recently, Hong [11] established the global existence of smooth solutions to heat flow for the Hermitian Yang–Mills–Higgs metric on E, and proved the existence of the Hermitian Yang–Mills–Higgs metric on the ϕ-stable holomorphic bundle by studying the limiting behaviour of the gauge flow.

It is natural to hope that geometric results dealing with closed manifolds will extend to yield interesting information for manifolds with boundary. In [5], Donaldson solved the Dirichlet boundary value problem for Hermitian Einstein metrics on Kähler manifolds. In the first part of this paper, we want to consider the Dirichlet boundary value problem for Hermitian Yang–Mills–Higgs metric (or the above vortex equation (1.1)). We obtain the following theorem.

Theorem 1.1 *Let E be a holomorphic vector bundle over the compact Kähler manifold $\overline{M}$ with non-empty boundary ∂M, and ϕ be a holomorphic section of E. For any Hermitian metric f on the restriction of E to ∂M there is a unique Hermitian Yang–Mills–Higgs metric H on E such that $H = f$ over ∂M.*

We will use the heat equation method to prove Theorem 1.1, and adapt the techniques which already appear in the literature on the Hermitian Yang–Mills flow [5, 20, 21]. Let H_0 be a Hermitian metric on E, and satisfying $H_0 = f$ over ∂M. Consider a family of Hermitian metric $H(t)$ on E with initial metric $H(0) = H_0$. Denote by $A_{H(t)}$ and $F_{H(t)}$ the corresponding connections and curvature forms. When there is no confusion, we will omit the parameter t and simply write H, A_H, F_H for $H(t), A_{H(t)}, F_{H(t)}$ respectively. The heat equation of (1.1) is following:

$$H^{-1}\frac{\partial H}{\partial t} = -2\sqrt{-1}\big(\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}}{2}\lambda \,\mathrm{Id}\big). \tag{1.2}$$

One can easily see that, written in local coordinates, this is a parabolic semilinear system. In fact, we proved that the heat equation (1.2) for the Dirichlet problem has a long time solution $H(t)$ for any initial metric H_0 such that $H_0|_{\partial M} = f$. We obtain the following theorem.

Theorem 1.2 *Let E be a holomorphic vector bundle over the compact Kähler manifold $\overline{M}$ with non-empty boundary ∂M, and ϕ be a holomorphic section of E. For any Hermitian metric f on the restriction of E to ∂M and any initial metric H_0 satisfying $H_0|_{\partial M} = f$ there exists a unique solution metric $H(\,\cdot\,,t)$ on E such that*

$$\begin{gathered} H^{-1}\frac{\partial H}{\partial t} = -2\sqrt{-1}\big(\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \lambda\frac{\sqrt{-1}}{2}\,\mathrm{Id}\big), \\ H(x,0) = H_0, \quad H(x,t)|_{\partial M} = f(x). \end{gathered} \tag{1.3}$$

In the second part of this paper, we study the existence of Hermitian Yang–Mills–Higgs metrics for a holomorphic vector bundle over a class of complete noncompact Kähler manifolds. We would like to point out that Ni and Ren [16, 17] had discussed the existence of a Hermitian Einstein metric on a complete Kähler manifold, and we would adapt the techniques used by them. In section 4, we prove a long-time existence of the Hermitian Yang–Mills–Higgs heat equation on any complete Kähler manifold, under some assumptions on the initial metric and the holomorphic section ϕ. We obtain:

Theorem 1.3 *Let M be a complete noncompact Kähler manifold without boundary, let E be a holomorphic vector bundle over M with Hermitian metric H_0, and ϕ be a holomorphic section of E. Suppose that there exists a positive number Θ such that $|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \,\mathrm{Id}\,| \leq \Theta$ everywhere, where λ is a real number. Then the Hermitian Yang–Mills–Higgs flow*

$$\begin{aligned}H^{-1}\frac{\partial H}{\partial t} &= -2\sqrt{-1}\big(\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \lambda\frac{\sqrt{-1}}{2}\,\mathrm{Id}\big),\\ H(x,0) &= H_0,\end{aligned} \tag{1.4}$$

has a long-time solution on $M \times [0,\infty)$.

In section 5, by studying the limiting behaviour of the solution of the above heat equation, we prove the existence of Hermitian Yang–Mills–Higgs metric over a class of complete Kähler manifolds under some assumptions. The method we use is similar to that used by Li [13] in the heat flow of the harmonic map.

2 Preliminary Results

Let M be a compact Kähler manifold and E a rank r complex vector bundle over M. Denote by ω the Kähler form, and define the map

$$L\colon \Omega^{p,q}(M,E) \to \Omega^{p+1,q+1}(M,E)$$

by

$$L(\alpha) = \alpha \wedge \omega. \tag{2.1}$$

Here $\Omega^{p,q}(M,E)$ is the space of global sections of $\wedge^{p,q}(M,E)$, where $\wedge^{p,q}(M,E)$ is the sheaf of germs of smooth (p,q) forms on M with values in E. The L^2-adjoint of this map is denoted by

$$\Lambda = L^*\colon \Omega^{p,q}(M,E) \to \Omega^{p-1,q-1}(M,E). \tag{2.2}$$

If $\alpha \in \Omega^{1,1}(M,E)$, then

$$\Lambda\alpha = \langle \alpha, \omega\rangle. \tag{2.3}$$

where $\langle\,,\rangle$ denotes the point-wise inner product on $(1,1)$ forms induced by the Kähler metric on M. Let A be a connection on E; through the definition of Λ, we have the following Kähler identities:

$$\begin{aligned}\bar\partial_A^* &= \sqrt{-1}[\partial_A, \Lambda] = \sqrt{-1}(\partial_A\Lambda - \Lambda\partial_A),\\ \partial_A^* &= -\sqrt{-1}[\bar\partial_A, \Lambda] = -\sqrt{-1}(\bar\partial_A\Lambda - \Lambda\bar\partial_A),\end{aligned} \tag{2.4}$$

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on general (p,q) forms $\Omega^{p,q}(M,E)$. For the special cases of the above Kähler identities are

$$\bar\partial_A^* = -\sqrt{-1}\Lambda\partial_A; \quad \partial_A^* = \sqrt{-1}\Lambda\bar\partial_A, \tag{2.5}$$

on $\Omega^{1,0}(M,E)$, $\Omega^{0,1}(M,E)$. Let $\rho \in \Omega^0(M,\mathrm{End}(E))$, one can check the fact that the covariant Laplacian $\triangle_A = -\nabla_A^*\nabla$ can be written:

$$\begin{aligned}\triangle_A\rho &= -2\bar\partial_A^*\bar\partial_A\rho - \sqrt{-1}[\Lambda F_A,\rho],\\ \triangle_A\rho &= -2\partial_A^*\partial_A\rho + \sqrt{-1}[\Lambda F_A,\rho].\end{aligned} \tag{2.6}$$

Here F_A is the curvature form with respect to connection A.

Definition 2.1 We define the Yang–Mills–Higgs functional

$$\mathrm{YMH}_\lambda\colon \mathbf{A}(H)\times\Omega^0(M,E)\to R$$

by

$$\mathrm{YMH}_\lambda(A,\phi) = \|F_A\|_{L^2}^2 + \|d_A\phi\|_{L^2}^2 + \frac{1}{4}\|\phi\otimes\phi^* - \lambda\,\mathrm{Id}\,\|_{L^2}^2. \tag{2.7}$$

Here, using the metric on E, we get identifications $E\approx E^*$ and also $E\otimes E^*\approx \mathrm{End}(E)$, $\mathbf{A}(H)$ denotes connections on E that are compatible with H, and ϕ^* is the adjoint of ϕ taken with respect to H and λ is a real parameter.

In the case of closed Kähler manifold M, Bradlow proved the following proposition:

Proposition 2.2 *The functional* $\mathrm{YMH}_\lambda\colon \mathbf{A}(H)\times\Omega^0(M,E)\to R$ *can be written as*

$$\begin{aligned}\mathrm{YMH}_\lambda(A,\phi) = {}& 4\|F_A^{0,2}\|_{L^2}^2 + 2\|\bar\partial_A\phi\|_{L^2}^2 + \|\sqrt{-1}\Lambda F_A + \frac{1}{2}\phi\otimes\phi^* - \frac{\lambda}{2}\,\mathrm{Id}\,\|_{L^2}^2\\ &+ \lambda\int_M \sqrt{-1}\,\mathrm{Tr}(F_A)\wedge\omega^{[n-1]} + \int_M \mathrm{Tr}(F_A\wedge F_A)\wedge\omega^{[n-2]}.\end{aligned} \tag{2.8}$$

Here $\omega^{[m]} = \frac{\omega^m}{(m)!}$ *and* $F_A^{0,2}$ *is the component of* F_A *of type* $(0,2)$.

An immediate corollary is that the functional YMH_λ is bounded below by

$$2\pi\lambda C_1(E,\omega) - 8\pi^2 Ch_2(E,\omega),$$

and this lower bound is attained at $(A,\phi)\in\mathbf{A}(H)\times\Omega^0(M,E)$ if and only if

$$F_A^{0,2} = 0, \tag{2.9}$$

$$\bar\partial_A\phi = 0, \tag{2.10}$$

$$\sqrt{-1}\Lambda F_A + \frac{1}{2}\phi\otimes\phi^* = \frac{\lambda}{2}\,\mathrm{Id}\,. \tag{2.11}$$

The third equation generalizes the Hermitian Yang–Mills equation (which is recovered by taking $\phi = 0$) and is the analog of the classical vortex equation over R^2. For this reasons we call the equation (2.11) the Hermitian Yang–Mills–Higgs or the vortex equation.

