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Linquan Ma, Karl Schwede and Kazuma Shimomoto

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Abstract

In this paper we study the local cohomology modules of Du Bois singularities. Let $(R, \mathfrak{m})$ be a local ring; we prove that if R_{red} is Du Bois, then $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R_{\mathrm{red}})$ is surjective for every i. We find many applications of this result. For example, we answer a question of Kovács and Schwede [*Inversion of adjunction for rational and Du Bois pairs*, Algebra Number Theory **10** (2016), 969–1000; MR 3531359] on the Cohen–Macaulay property of Du Bois singularities. We obtain results on the injectivity of Ext that provide substantial partial answers to questions in Eisenbud *et al.* [*Cohomology on toric varieties and local cohomology with monomial supports*, J. Symbolic Comput. **29** (2000), 583–600] in characteristic 0. These results can also be viewed as generalizations of the Kodaira vanishing theorem for Cohen–Macaulay Du Bois varieties. We prove results on the set-theoretic Cohen–Macaulayness of the defining ideal of Du Bois singularities, which are characteristic-0 analogs and generalizations of results of Singh–Walther and answer some of their questions in Singh and Walther [*On the arithmetic rank of certain Segre products*, in *Commutative algebra and algebraic geometry*, Contemporary Mathematics, vol. 390 (American Mathematical Society, Providence, RI, 2005), 147–155]. We extend results on the relation between Koszul cohomology and local cohomology for F-injective and Du Bois singularities first shown in Hochster and Roberts [*The purity of the Frobenius and local cohomology*, Adv. Math. **21** (1976), 117–172; MR 0417172 (54 #5230)]. We also prove that singularities of dense F-injective type deform.

1. Introduction

The notion of Du Bois singularities was introduced by Steenbrink based on work of Du Bois [DBoi81], which itself was an attempt to localize Deligne's Hodge theory of singular varieties [Del74]. Steenbrink studied them initially because families with Du Bois fibers have remarkable base-change properties [Ste81]. On the other hand, Du Bois singularities have recently found new connections across several areas of mathematics [KK10, KS16a, Lee09, CHWW11, HJ14]. In this paper we find numerous applications of Du Bois singularities especially to questions of local cohomology. Our key observation is as follows.

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Key Lemma (Lemma 3.3). *Suppose that $(S, \mathfrak{m})$ is a local ring essentially of finite type over $\mathbb{C}$ and that $S_{\mathrm{red}} = S/\sqrt{0}$ has Du Bois singularities. Then*

$$H^i_{\mathfrak{m}}(S) \to H^i_{\mathfrak{m}}(S_{\mathrm{red}})$$

is surjective for every i.

In fact, we obtain this surjectivity for Du Bois pairs, but we leave the statement simple in the introduction. Utilizing this lemma and related methods, we prove several results.

Theorem A (Corollary 3.9). *Let X be a reduced scheme and let $H \subseteq X$ be a Cartier divisor. If H is Du Bois and $X \backslash H$ is Cohen–Macaulay, then X is Cohen–Macaulay and hence so is H.*

This first consequence answers a question of Kovács and the second author and should be viewed as a generalization of several results in [KK10, § 7]. In particular, we do not need a projective family. This result holds if one replaces Cohen–Macaulay with S_k and also generalizes to the context of Du Bois pairs as in [KS16b]. Theorem A should also be viewed as a characteristic-0 analog of [FW89, Proposition 2.13] and [HMS14, Corollary A.4]. Theorem A also implies that if $X \backslash H$ has rational singularities and H has Du Bois singularities, then X has rational singularities (see Corollary 3.10), generalizing [KS16b, Theorem E].

Our next consequence of the key lemma is the following.

Theorem B (Proposition 4.1). *Let $(R, \mathfrak{m})$ be a Gorenstein local ring essentially of finite type over $\mathbb{C}$, and let $I \subseteq R$ be an ideal such that R/I has Du Bois singularities. Then the natural map $\mathrm{Ext}^j_R(R/I, R) \to H^j_I(R)$ is injective for every j.*

This gives a partial characteristic-0 answer to [EMS00, Question 6.2], asking when such maps are injective. A special case of the analogous characteristic $p > 0$ result was obtained by Singh–Walther [SW07]. This result leads to an answer of [DN16, Question 7.5] on the bounds on the projective dimension of Du Bois and log canonical singularities. In the graded setting, we can prove a much stronger result on the injectivity of Ext.