Let H be a Hermitian metric on holomorphic vector bundle E, and denote the holomorphic structure by $\bar{\partial}_E$. Then there exists one and only one complex metric connection which is denoted by A_H. By taking a local holomorphic basis e_α $(1 \leq \alpha \leq r)$, the Hermitian metric H is a positive Hermitian matrix $(H_{\alpha\bar{\beta}})_{1\leq\alpha,\beta\leq r}$ which also will be denoted by H for simplicity. In fact, the complex metric connection can be written as

$$A_H = H^{-1}\partial H, \tag{2.12}$$

and the curvature form as

$$F_H = \bar{\partial} A_H = \bar{\partial}(H^{-1}\partial H). \tag{2.13}$$

In the literature sometimes the connection is written as $(\partial H)H^{-1}$ because of the reversal of the roles of the row and column indices.

Definition 2.3 If the Hermitian metric H satisfying the vortex equation

$$\sqrt{-1}\Lambda F_H + \frac{1}{2}\phi \otimes \phi^{*H} = \frac{\lambda}{2}\,\mathrm{Id}, \tag{2.14}$$

then we will call it Hermitian Yang–Mills–Higgs metric. The notation ϕ^{*H} emphasizes that the adjoint is taken with respect to the metric H.

Bradlow [1] showed that the problem of minimizing the functional $\mathrm{YMH}_\lambda(A,\phi)$ defined on (E,H), a complex bundle with fixed metric, into the problem of finding a special metric on $(E,\bar{\partial}_E,\phi)$, *i.e.,* on a holomorphic bundle with a prescribed holomorphic section. In fact these two problems are equivalent.

It is well known that any two Hermitian metrics H and K are related by $H = Kh$, where $h = K^{-1}H \in \Omega^0(M, \mathrm{End}(E))$ is positive and self adjoint with respect to K. It is easy to check that

$$A_H - A_K = h^{-1}\partial_K h, \tag{2.15}$$

$$F_H - F_K = \bar{\partial}(h^{-1}\partial_K h), \tag{2.16}$$

$$\triangle \lg \mathrm{Tr}\, h \geq -(|\Lambda F_K| + |\Lambda F_H|). \tag{2.17}$$

Let H_0 be a Hermitian metric on E. Consider a family of Hermitian metrics $H(t)$ on E with initial metric $H(0) = H_0$. Denote by $A_{H(t)}$ and $F_{H(t)}$ the corresponding connections and curvature forms, and denote $h(t) = H_0^{-1}H(t)$. When there is no confusion, we will omit the parameter t and simply write H, A_H, F_H, h for $H(t), A_{H(t)}, F_{H(t)}, h(t)$ respectively. The heat equation of (2.14) is

$$H^{-1}\frac{\partial H}{\partial t} = -2\sqrt{-1}\big(\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}}{2}\lambda\,\mathrm{Id}\big). \tag{2.18}$$

It is completely equivalent to the following evolution equation:

$$\frac{\partial h}{\partial t} = \triangle_0 h + 2\sqrt{-1}\Lambda(\bar{\partial}_E h h^{-1}\partial_0 h) - \sqrt{-1}(\Lambda F_0 h + h\Lambda F_0) + \lambda h - h\phi\otimes\phi^{*H_0} h, \tag{2.19}$$

where $\triangle_0 = \triangle_{H_0}, \partial_0 = \partial_{H_0}$. We know that the above equation is a nonlinear parabolic equation, as in [4],and $h(t)$ are self adjoint with respect to H_0 for $t > 0$ since $h(0) = \mathrm{Id}$. Proceeding as in [4], we have

Proposition 2.4 *Let $H(t)$ be a solution of Hermitian Yang–Mills–Higgs flow* (2.18), *ϕ be a holomorphic section of E. Then*

$$\Big(\frac{\partial}{\partial t} - \triangle\Big) |\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}}{2}\lambda\,\mathrm{Id}\,|_H^2 \leq 0. \tag{2.20}$$

Proof By calculating directly, we have

$$\begin{aligned}
\frac{\partial}{\partial t}(\Lambda F_H) &= \frac{\partial}{\partial t}(\Lambda\bar{\partial}_E(h^{-1}\partial_0 h)) \\
&= \Lambda\bar{\partial}_E(\partial_H(h^{-1}\frac{\partial h}{\partial t})) \\
&= -2\sqrt{-1}\Lambda\bar{\partial}_E(\partial_H(\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id})) \\
&= \triangle_H(\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id}) \\
&\quad - \big[\sqrt{-1}\Lambda F_H, \Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id}\big],
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
&\triangle|\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id}\,|_H^2 \\
&= -\nabla^*\nabla|\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id}\,|_H^2 \\
&= 2\,\mathrm{Re}\Big\langle -\nabla_H^*\nabla_H(\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id})\,, \\
&\qquad \Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id}\Big\rangle_H \\
&\quad + |\nabla_H(\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id})|_H^2.
\end{aligned} \tag{2.22}$$

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It is easy to check the following formulas,

$$\mathrm{Re}\Big\langle \Lambda F_H\phi\otimes\phi^{*H}+\phi\otimes\phi^{*H}\Lambda F_H,\Lambda F_H\Big\rangle_H=2|\Lambda F_H\phi|_H^2,$$

$$\mathrm{Re}\Big\langle \Lambda F_H\phi\otimes\phi^{*H}+\phi\otimes\phi^{*H}\Lambda F_H,-\frac{\sqrt{-1}}{2}(\phi\otimes\phi^{*H}-\lambda\,\mathrm{Id})\Big\rangle_H$$
$$=(|\phi|^2-\lambda)\,\mathrm{Re}\langle-\sqrt{-1}\phi,\Lambda F_H\phi\rangle_H,$$

$$\mathrm{Re}\Big\langle\sqrt{-1}\phi\otimes\phi^{*H}(\lambda Id-\phi\otimes\phi^{*H}),\Lambda F_H\Big\rangle_H=(|\phi|^2-\lambda)\,\mathrm{Re}\langle-\sqrt{-1}\phi,\Lambda F_H\phi\rangle_H,$$

$$\mathrm{Re}\Big\langle\sqrt{-1}\phi\otimes\phi^{*H}(\lambda\,\mathrm{Id}-\phi\otimes\phi^{*H}),\frac{\sqrt{-1}}{2}(\lambda\,\mathrm{Id}-\phi\otimes\phi^{*H})\Big\rangle_H=\frac{1}{2}|\phi|_H^2(\lambda-|\phi|_H^2)^2.$$

Using the above formulas, we have

$$\begin{aligned}
&(\triangle-\frac{\partial}{\partial t})\Big|\Lambda F_H-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id}\Big|_H^2\\
&=2\Big|\nabla_H(\Lambda F_H-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id})\Big|_H^2\\
&\quad+2\,\mathrm{Re}\Big\langle\Big[\sqrt{-1}\Lambda F_H,\Lambda F_H-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id}\Big],\\
&\qquad\quad\Lambda F_H-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id}\Big\rangle_H\\
&\quad+2\,\mathrm{Re}\Big\langle\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}h^{-1}\frac{\partial h}{\partial t},\Lambda F_H-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id}\Big\rangle_H\\
&=2\Big|\nabla_H(\Lambda F_H-\frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H}+\frac{\sqrt{-1}\lambda}{2}\,\mathrm{Id})\Big|_H^2\\
&\quad+\frac{1}{2}\Big|-\sqrt{-1}(|\phi|^2-\lambda)\phi+2\Lambda F_H\phi\Big|_H^2\\
&\geq 0.
\end{aligned}$$

■

Proposition 2.5 *Let $H(t)$ be a solution of Hermitian Yang–Mills–Higgs flow* (2.18) *on $M\times[0,T)$, and let ϕ be a holomorphic section of E. Then*

$$\Big(\frac{\partial}{\partial t}-\triangle\Big)|\phi|_H^2=(\lambda-|\phi|_H^2)|\phi|_H^2-2|\nabla_H\phi|_H^2. \tag{2.23}$$

Proof By calculating directly, we have

$$\begin{aligned}
\frac{\partial}{\partial t}|\phi|_H^2&=\frac{\partial}{\partial t}\langle\phi\otimes\phi^{*H},\mathrm{Id}\rangle_H\\
&=\langle\phi\otimes\phi^{*H}h^{-1}\frac{\partial h}{\partial t},\mathrm{Id}\rangle_H\\
&\qquad+\Big\langle\phi\otimes\phi^{*H}(2\sqrt{-1}\Lambda F_H),\mathrm{Id}\Big\rangle_H-\Big\langle(2\sqrt{-1}\Lambda F_H)\phi\otimes\phi^{*H},\mathrm{Id}\Big\rangle_H\\
&=\Big\langle\phi\otimes\phi^{*H}(-\phi\otimes\phi^{*H}+\lambda Id)-2\sqrt{-1}\Lambda F_H\phi\otimes\phi^{*H},\mathrm{Id}\Big\rangle_H
\end{aligned}$$

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and

$$\begin{aligned}\triangle|\phi|_H^2 &= 2|\nabla_H\phi|_H^2 - 2Re\langle\nabla_H^*\nabla_H\phi,\phi\rangle_H\\ &= 2|\nabla_H\phi|_H^2 - 2\langle\sqrt{-1}\Lambda F_H\phi,\phi\rangle_H,\end{aligned}$$

where we have used $\partial_E\phi = 0$. From the above equalities we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \triangle\Big)|\phi|_H^2 &= -2|\nabla_H\phi|^2 + \big\langle \phi\otimes\phi^{*H}(-\phi\otimes\phi^{*H} + \lambda\,\mathrm{Id}), \mathrm{Id}\big\rangle_H\\ &= -2|\nabla_H\phi|^2 + (\lambda - |\phi|_H^2)|\phi|_H^2.\end{aligned}$$

■

Next, we will introduce Donaldson's "distance" on the space of Hermitian metrics as follows.