Theorem C (Theorem 4.5). *Let $(R, \mathfrak{m})$ be a reduced Noetherian $\mathbb{N}$-graded $(R_0 = \mathbb{C})$-algebra with $\mathfrak{m}$ the unique homogeneous maximal ideal. Suppose that R_P is Du Bois for all $P \neq \mathfrak{m}$. Write $R = A/I$, where $A = \mathbb{C}[x_1, \ldots, x_n]$ is a polynomial ring with $\deg x_i = d_i > 0$ and I is a homogeneous ideal. Then the natural degree-preserving map $\mathrm{Ext}^j_A(R, A) \to H^j_I(A)$ induces an injection*

$$[\mathrm{Ext}^j_A(R, A)]_{\geqslant -d} \hookrightarrow [H^j_I(A)]_{\geqslant -d}$$

for every j, where $d = \sum d_i$.

Theorem C leads to new vanishing for local cohomology for $\mathbb{N}$-graded isolated non-Du Bois singularities (Theorem 4.8), a generalization of the Kodaira vanishing theorem for Cohen–Macaulay Du Bois projective varieties.

Our next consequence of the key lemma answers a longstanding question, but first we give some background. Du Bois singularities are closely related to the notion of F-injective singularities (rings where the Frobenius acts injectively on local cohomology of maximal ideals). In particular, it is conjectured that X has Du Bois singularities if and only if its reduction to characteristic $p > 0$ has F-injective singularities for a Zariski dense set of primes $p \gg 0$

(this is called *dense F-injective type*). This conjecture is expected to be quite difficult, as it is closely related to asking for infinitely many ordinary characteristic $p > 0$ reductions for smooth projective varieties over $\mathbb{C}$ [BST13]. On the other hand, several recent papers have studied how Du Bois and F-injective singularities deform [KS16a, HMS14]. We know that if a Cartier divisor $H \subseteq X$ has Du Bois singularities, then X also has Du Bois singularities near H. However, the analogous statement for F-injective singularities in characteristic $p > 0$ is open and has been since [Fed83]. In fact, F-injective singularities were introduced because it was observed that Cohen–Macaulay F-injective singularities deform in this way but F-pure singularities[1] do not. We show that at least the property of having dense F-injective type satisfies such a deformation result, in other words that F-injectivity deforms when p is large.

THEOREM D (Theorem 5.3). *Let $(R, \mathfrak{m})$ be a local ring essentially of finite type over $\mathbb{C}$ and let x be a non-zero divisor on R. Suppose that R/xR has dense F-injective type. Then, for infinitely many $p > 0$, the Frobenius action $x^{p-1}F$ on $H^i_{\mathfrak{m}_p}(R_p)$ is injective for every i, where $(R_p, \mathfrak{m}_p)$ is the reduction mod p of R. In particular, R has dense F-injective type.*

Our final result is a characteristic-0 analog of a strengthening of the main result of [HR76], and we can give a characteristic $p > 0$ generalization as well.

THEOREM E (Corollary 4.13). *Let R be a Noetherian $\mathbb{N}$-graded k-algebra, with $\mathfrak{m}$ the unique homogeneous maximal ideal. Suppose that R is equidimensional and Cohen–Macaulay on $\operatorname{Spec} R - \{\mathfrak{m}\}$. Assume one of the following:*

(a) *k has characteristic $p > 0$ and R is F-injective;*
(b) *k has characteristic 0 and R is Du Bois.*

Then $[H^r(\underline{x}, R)]_0 \cong H^r_{\mathfrak{m}}(R)$ for every $r < n = \dim R$ and every homogeneous system of parameters $\underline{x} = x_1, \ldots, x_n$, where $H^r(\underline{x}, R)$ denotes the rth Koszul cohomology of $\underline{x}$. In other words, it is not necessary to take a direct limit when computing the local cohomology.

In fact, we prove a more general result (Theorem 4.12), which does not require any F-injective or Du Bois conditions, from which Theorem E follows immediately thanks to our injectivity and vanishing results (Theorems 4.5 and 4.8).