Definition 2.6 For any two Hermitian metrics H, K on bundle E set

$$\sigma(H,K) = \mathrm{Tr}\, H^{-1}K + \mathrm{Tr}\, K^{-1}H - 2\,\mathrm{rank}\, E. \tag{2.24}$$

It is obvious that $\sigma(H,K) \geq 0$ with equality if and only if $H = K$. The function σ is not quite a metric, but it serves almost equally well in our problem. Moreover the function σ compares uniformly with $d(,)$, where d is the Riemannian distance function on the metric space, in that $f_1(d) \leq \sigma \leq f_2(d)$ for monotone functions f_1, f_2. In particular, a sequence of metrics H_i converges to H in the usual C^0 topology if and only if $\sup_M \sigma(H_i, H) \longrightarrow 0$.

Proposition 2.7 *Let H and K be two Hermitian Yang–Mills–Higgs metrics, then $\sigma(H,K)$ is sub-harmonic:*

$$\triangle\sigma \geq 0. \tag{2.25}$$

Proof Denote $h = K^{-1}H$, applying $-i\Lambda$ to (2.16) and also taking the trace in the bundle E, we have

$$\mathrm{Tr}(\sqrt{-1}h(\Lambda F_H - \Lambda F_K)) = -\frac{1}{2}\triangle\,\mathrm{Tr}\, h + \mathrm{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E h h^{-1}\partial_K h). \tag{2.26}$$

By formula (2.14), then

$$\triangle\,\mathrm{Tr}\, h = 2\,\mathrm{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E h h^{-1}\partial_K h) + \mathrm{Tr}(h\phi\otimes\phi^{*H} - h\phi\otimes\phi^{*K}).$$

Similarly, let $k = H^{-1}K$; we have

$$\triangle\,\mathrm{Tr}\, k = 2\,\mathrm{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E k k^{-1}\partial_H k) + \mathrm{Tr}(k\phi\otimes\phi^{*K} - k\phi\otimes\phi^{*H}).$$

Let $\{e_i\}$ be unitary basis with respect to metric K at the point under consideration, and suppose that $h(e_i) = \lambda_i e_i$.

(2.27)
$$\begin{aligned}
&\operatorname{Tr}\big(h\phi\otimes\phi^{*H} - h\phi\otimes\phi^{*K} + h^{-1}\phi\otimes\phi^{*K} - h^{-1}\phi\otimes\phi^{*H})\\
&\qquad = \operatorname{Tr}((h-h^{-1})\phi\otimes\phi^{*K}(h-\mathrm{Id})\big)\\
&\qquad = \sum_{i=1}^{r}\big\langle (h-h^{-1})\phi\otimes\phi^{*K}(h-\mathrm{Id})(e_i), e_i\big\rangle\\
&\qquad = \sum_{i=1}^{r}(\lambda_i - 1)(\lambda_i - \lambda_i^{-1})|\langle\phi, e_i\rangle_K|^2\\
&\qquad = \sum_{i=1}^{r}(\lambda_i - 1)^2(\lambda_i + 1)(\lambda_i)^{-1}|\langle\phi, e_i\rangle_K|^2\\
&\qquad \geq 0.
\end{aligned}$$

Using the above formula and the facts[4, 21]

$$\operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E h h^{-1}\partial_K h) \geq 0, \operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E k k^{-1}\partial_H k) \geq 0,$$

$h^{-1} = k$, we have

$$\begin{aligned}
\triangle\sigma(H,K) &= \triangle(\operatorname{Tr} h + \operatorname{Tr} h^{-1} - 2r)\\
&= 2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E h h^{-1}\partial_K h) + 2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E k k^{-1}\partial_H k)\\
&\qquad + \operatorname{Tr}\big(h\phi\otimes\phi^{*H} - h\phi\otimes\phi^{*K} + h^{-1}\phi\otimes\phi^{*K} - h^{-1}\phi\otimes\phi^{*H}\big)\\
&\geq 0.
\end{aligned}$$

■

Let $H(t)$, $K(t)$ be two solutions of Hermitian Yang–Mills–Higgs flow (2.18), and denote $h(t) = K(t)^{-1}H(t)$. Applying $-i\Lambda$ to (2.16) and taking the trace in the bundle E, we have

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)\operatorname{Tr} h(t) = 2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E h h^{-1}\partial_K h) + \operatorname{Tr}(h\phi\otimes\phi^{*H} - h\phi\otimes\phi^{*K})$$

and

$$(\triangle - \frac{\partial}{\partial t})\operatorname{Tr} h^{-1}(t) = 2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E h^{-1} h\partial_K h^{-1}) + \operatorname{Tr}(h^{-1}\phi\otimes\phi^{*K} - h^{-1}\phi\otimes\phi^{*H}).$$

Using (2.27) again, then

$$\begin{aligned}
&\Big(\triangle - \frac{\partial}{\partial t}\Big)(\operatorname{Tr} h(t) + \operatorname{Tr} h^{-1}(t))\\
&\qquad = 2Tr(-\sqrt{-1}\Lambda\bar{\partial}_E h h^{-1}\partial_K h) + 2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_E h^{-1} h\partial_H h^{-1})\\
&\qquad\qquad + \operatorname{Tr}\big(h\phi\otimes\phi^{*H} - h\phi\otimes\phi^{*K} + h^{-1}\phi\otimes\phi^{*K} - h^{-1}\phi\otimes\phi^{*H}\big)\\
&\qquad \geq 0.
\end{aligned}$$

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So we have proved the following proposition.

Proposition 2.8 *Let $H(t)$, $K(t)$ be two solutions of Hermitian Yang–Mills–Higgs flow* (2.18). *Then*

$$\Big(\triangle-\frac{\partial}{\partial t}\Big)\sigma(H(t),K(t))\geq 0. \tag{2.28}$$

Proposition 2.9 *Let $H(x,t)$ be a solution of Hermitian Yang–Mills–Higgs flow* (2.18) *with the initial metric H_0. Then*

$$\begin{aligned}\Big(\triangle-\frac{\partial}{\partial t}\Big)\lg\{\operatorname{Tr}(H_0^{-1}H)+\operatorname{Tr}(H^{-1}H_0)\}\\ \geq -\big|2\sqrt{-1}\Lambda F_{H_0}+\phi\otimes\phi^{*H_0}-\lambda\operatorname{Id}\big|_{H_0}.\end{aligned} \tag{2.29}$$

Proof Let $h=H_0^{-1}H$. Applying (2.18) and (2.26), we have

$$\begin{aligned}\Big(\triangle-\frac{\partial}{\partial t}\Big)\operatorname{Tr}h=\operatorname{Tr}\big(2\sqrt{-1}h\Lambda F_{H_0}+h\phi\otimes\phi^{*H}-\lambda h\big)\\ +2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_E hh^{-1}\partial_0 h).\end{aligned} \tag{2.30}$$

$$\begin{aligned}\Big(\triangle-\frac{\partial}{\partial t}\Big)\operatorname{Tr}h^{-1}=-\operatorname{Tr}\big(2\sqrt{-1}h^{-1}\Lambda F_{H_0}+\phi\otimes\phi^{*H}h^{-1}-\lambda h^{-1}\big)\\ +2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_E h^{-1}h\partial_H h^{-1}).\end{aligned} \tag{2.31}$$

Direct calculation shows that [21]

$$\begin{aligned}2(\operatorname{Tr}h)^{-1}\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_E hh^{-1}\partial_0 h)-(\operatorname{Tr}h)^{-2}|\nabla\operatorname{Tr}h|^2\geq 0,\\ 2(\operatorname{Tr}h^{-1})^{-1}\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_E h^{-1}h\partial_H h^{-1})-(\operatorname{Tr}h^{-1})^{-2}|\nabla\operatorname{Tr}h^{-1}|^2\geq 0.\end{aligned} \tag{2.32}$$

From the above two inequalities, it is easy to check

$$\begin{aligned}(\operatorname{Tr}h+\operatorname{Tr}h^{-1})^{-1}\big\{-2\sqrt{-1}\Lambda\bar\partial_E hh^{-1}\partial_0 h-2\sqrt{-1}\Lambda\bar\partial_E h^{-1}h\partial_H h^{-1}\big\}\\ \geq(\operatorname{Tr}h+\operatorname{Tr}h^{-1})^{-2}|\nabla\operatorname{Tr}h+\nabla\operatorname{Tr}h^{-1}|^2.\end{aligned} \tag{2.33}$$

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Then, we have

$$
\begin{aligned}
(\triangle - \frac{\partial}{\partial t}) \lg\{\operatorname{Tr} h + \operatorname{Tr} h^{-1}\}& \\
&= (\operatorname{Tr} h + \operatorname{Tr} h^{-1})^{-1}(\triangle - \frac{\partial}{\partial t})\{\operatorname{Tr} h + \operatorname{Tr} h^{-1}\} \\
&\quad - (\operatorname{Tr} h + \operatorname{Tr} h^{-1})^{-2}|\nabla \operatorname{Tr} h + \nabla \operatorname{Tr} h^{-1}|^2 \\
&= (\operatorname{Tr} h + \operatorname{Tr} h^{-1})^{-1} \operatorname{Tr}(2\sqrt{-1}h\Lambda F_{H_0} + h\phi \otimes \phi^{*H_0} - \lambda h) \\
&\quad - (\operatorname{Tr} h + \operatorname{Tr} h^{-1})^{-1} \\
&\qquad \times \operatorname{Tr}(2\sqrt{-1}h^{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} h^{-1} - \lambda h^{-1}) \\
&\quad + (\operatorname{Tr} h + \operatorname{Tr} h^{-1})^{-1}\big(\operatorname{Tr}(h - h^{-1})\phi \otimes \phi^{*H_0}(h - \mathrm{Id})\big) \\
&\quad + (\operatorname{Tr} h + \operatorname{Tr} h^{-1})^{-1} \\
&\qquad \times \left\{ -2\sqrt{-1}\Lambda\bar{\partial}_E h h^{-1}\partial_0 h - 2\sqrt{-1}\Lambda\bar{\partial}_E h^{-1} h\partial_H h^{-1}\right\} \\
&\quad - (\operatorname{Tr} h + \operatorname{Tr} h^{-1})^{-2}|\nabla \operatorname{Tr} h + \nabla \operatorname{Tr} h^{-1}|^2 \\
&\geq -|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \,\mathrm{Id}\,|_{H_0}.
\end{aligned}
$$

where we have used formula (2.27) and (2.33). ■

Using (2.26), (2.32), (2.33), and proceeding as in the above proposition, we have:

Proposition 2.10 *If $H(x)$ and $H_0(x)$ are two Hermitian metrics, then*

$$
\begin{aligned}
(2.34)\quad \triangle \lg\{\operatorname{Tr} H_0^{-1}H + \operatorname{Tr} H^{-1}H_0\} \geq -|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \,\mathrm{Id}\,|_{H_0} \\
- |2\sqrt{-1}\Lambda F_H + \phi \otimes \phi^{*H} - \lambda \,\mathrm{Id}\,|_H.
\end{aligned}
$$

Corollary 2.11 *Let H be an Hermitian Yang–Mills–Higgs metric, and H_0 a Hermitian metric. Then*

$$
(2.35)\quad \triangle \lg\{\operatorname{Tr}(H_0^{-1}H) + Tr(H^{-1}H_0)\} \geq -\big|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \,\mathrm{Id}\big|_{H_0}.
$$

3 The Dirichlet Boundary Problem for HYMH Equations

In this section we will consider the case when M is the interior of compact Kähler manifold $\overline{M}$ with non-empty boundary ∂M, and the Kähler metric is smooth and non-degenerate on the boundary. The holomorphic vector bundle E is defined over $\overline{M}$. Let ϕ be a holomorphic section of E. We will discuss the Dirichlet boundary problem for the Hermitian Yang–Mills–Higgs metric by using the heat equation method to deform an arbitrary initial metric to the desired solution. The main points in the discussion are similar to that in [5, 20]. For given data f on ∂M we consider the evolution equation:

$$
\begin{aligned}
(3.1)\qquad & H^{-1}\frac{\partial H}{\partial t} = -2\sqrt{-1}(\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\sqrt{-1}}{2}\lambda\,\mathrm{Id}),\\
& H(t)|_{t=0} = H_0, \quad H|_{\partial M} = f.
\end{aligned}
$$

Here H_0 is an arbitrary smooth initial Hermitian metric satisfying the boundary condition. Denote $h(t) = H_0^{-1}H(t)$. Then the evolution equation (3.1) is completely equivalent to the following equation:

$$
\begin{aligned}
(3.2)\qquad \frac{\partial h}{\partial t} &= \triangle_0 h + 2\sqrt{-1}\Lambda(\bar{\partial}_E h h^{-1}\partial_0 h) - \sqrt{-1}(\Lambda F_0 h + h\Lambda F_0) + \lambda h\\
&\quad - h\phi\otimes\phi^{*H_0}h,\\
& h(0) = \mathrm{Id}, \quad h|_{\partial M} = \mathrm{Id}
\end{aligned}
$$

where $\triangle_0 = \triangle_{H_0}$, $\partial_0 = \partial_{H_0}$. We know that the above equation is a parabolic equation, so standard theory gives short-time existence.

Proposition 3.1 *For sufficiently small $\epsilon > 0$, the equation* (3.2)*, and so also equation* (3.1)*, have a smooth solution defined for $0 \le t < \epsilon$.*

The main point of the proof is to show that the solution of equation (3.1) persists for all time and converges to a limit. First we want to prove the long-time existence of the evolution equation. Let $H(t)$ be a solution of the evolution equation (3.1), and $h(t) = H_0^{-1}H(t)$, then

$$
\begin{aligned}
(3.3)\qquad \frac{\partial}{\partial t}(\lg \mathrm{Tr}\, h) &= \frac{\mathrm{Tr}\left(\frac{\partial h}{\partial t}\right)}{\mathrm{Tr}\, h}\\
&= \frac{\mathrm{Tr}\, h(-2\sqrt{-1}\Lambda F_H - \phi\otimes\phi^{*H} + \lambda\,\mathrm{Id})}{\mathrm{Tr}\, h}\\
&\le 2|\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\lambda\sqrt{-1}}{2}\,\mathrm{Id}\,|_H,
\end{aligned}
$$

and similarly

$$
(3.4)\qquad \frac{\partial}{\partial t}(\lg \mathrm{Tr}\, h^{-1}) \le 2|\Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H} + \frac{\lambda\sqrt{-1}}{2}\,\mathrm{Id}\,|_H.
$$

Theorem 3.2 *Suppose that a smooth solution H_t to the evolution equation* (3.1) *is defined for $0 \le t < T$. Then H_t converges in C^0 to some continuous non-degenerate metric H_T as $t \to T$.*

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Proof It is well known that the space of metrics on the given bundle E is complete. Given $\epsilon > 0$, by continuity at $t = 0$ we can find a δ such that

$$\sup_M \sigma(H_t, H_{t'}) < \epsilon,$$

for $0 < t, t' < \delta$. Then Proposition 2.8 and the maximum principle imply that

$$\sup_M \sigma(H_t, H_{t'}) < \epsilon,$$

for all $t, t' > T - \delta$. This implies that H_t are a uniformly Cauchy sequence and converge to a continuous limiting metric H_T. On the other hand, by Proposition 2.4, we know that $|\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\lambda\sqrt{-1}}{2} Id|_H$ are bounded uniformly. Using formulas (3.3) and (3.4), one can conclude that $\sigma(H, H_0)$ are bounded uniformly, therefore $H(T)$ is a non-degenerate metric. ■

We take the following lemma from [4, Lemma 19]and [20, Lemma 6.4].

Lemma 3.3 *Let $H(t)$, $0 \le t < T$, be any one-parameter family of Hermitian metrics on a holomorphic bundle E over compact Kähler manifold and satisfying a Dirichlet boundary condition. If $H(t)$ converges in C^0 topology to some continuous metric H_T as $t \to T$, and if $\sup_M |\Lambda F_H|$ is bounded uniformly in t, then $H(t)$ are bounded in C^1 and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t.*

Theorem 3.4 *Given data f on the boundary ∂M and initial Hermitian metric H_0, then the evolution equation (3.1) has a unique solution $H(t)$ which exists for $0 \le t < \infty$.*

Proof Proposition 3.1 guarantees that a solution exists for a short time. Suppose that the solution $H(t)$ exists for $0 \le t < T$. By Lemma 3.3, $H(t)$ converges in C^0 to a non-degenerate continuous limit metric $H(T)$ as $t \to t$. From Proposition 2.4 and the maximum principle, we conclude that $|\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}\lambda}{2} \,\mathrm{Id}\,|_H$ is bounded independently of t. Moreover, from Proposition 2.5, we have

$$\Big(\frac{\partial}{\partial t} - \triangle\Big) |\phi|_H^2 \le (\lambda - |\phi|_H^2)|\phi|_H^2.$$

Assume that $|\phi|_H^2$ attains its maximum on $\overline{M} \times [0, \infty)$ at the point (x_0, t_0) with $0 < t_0 < T$, $x_0 \in M$. If $|\phi|_H^2(x_0, t_0) > |\lambda|$, then

$$\Big(\frac{\partial}{\partial t} - \triangle\Big) |\phi|_H^2(x_0, t_0) \le 0.$$

This is contradicted with the maximum principle of the heat operator. Then $|\phi|_H^2$ must attain its maximum point at $t = 0$ or on ∂M. So we have

$$|\phi|_H^2 \le \max\big\{ \sup_{\overline{M}} |\phi|_{H_0}^2, |\lambda| \big\}. \tag{3.5}$$

Moreover, $\sup_{\overline{M}} |\Lambda F_H|^2_{H_0}$ is bounded independently of t. Hence by Lemma 3.3, $H(t)$ are bounded in C^1 and also bounded in L^p_2 (for any $1 < p < \infty$) uniformly in t. Since the evolution (3.2) is quadratic in the first derivative of h, we can apply Hamilton's method [9] to deduce that $H(t) \to H(T)$ in C^∞, and the solution can be continued past T. Then the evolution equation (3.1) has a solution $H(t)$ defined for all time.

Next, we want to show the uniqueness of the solution. Suppose that $K(t)$ is another solution of equation (3.1). From Proposition 2.8, we have

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)\sigma(H(t), K(t)) \geq 0,$$

and $\sigma(H,K)|_{t=0} = 0$, $\sigma(H,K)|_{\partial M} = 0$. By the maximum principle, we have

$$\sigma(H(t), K(t)) \equiv 0, \ \textit{i.e.}, H(t) \equiv K(t). \qquad \blacksquare$$

From Theorem 3.4, we can discuss as in [5] to deduce the existence of Hermitian Yang–Mills–Higgs metric. First, we shall need the following lemma.

Lemma 3.5 ([5]) *Suppose $g \geq 0$ is a sub-solution of the heat equation on $\overline{M} \times [0, \infty)$,* i.e., $\frac{\partial g}{\partial t} - \triangle g \leq 0$. *If $g = 0$ on ∂M for all time, then g decays exponentially:*

$$\sup_M g(\,\cdot\,, t) \leq Ce^{-\mu t}, \tag{3.6}$$

where $\mu > 0$ depends only on M, and C depends on the initial value of g.

Let H_t be a solution of the evolution equation (3.1) for $0 \leq t < \infty$. We consider the function $g = \big|\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}}{2}\lambda\,\mathrm{Id}\big|^2_H$ on $\overline{M} \times [0, \infty)$. By Proposition 2.4, we know that $\frac{\partial g}{\partial t} - \triangle g \leq 0$, and the Dirichlet boundary condition satisfied by $H(t)$ implies that, for $t > 0$, g vanishes on the boundary of M. Thus Lemma 3.5 tells us that g decays exponentially, and in particular that

$$\int_0^\infty \sqrt{g}\, dt < \infty. \tag{3.7}$$

Directly calculated, we have

$$\begin{aligned} |\lg \operatorname{Tr} h(t)| - \lg \operatorname{rank} E &= \int_0^t \frac{\partial}{\partial s} |\lg \operatorname{Tr} h(s)|\, ds \\ &\leq \int_0^t \Big| \frac{\operatorname{Tr} h h^{-1} \frac{\partial h}{\partial t}}{\operatorname{Tr} h} \Big|\, ds \leq \int_0^t \sqrt{g}\, dt \end{aligned} \tag{3.8}$$

and similarly

$$|\lg \operatorname{Tr} h^{-1}(t)| \leq \lg \operatorname{rank} E + \int_0^t \sqrt{g}\, dt. \tag{3.9}$$

Using (3.8), (3.9), we know that $\sup_{\overline{M}} \sigma(H_0, H(t))$ is uniformly bounded for $t \in [0, \infty)$. Then there exists a subsequence of the $H(t)$ converging in C^0 to some continuous metric H_∞. It is then easy to show, as in Theorem 3.4, that a subsequence of the H_t converges in C^∞ to a smooth metric H_∞, and since $\left| \Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}\lambda}{2} \operatorname{Id} \right|_H^2$ tends to zero with $t \to \infty$, this limit is the desired Hermitian Yang–Mills–Higgs metric. By Proposition 2.7, it is easy to prove the uniqueness of Hermitian Yang–Mills–Higgs metric satisfying the same data f on boundary ∂M. So we have proved Theorem 1.1.

4 HYMH Flow over Complete Kähler Manifolds

Let M be a complete, noncompact Kähler manifold without boundary, in this case, we will simply say M is a complete Kähler manifold. Let E be a holomorphic vector bundle of rank r over M with metric H_0, and ϕ be a holomorphic section of E. In this section we are going to prove a long-time existence for the Hermitian Yang–Mills–Higgs flow under some conditions on initial metric H_0 and section ϕ. As in [16], we use the compact exhaustion construction to prove the long-time existence.

Let $\{\Omega_i\}_{i=1}^\infty$ be an exhausting sequence of compact sub-domains of M, *i.e.,* they satisfy $\Omega_i \subset \Omega_{i+1}$ and $\bigcup_{i=1}^\infty \Omega_i = M$. By Theorem 3.4, we can find a Hermitian metric $H_i(x,t)$ on $E|_{\Omega_i}$ for each i such that

$$(4.1)\qquad \begin{gathered} H_i^{-1}\frac{\partial H_i}{\partial t} = -2\sqrt{-1}\Big(\Lambda F_{H_i} - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H_i} + \frac{\sqrt{-1}}{2}\lambda \operatorname{Id}\Big), \\ H_i(x,0) = H_0(x), \\ H_i(x,t)|_{\partial\Omega_i} = H_0(x), \\ \lim_{t\to\infty}\Big(\Lambda F_{H_i} - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*H_i} + \frac{\sqrt{-1}}{2}\lambda \operatorname{Id}\Big) = 0. \end{gathered}$$

Suppose that there exists a positive number Θ such that

$$\left| 2\sqrt{-1}\Lambda F_{H_0} + \phi\otimes\phi^{*H_0} - \lambda \operatorname{Id}\right|_{H_0} \le \Theta$$

on any points of M. Denote $h_i = H_0^{-1}H_i$. Direct calculation shows that

$$(4.2)\qquad \begin{aligned} \Big|\frac{\partial}{\partial t}\lg \operatorname{Tr} h_i\Big| &= \Big|\frac{\operatorname{Tr}(h_i h_{i^{-1}}\frac{\partial}{\partial t}h_i)}{\operatorname{Tr} h_i}\Big| \le \left|2\sqrt{-1}\Lambda F_{H_i} + \phi\otimes\phi^{*H_i} - \lambda\operatorname{Id}\right|_{H_i}, \\ \Big|\frac{\partial}{\partial t}\lg \operatorname{Tr} h_i^{-1}\Big| &= \Big|-\frac{\operatorname{Tr}(h_{i^{-1}}\frac{\partial}{\partial t}h_i h_i^{-1})}{\operatorname{Tr} h_i^{-1}}\Big| \le \left|2\sqrt{-1}\Lambda F_{H_i} + \phi\otimes\phi^{*H_i} - \lambda\operatorname{Id}\right|_{H_i}, \end{aligned}$$

By Proposition 2.4 and the Maximum principle, we have

$$(4.3)\qquad \sup_{\Omega_i\times[0,\infty)}\left|2\sqrt{-1}\Lambda F_{H_i} + \phi\otimes\phi^{*H_i} - \lambda\operatorname{Id}\right|_{H_i} \le \Theta.$$

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Integrating (4.2) along the time direction,

$$|\lg \operatorname{Tr} h_i(x,t) - \lg r| = \Big| \int_0^t \frac{\partial}{\partial s}(\lg \operatorname{Tr} h_i(x,s))\, ds \Big| \le \Theta t.$$

Then we have

$$\sup_{\Omega_i\times[0,T]} \operatorname{Tr} h_i \le r\exp(\Theta T), \qquad \inf_{\Omega_i\times[0,T]} \operatorname{Tr} h_i \ge r\exp(-\Theta T), \tag{4.4}$$

and

$$\sup_{\Omega_i\times[0,T]} \operatorname{Tr} h_i^{-1} \le r\exp(\Theta T), \qquad \inf_{\Omega_i\times[0,T]} \operatorname{Tr} h_i^{-1} \ge r\exp(-\Theta T), \tag{4.5}$$

This implies that

$$\sup_{\Omega_i\times[0,T]} \sigma(H_0, H_i) \le 2r\exp(\Theta T), \tag{4.6}$$

and

$$(r\exp(\Theta T))^{-1}\operatorname{Id} \le h_i(x,t) \le r\exp(\Theta T)\operatorname{Id} \tag{4.7}$$

for any $(x,t) \in \Omega_i \times [0,T]$. In particular, over any compact subset Ω, for i sufficiently large such that $\Omega \subset \Omega_i$, we have the C^0-estimate

$$\sup_{\Omega\times[0,T]} \sigma(H_0, H_i) \le 2r\exp(\Theta T). \tag{4.8}$$

Without loss of generality we can assume that $\Omega = B_o(R)$. Here $B_0(R)$ denotes the geodesic ball of radius R with center at fixed point $o \in M$. First, we want to show that there exists a subsequence of $\{H_i\}$ converging uniformly to an Hermitian metric H_∞ on $B_o(R) \times [0, \frac{T}{2}]$.

Denote $\tau_i = \operatorname{Tr}(h_i) + \operatorname{Tr}(h_i^{-1})$. Direct calculation as before shows that over $\Omega_i\times[0,T]$

$$\begin{aligned}
\triangle\tau_i &= \triangle \operatorname{Tr}(h_i) + \triangle \operatorname{Tr}(h_i^{-1}) \\
&= -\operatorname{Tr}(h_i(2\sqrt{-1}\Lambda F_{H_i} + \phi\otimes\phi^{*H_i} - \lambda \operatorname{Id})) \\
&\qquad + \operatorname{Tr}(h_i(2\sqrt{-1}\Lambda F_{H_0} + \phi\otimes\phi^{*H_0} - \lambda \operatorname{Id})) \\
&\qquad - \operatorname{Tr}(h_i^{-1}(2\sqrt{-1}\Lambda F_{H_0} + \phi\otimes\phi^{*H_0} - \lambda \operatorname{Id})) \\
&\qquad + Tr(h_i^{-1}(2\sqrt{-1}\Lambda F_{H_i} + \phi\otimes\phi^{*H_i} - \lambda \operatorname{Id})) \\
&\qquad - \operatorname{Tr}(2\sqrt{-1}\Lambda\bar\partial_E h_i h_i^{-1}\partial_{H_0} h_i) - \operatorname{Tr}(2\sqrt{-1}\Lambda\bar\partial_E h_i^{-1} h_i \partial_{H_i} h_i^{-1}) \\
&\qquad + \operatorname{Tr}((h_i - h_i^{-1})\phi\otimes\phi^{*H_0}(h_i - \operatorname{Id})) \\
&\ge -C_1 + C_2 e(h_i).
\end{aligned} \tag{4.9}$$

Here $e(h_i) = -\operatorname{Tr}(2\sqrt{-1}\Lambda\bar{\partial}_E h_i \partial_{H_0} h_i)$, C_1 and C_2 are positive constants depending only on Θ and T, and we have used formulas (2.26), (4.3)–(4.5) and (4.7). Choosing i sufficiently larger such that $B_o(4R) \subset \Omega_i$, let ψ be a cutoff function which equals 1 in $B_o(2R)$ and is supported in $B_o(4R)$. Now multiply the above inequality by $\tau_i \psi^2$ and integrate it over M. Then

$$\int_M \tau_i \psi^2 \triangle \tau_i \geq -C_1 \int_M \tau_i \psi^2 + C_2 \int_M \tau_i \psi^2 e(h_i).$$

Integrating by parts, and then integrating along the time direction, we have

$$C_2 \int_0^T \int_M \tau_i \psi^2 e(h_i) \leq C_1 \int_0^T \int_M \tau_i \psi^2 + \int_0^T \int_M |\nabla \psi|^2 \tau_i^2.$$

Using (4.4) and (4.5) again, we obtain the following estimate:

$$\int_0^T \int_{B_o(2R)} e(h_i) \leq C_3. \tag{4.10}$$

Here C_3 is a positive number depending only on Θ, T and R.

Because $e(h_i)$ contain all the squares of the first order derivatives (space direction) of h_i, h_i have uniform C^0 bound, and also $\frac{\partial}{\partial t} h_i$ are uniformly bounded. So, the above inequality implies that h_i are uniformly bounded in $L_1^2(B_o(2R) \times [0, T])$. Using the fact that $L_1^2(B_o(2R) \times [0, T])$ is compact in $L^2(B_o(2R) \times [0, T])$, by passing to a subsequence which we also denote by $\{H_i\}$, we have that the H_i converge in $L^2(B_o(2R) \times [0, T])$. Given any positive number ϵ, we have

$$\int_0^T \int_{B_o(2R)} \sigma^2(H_j, H_k) \leq \epsilon, \tag{4.11}$$

for j, k sufficiently large. By Proposition 2.8, we know that $\sigma(H_j, H_k)$ is a sub-solution of the heat equation. Applying the mean value inequality of Li and Tam [12] for the nonnegative sub-solution to the heat equation, we have

$$\sup_{B_o(R) \times [0, \frac{T}{2}]} \sigma^2(H_j, H_k) \leq C_4 \epsilon. \tag{4.12}$$

Here C_4 is a positive constant depending only on Θ, R, T, and the bound of sectional curvature on $B_o(2R)$. From (4.12), we can conclude that H_i converges uniformly to a continuous Hermitian metric H_∞ on $B_o(R) \times [0, \frac{T}{2}]$.

Next, we will proceed as in [5, §2.3] to obtain the C^1-estimate on $B_o(R) \times [0, \frac{T}{2}]$. We need the following lemmas. The following Sobolev inequality was proved by Saloff-Coste [19, Theorem 3.1].

Lemma 4.1 ([19]) *Let M^m be an m-dimensional complete noncompact Riemannian manifold, and $B_x(r)$ be a geodesic ball of radius r and centered at x. Suppose that*

$-K \leq 0$ is the lower bound of the Ricci curvature of $B_x(r)$. If $m > 2$, there exists C, depending only on m, such that

$$(4.13)\quad \Big(\int_{B_x(r)} |f|^{2q}\Big)^{\frac{1}{q}} \leq \exp\big(C(1+\sqrt{K}r)\big)\,\mathrm{Vol}(B_x(r))^{-\frac{2}{m}} r^2 \times \int_{B_x(r)} \big(|\nabla f|^2 + r^{-2}|f|^2\big),$$

for any $f \in C_0^\infty(B_x(r))$, where $q = \frac{m}{m-2}$. For $m \leq 2$, the above inequality holds with a replaced by any fixed $\mu > 2$.

Using the above Sobolev inequality, and Moser's iteration [12, 14], we have the following mean-value inequalities.

Lemma 4.2 *Let M^m be an m-dimensional complete noncompact Riemannian manifold without boundary, and $B_o(2R)$ a geodesic ball centered at $o \in M$ of radius $2R$. Suppose that $f(x,t)$ is a nonnegative function satisfying*

$$(4.14)\quad (\triangle - \frac{\partial}{\partial t}) f \geq -C_1 f$$

on $B_o(2R) \times [0,T]$. If $-K \leq 0$ is the lower bound of the Ricci curvature of $B_o(2R)$, then for $p > 0$, there exist positive constants C_2 and C_3 depending only on m, R, K, p, and T such that

$$(4.15)\quad \sup_{B_o(\frac{1}{4}R)\times[0,\frac{T}{4}]} f^p \leq C_2 \int_0^T \int_{B_o(R)} f^p(y,t)\,dydt + C_3 \sup_{B_o(R)} f^p(\,\cdot\,,0).$$

Lemma 4.3 *Let M^m be an m-dimensional complete noncompact Riemannian manifold without boundary, and $B_o(2R)$ a geodesic ball centered at $o \in M$ of radius $2R$. Suppose that $f(x,t)$ is a nonnegative function satisfying (4.14) on $B_o(2R) \times [0,T]$. If $-K \leq 0$ is the lower bound of the Ricci curvature of $B_o(2R)$, then for $p > 0$, there exists a positive constant C depending only on m, R, K, p, δ, η, and T such that*

$$(4.16)\quad \sup_{B_o((1-\delta)R)\times[\eta T,(1-\eta)T]} f^p \leq C \int_0^T \int_{B_o(R)} f^p(y,t)\,dydt,$$

where $0 < \delta, \eta < 1$.

Proof of Theorem 1.3 For any point $x \in B_o(2R)$, choose a small ball $B_x(r)$ such that the bundle E can be trivialized locally, and let $\{e_\alpha\}$ be the holomorphic frame of E. So, a metric H_i can be written as a matrix which also is denoted by H_i on $B_x(r)$. The complex metric connection with respect to H_i can be written as follows:

$$A_i = H_i^{-1} \partial H_i$$

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and the curvature form

$$F_{H_i} = \bar{\partial}(H_i^{-1}\partial H_i).$$

Choose a normal coordinate $\{y_l\}$ on $B_x(r)$ and centered at x, and denote $\rho_l = H_i^{-1} dH_i(\frac{\partial}{\partial y_l})$. It is easy to check that

$$\left(\triangle_{H_i} - \frac{\partial}{\partial t}\right)\rho_l = -\rho_l H_i^{-1}\frac{\partial H_i}{\partial t} + H_i^{-1}\frac{\partial H_i}{\partial t}\rho_l + \frac{\partial}{\partial t}(\phi\otimes\phi^{*H_i}) \tag{4.17}$$

on $B_x(\frac{r}{2})$. In fact, this follows from (4.1) by considering the one-parameter family of solutions obtained by translating in the direction of $\frac{\partial}{\partial y_l}$, $H_i^s(y_1,\dots,y_m) = H_i(y_1,\dots,y_l+s,\dots,y_m)$. It follows that the square norm $|\rho_l|^2_{H_i} = \operatorname{Tr}\rho_l H_i^{-1}\bar{\rho}_l{}^* H_i$ satisfies

$$\left(\triangle - \frac{\partial}{\partial t}\right)|\rho_l|^2_{H_i} \geq -C_4|\rho_l|^2_{H_i} - c_{11}|\rho_l|_{H_i} \tag{4.18}$$

on $B_x(\frac{r}{2})$, where C_4, C_5 are positive constants depending only on Θ, T, and ϕ. Let $f = 1 + |\rho_l|^2_{H_i}$, from (4.18), we have

$$\left(\triangle - \frac{\partial}{\partial t}\right) f \geq -C_6 f.$$

On the other hand, there must exist constants C_7 and C_8 such that

$$C_7\,\mathrm{Id} \leq \left\{ g\left(\frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_l}\right)\right\} \leq C_8\,\mathrm{Id}$$

on $B_x(\frac{r}{2})$, where g is the Kähler metric of M. So, we have

$$C_7\sum_l \left|H_i^{-1}\frac{\partial H_i}{\partial y_l}\right|_{H_i} \leq |H_i^{-1}\nabla H_i|^2_{H_i} \leq C_8\sum_l\left|H_i^{-1}\frac{\partial H_i}{\partial y_l}\right|_{H_i}.$$

Using formula (4.15) in Lemma 4.2, and (4.10), we conclude that there exists a positive constant C_9 which is independent of i such that

$$\sup_{B_o(r/4)\times[0,T/4]} |H_i^{-1}\nabla H_i|^2_{H_0} \leq C_9. \tag{4.19}$$

Since x is arbitrary, we can conclude that the C^1-norm of H_i is bounded uniformly on any $B_o(R)\times[0,\frac{T}{4}]$. By the C^0-estimate (4.6) and the above C^1-estimate, the standard parabolic theory shows that, by passing to a subsequence, H_i converges uniformly over any compact subset of $M\times[0.\infty)$ to a smooth H_∞ which is a solution of the Hermitian Yang–Mills–Higgs flow (1.4) on the whole manifold. Therefore we complete the proof of Theorem 1.3 ∎

5 HYMH Metric Over Complete Kähler Manifolds

In this section, we consider the existence of the Hermitian Yang–Mills–Higgs (HYMH) metrics on some complete Kähler manifolds. As above, here complete means complete, noncompact, without boundary. Since we have established the global existence of the HYMH flow on complete Kähler manifolds, in this section we aim to show that the HYMH flow can converge to a HYMH metric under some assumptions. First, we will give a proof of the following existence theorem. The method we used here is similar to that used by Li in the harmonic map case [13].

Theorem 5.1 *Let M be an m-dimensional complete Kähler manifold, (E, H_0) be a holomorphic vector bundle with Hermitian metric H_0, and ϕ a holomorphic section of E. Assume that $\lambda_1(M) > 0$, where $\lambda_1(M)$ denotes the lower bound of the spectrum of the Laplacian operator, and that $\|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \operatorname{Id}\|_{L^p(M)} < \infty$ for some $p > 1$ and real number λ. Then there exists a Hermitian metric H on E such that*

$$2\sqrt{-1}\Lambda F_H + \phi \otimes \phi^{*H} - \lambda \operatorname{Id} = 0.$$

In addition, if $p > \frac{m}{2}$, $\operatorname{Ric} M \geq -K$ $(K \geq 0)$, and $\inf_{x \in M} V_x(1) = a > 0$, where $V_x(1)$ denote the volume of the geodesic ball $B_x(1)$ centered at x of radius 1, then the Hermitian Yang–Mills–Higgs metric H given above must satisfy

$$\sigma(H(x), H_0(x)) \longrightarrow 0$$

as $x \longrightarrow \infty$.

Proof Let $\{\Omega_i\}_{i=1}^{\infty}$ be the exhausting sequence of compact sub-domain which we chose in the above section. Let $H_i(x,t)$ be a solution of (4.1), and let $H_i(x,t) = H_0(x)$ outside Ω_i for all $t > 0$. From the proof of Proposition 2.4, we have

$$\text{(5.1)} \quad \Big(\triangle - \frac{\partial}{\partial t}\Big)\big|2\sqrt{-1}\Lambda F_{H_i} + \phi \otimes \phi^{*H_i} - \lambda \operatorname{Id}\big|^2_{H_i} \geq 2\big|\nabla_{H_i}(2\sqrt{-1}\Lambda F_{H_i} + \phi \otimes \phi^{*H_i} - \lambda \operatorname{Id})\big|^2_{H_i}.$$

By direct calculation, one can check that

$$|\nabla_H \theta|^2_H \geq |\nabla|\theta|_H|^2$$

for any section θ in $\operatorname{End}(E)$. Then, we have

$$\text{(5.2)} \qquad \Big(\triangle - \frac{\partial}{\partial t}\Big)\big|2\sqrt{-1}\Lambda F_{H_i} + \phi \otimes \phi^{*H_i} - \lambda \operatorname{Id}\big|_{H_i} \geq 0.$$

Since $|H_i^{-1}\frac{\partial}{\partial t}H_i| = 0$ outside Ω_i, using the maximum principle, we have

$$\begin{aligned}\text{(5.3)} \quad & |2\sqrt{-1}\Lambda F_{H_i} + \phi \otimes \phi^{*H_i} - \lambda \operatorname{Id}|_{H_i}(x,t) \\ &\leq \int_{\Omega_i} K_{\Omega_i}(x,y,t)\big|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \operatorname{Id}\big|_{H_0}(y,t)\,dy \\ &\leq \int_M K(x,y,t)\big|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \operatorname{Id}\big|_{H_0}(y,t)\,dy,\end{aligned}$$

where $K_{\Omega_i}(x, y, t)$ is the Dirichlet heat kernel of Ω_i and $K(x, y, t)$ is the heat kernel of M. For simplicity, we denote $\tau(H) = |2\sqrt{-1}\Lambda F_H + \phi \otimes \phi^{*H} - \lambda \,\mathrm{Id}\,|_H$. Therefore

$$\int_M \tau(H_i)^p(x,t)\,dx \leq \int_M \tau(H_0)^p(x)\,dx. \tag{5.4}$$

for any $p > 1$. From (5.3), we have

$$\begin{aligned}\tau(H_i)(x,t) &\leq \int_M K(x,y,t)\tau(H_0)(y)\,dy \\ &\leq \Big(\int_M K(x,y,t)^q\,dy\Big)^{\frac{1}{q}}\Big(\int_M \tau(H_0)^p(y)\,dy\Big)^{\frac{1}{p}}.\end{aligned} \tag{5.5}$$

By the estimate of the heat kernel [7], from the condition $\lambda_1(M) > 0$, we have

$$\int_M K^q(x,y,t)\,dy \leq C^* \exp\Big(-\frac{4\lambda_1(M)(q-1)}{q}(t-1)\Big). \tag{5.6}$$

Proceeding as (4.2)–(4.4), we have

$$\sigma(H_i(x,t), H_0(x)) \leq 2\,\mathrm{rank}\,E\Big(\exp\Big(\int_0^t \tau(H_i)(x,t)\,dt\Big) - 1\Big). \tag{5.7}$$

So, we have obtained the uniformly C^0 estimates on H_i. From §4, we know that, by passing to a subsequence, H_i converges uniformly over any compact subset of $M \times [0.\infty)$ to a smooth $H(x,t)$ which is a solution of the Hermitian Yang–Mills–Higgs flow (1.4) on the whole manifold.

From (5.4)–(5.7), we have

$$\int_M \tau(H)^p(x,t)\,dx \leq \int_M \tau(H_0)^p(x)\,dx, \tag{5.4$'$}$$

$$\tau(H)(x,t) \leq \Big(\int_M K(x,y,t)^q\,dy\Big)^{\frac{1}{q}}\Big(\int_M \tau(H_0)^p(y)\,dy\Big)^{\frac{1}{p}}, \tag{5.5$'$}$$

and

$$\sigma(H(x,t_1), H(x,t_2)) \leq 2\,\mathrm{rank}\,E\Big(\exp\Big(\int_{t_1}^{t_2} \tau(H)(x,t)\,dt\Big) - 1\Big). \tag{5.8}$$

Combining the above inequality with (5.5$'$), (5.6), we have proved that the $H(x,t)$ converge uniformly to a continuous Hermitian metric H_∞.

From the C^0-estimate (5.8), proceeding as in the proof of Theorem 1.3, we can obtain a C^1-estimate of $H(\cdot,t)$. Then the standard parabolic theory shows that a $H(\cdot,t)$ converges uniformly over any compact subset of M to a smooth Hermitian metric H_∞ as $t \to \infty$. From (5.5$'$), and (5.6), we have proved that H_∞ must satisfy

$$2\sqrt{-1}\Lambda F_{H_\infty} + \phi \otimes \phi^{*H_\infty} - \lambda\,\mathrm{Id} = 0. \tag{5.9}$$

Let $h_i(x,t) = H_0^{-1}H_i$, and $\rho_i(x,t) = \lg(\operatorname{Tr} h_i + \operatorname{Tr} h_i^{-1}) - \lg 2 \operatorname{rank} E$. By (2.34), we have

$$\triangle \rho_i \geq -\tau(H_i) - \tau(H_0). \tag{5.10}$$

Then, the above inequality and the Poincaré inequality imply

$$\begin{aligned}\int_{\Omega_i} \rho_i^p(x,t)\,dx &\leq \frac{1}{\lambda_1(M)} \int_{\Omega_i} |\nabla \rho_i^{\frac{p}{2}}|^2\,dx \\ &\leq \frac{p^2}{4(p-1)\lambda_1(M)} \int_{\Omega_i} \rho_i^{p-1}(\tau(H_0)+\tau(H_i))\,dx \\ &\leq \Big(\int_{\Omega_i} \rho_i^p\,dx\Big)^{\frac{p-1}{p}} \Big(\int_{\Omega_i} (\tau(H_0)+\tau(H_i))^p\,dx\Big)^{\frac{1}{p}}.\end{aligned}$$

By (5.4), we have

$$\int_{\Omega_i} \rho_i^p(x,t) \leq C(p,\lambda_1(M))\|\tau(H_0)\|^p_{L^p(M)}. \tag{5.11}$$

where $C(p,\lambda_1(M))$ denote a positive constant depending only on p and $\lambda_1(M)$.

Denote $h = H_0^{-1}H_\infty$, and $\rho = \lg(\operatorname{Tr} h + \operatorname{Tr} h^{-1}) - \lg 2 \operatorname{rank} E$. From (5.11), we have

$$\int_M \rho^p(x)\,dx \leq C(p,\lambda_1(M))\|\tau(H_0)\|^p_{L^p(M)}. \tag{5.12}$$

From Corollary 2.11, we have

$$\triangle \rho \geq -\tau(H_0). \tag{5.13}$$

If $\dim M = m > 2$, we denote the Sobolev constant by

$$S_2(B_x(R)) = \inf_{0\neq u \in W_0^{1,2}(B_x(R))} \frac{\|\nabla u\|_2^2}{\|u\|^2_{\frac{2m}{m-2}}} \tag{5.14}$$

and if $\dim M = m = 2$, we denote the Sobolev constant by

$$S_1(B_x(R)) = \inf_{0\neq u \in W_0^{1,1}(B_x(R))} \frac{\|\nabla u\|_1^2}{\|u\|^2_{\frac{m}{m-1}}} \tag{5.15}$$

It was shown in [6] that under the additional assumption that $\operatorname{Ric}_M \geq -K$ and $\inf_{x\in M} V_x(1) = a > 0$, the Sobolev constant on $B_x(1)$ has a positive lower bound depending only on K, a, and $\lambda_1(M)$. By Moser's iterative argument, we have

$$\rho(x) \leq C(p,a,K,\lambda_1(M))\Big\{ \|\rho\|_{L^p(B_x(1))} + V_x(1)^{\frac{1}{p}} \|\tau(H_0)\|_{L^p(B_x(1))} \Big\}. \tag{5.16}$$

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By the volume comparison theorem and (5.12), we have

$$\rho(x) \to 0.$$

as $x \to \infty$. Equivalently,

$$\sigma(H_\infty(x), H_0(x)) = \operatorname{Tr} h(x) + \operatorname{Tr} h^{-1}(x) - 2 \operatorname{rank} E \to 0,$$

as $x \to \infty$. ∎

Theorem 5.2 *Let M be an m-dimensional complete Kähler manifold ($m > 2$), (E, H_0) be a holomorphic vector bundle with Hermitian metric H_0, and ϕ be a holomorphic section of E. Assume that the Sobolev constant $S_2(M) > 0$, and that*

$$\|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \operatorname{Id} \|_{L^p(M)} < \infty$$

for some $p \in (1, \frac{m}{2})$ and real number λ. Then there exists a Hermitian metric H on E such that

$$2\sqrt{-1}\Lambda F_H + \phi \otimes \phi^{*H} - \lambda \operatorname{Id} = 0.$$

Furthermore, if we assume that $\left|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \operatorname{Id}\right| \in L^p(M) \cap L^r(M)$ for some $r > \frac{m}{2}$ and

$$V_x(1)^{\frac{r}{q}} \int_{B_x(1)} \left|2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \operatorname{Id}\right|^r dy \to 0$$

as $x \to \infty$, where $q = \frac{mp}{m-2p}$, then the Hermitian Yang–Mills–Higgs metric H given in the above must satisfy

$$\sigma(H(x), H_0(x)) \to 0$$

as $x \to \infty$.

Proof The proof is completely similar to that of Theorem 5.1. Using the condition $S_2(M) > 0$, we have [3, 15]

$$\left(\int_M K^q(x, y, t)\, dy\right)^{\frac{1}{q}} \leq C(m, S_2(M)) t^{-\frac{m}{2p}}. \tag{5.17}$$

Here $q = \frac{p}{p-1}$. Then, from (5.5), (5.5′), (5.7) and (5.8), we can conclude that there exists a global Hermitian Yang–Mills–Higgs flow $H(\cdot, t)$, and $H(\cdot, t)$ converge to a smooth Hermitian Yang–Mills–Higgs metric H_∞ as $t \to \infty$.

On the other hand, we only need to replace the Poincaré inequality by the Sobolev inequality (5.14) when we apply the inequality (5.10) $\triangle \rho_i \geq -\tau(H_i) - \tau(H_0)$. Then, we have

$$\left(\int_M \rho^q(x)\, dx\right)^{\frac{1}{q}} \leq C(p, S_2(M)) \|\tau(H_0)\|_{L^p(M)}, \tag{5.18}$$

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where $q = \frac{mp}{m-2p}$. By Moser's iterative argument [8, Theorem 8.17] for (5.13), we have

$$\rho(x) \le C(p, m, S_2(M))\{\|\rho\|_{L^q(B_x(1))} + V_x(1)^{\frac{1}{q}}\|\tau(H_0)\|_{L^r(B_x(1))}\}. \tag{5.19}$$

Then, the conclusion that $\sigma(H(x), H_0(x)) \to 0$ as $x \to \infty$ follows. ∎

Lemma 5.3 ([16]) *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Let f be a nonnegative continuous function on M. Consider the heat equation*

$$\Big(\triangle - \frac{\partial}{\partial t}\Big) u(x,t) = 0, \quad u(x,0) = f(x).$$

Then it has a nonnegative solution $u(x,t)$ such that $\int_0^\infty u(x,s)\,ds < \infty$ if and only if

$$\int_0^\infty \frac{s}{V_x(s)} \int_{B_x(s)} f(y)\,dyds < \infty.$$

From Proposition 2.4, we know that the Hermitian Yang–Mills–Higgs flow $H_i(\,\cdot\,,t)$ which we constructed on the sub-domain Ω_i must satisfy

$$\begin{aligned} &\Big(\triangle - \frac{\partial}{\partial t}\Big)\tau(H_i) \ge 0,\\ &\tau(H_i)(x,0) = \tau(H_0)(x),\\ &\tau(H_i)(x,t)|_{x\in\partial\Omega_i}. \end{aligned} \tag{5.20}$$

If the initial metric H_0 satisfies $\int_0^\infty \frac{s}{V_x(s)} \int_{B_x(s)} \tau(H_0)(y)\,dyds < \infty$, by Lemma 5.3 and the maximum principle, we have

$$\tau_{H_i}(x,t) \le u(x,t).$$

Proceeding as in Theorem 5.1, we can conclude there exists a global HYMH flow $H(x,t)$ satisfying

$$\begin{aligned} \sigma(H(x,t_1), H(x,t_2)) &\le 2\,\mathrm{rank}\,E\Big(\exp\Big(\int_{t_1}^{t_2} \tau(H)(x,t)\,dt\Big) - 1\Big)\\ &\le 2\,\mathrm{rank}\,E\Big(\exp\Big(\int_{t_1}^{t_2} u(x,t)\,dt\Big) - 1\Big). \end{aligned}$$

Since $\int_0^\infty u(x,s)\,ds < \infty$, we can conclude that $H(\,\cdot\,,t)$ must converge to an Hermitian Yang–Mills–Higgs metric H_∞. So, we have proved the following theorem.

Theorem 5.4 *Let M be a complete Kähler manifold with nonnegative Ricci curvature, (E, H_0) a holomorphic vector bundle with Hermitian metric H_0, and ϕ a holomorphic section of E. Assume the initial metric H_0 satisfies, for every $x \in M$,*

$$\int_0^\infty \frac{s}{V_x(s)} \int_{B_x(s)} |2\sqrt{-1}\Lambda F_{H_0} + \phi \otimes \phi^{*H_0} - \lambda \,\mathrm{Id}\,|(y)\, dy ds < \infty.$$

Then there exists an Hermitian metric H on E such that

$$2\sqrt{-1}\Lambda F_H + \phi \otimes \phi^{*H} - \lambda \,\mathrm{Id} = 0.$$

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References

[1] S. B. Bradlow, *Vortices in holomorphic line bundles over closed Kähler manifolds.* Commun. Math. Phys. **135**(1990), 1–17.
[2] ———, *Special metrics and stability for holomorphic bundles with global sections.* J. Differential Geom. **33**(1991), 169–213.
[3] S. Y. Cheng and P. Li, *Heat kernel estimates and lower bound of eigenvalues.* Comment. Math. Helv. **56**(1981), 327–338.
[4] S. K. Donaldson, *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles.* Proc. London Math. Soc. (3) **50**(1985), 1–26.
[5] ———, *Boundary value problems for Yang-Mills fields.* J. Geom. Phys. **8**(1992), 89–122.
[6] W. Y. Ding and Y. Wang, *Harmonic maps of complete noncompact Riemannian manifolds.* Internat. J. Math. **2**(1991), 617–633.
[7] A. Grigor'yan, *Gaussian upper bounds for the heat kernel on arbitrary manifolds.* J. Differential Geom. **45**(1997), 33–52.
[8] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Second edition. Grundlehren der Mathematischen Wissenschaften 224, Springer-Verlag, Berlin, 1983.
[9] R. S. Hamilton, *Harmonic maps of manifolds with boundary.* Lecture Notes in Mathematics 471, Springer-Verlag, Berlin, 1975.
[10] N. J. Hitchin, *The self-duality equations on a Riemann surface.* Proc. London Math. Soc. (3) **55**(1987), 59–126.
[11] M. C. Hong, *Heat flow for the Yang-Mills-Higgs field and the Hermitian Yang-Mills-Higgs metric.* Ann. Global Anal. Geom. **20**(2001), 23–46.
[12] P. Li and L. F. Tam, *The heat equation and harmonic maps of complete manifolds.* Invent. Math. **105**(1991), 1–46.
[13] J. Y. Li, *The heat flows and harmonic maps of complete noncompact Riemannian manifolds.* Math. Z. **212**(1993), 161–173.
[14] J. Moser, *On Harnack's theorem for elliptic differential equations.* Commun. Pure Appl. Math. **14**(1961), 577–591.
[15] J. Nash, *Continuity of solutions of parabolic and elliptic equations.* Amer. J. Math. **80**(1958), 931–954.
[16] L. Ni, *The Poisson equation and Hermitian-Einstein metrics on holomorphic vector bundles over complete noncompact Kähler manifolds.* Indiana. Univ. Math. J. **51**(2002), 679–704.
[17] L. Ni and H. Ren, *Hermitian-Einstein metrics on vector bundles on complete Kähler manifolds.* Trans. Amer. Math. Soc. **353**(2001), 441–456.
[18] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface.* Ann. of Math. **82**(1965), 540–567.
[19] L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds.* J. Diff. Geom. **36**(1992), 417–450.
[20] C. T. Simpson, *Constructing variations of Hodge structures using Yang-Mills theory and applications to uniformization.* J. Amer. Math. Soc. **1**(1988), 867–918.

[21] Y. T. Siu, *Lectures on Hermitian-Einstein metrics for stable bundles and Kahler-Einstein metrics.* DMV Seminar 8, Birkhauser Verlag, Basel, 1987.

[22] K. K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian-Yang-Mills connection in stable vector bundles.* Comm. Pure Appl. Math. **39S**(1986), S257–S293.

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