2. Preliminaries

Throughout this paper, all rings will be Noetherian and all schemes will be Noetherian and separated. When in equal characteristic 0, we will work with rings and schemes that are essentially of finite type over $\mathbb{C}$. Of course, nearly all of our results also hold over all other fields of characteristic 0 by base change.

2.1 Du Bois singularities

We give a brief introduction to Du Bois singularities and pairs. For more details, see for instance [KS11] and [Kol13]. Frequently we will work in the setting of pairs in the Du Bois sense.

DEFINITION 2.1. Suppose that X and Z are schemes of essentially finite type over $\mathbb{C}$. By a *pair* we will mean the combined data of (X, Z). We will call the pair *reduced* if both X and Z are reduced.

---

[1] We now know that F-pure singularities are analogs of log canonical singularities [HW02].

Suppose that X is a scheme essentially of finite type over $\mathbb{C}$. Associated to X is an object $\underline{\Omega}^0_X \in D^b_{\mathrm{coh}}(X)$ with a map $\mathcal{O}_X \to \underline{\Omega}^0_X$ functorial in the following way. If $f : Y \to X$ is a map of schemes, then we have a commutative square

$$
\begin{array}{ccc}
\mathcal{O}_X & \longrightarrow & \underline{\Omega}^0_X \\
\downarrow & & \downarrow \\
\mathbf{R}f_*\mathcal{O}_Y & \longrightarrow & \mathbf{R}f_*\underline{\Omega}^0_Y
\end{array}
$$

If X is non-reduced, then $\underline{\Omega}^0_X = \underline{\Omega}^0_{X_{\mathrm{red}}}$ by definition. To define $\underline{\Omega}^0_X$ in general, let $\pi : X_\bullet \to X$ be a hyper-resolution and set $\underline{\Omega}^0_X = \mathbf{R}\pi_*\mathcal{O}_{X_\bullet}$ (alternatively, see [Sch07]).

If $Z \subseteq X$ is a closed subscheme, then we define $\underline{\Omega}^0_{X,Z}$ as the object completing the following distinguished triangle:

$$
\underline{\Omega}^0_{X,Z} \to \underline{\Omega}^0_X \to \underline{\Omega}^0_Z \xrightarrow{+1} .
$$

We observe that there is an induced map $\mathscr{I}_{Z \subseteq X} \to \underline{\Omega}^0_{X,Z}$. We also note that by this definition and the fact that $\underline{\Omega}^0_X = \underline{\Omega}^0_{X_{\mathrm{red}}}$, we have $\underline{\Omega}^0_{X,Z} = \underline{\Omega}^0_{X_{\mathrm{red}},Z_{\mathrm{red}}}$.

DEFINITION 2.2 (Du Bois singularities). We say that X has Du Bois singularities if the map $\mathcal{O}_X \to \underline{\Omega}^0_X$ is a quasi-isomorphism. If $Z \subseteq X$ is a closed subscheme, we say that (X, Z) has Du Bois singularities if the map $\mathscr{I}_{Z \subseteq X} \to \underline{\Omega}^0_{X,Z}$ is a quasi-isomorphism.

Remark 2.3. It is clear that Du Bois singularities are reduced. In general, a pair (X, Z) being Du Bois does not necessarily imply that X or Z is reduced. However, if a pair (X, Z) is Du Bois and X is reduced, then so is Z [KS16b, Lemma 2.9].

2.2 Cyclic covers of non-reduced schemes

In this paper we will need to take cyclic covers of non-reduced schemes. We will work in the following setting. We assume what follows is well known to experts but we do not know a reference in the generality we need (see [KM98, § 2.4] or [dFEM, § 2.1.1]). Note also that the reason we are able to work with non-reduced schemes is because our $\mathscr{L}$ is actually locally free and *not* just a reflexive rank-1 sheaf.

Setting 2.4. Suppose that X is a scheme of finite type over $\mathbb{C}$ (typically projective). Suppose also that $\mathscr{L}$ is a line bundle (typically semi-ample). Choose a (typically general) global section $s \in H^0(X, \mathscr{L}^n)$ for some $n > 0$ (typically $n \gg 0$) and form the sheaf of rings

$$
\mathscr{R}(X, \mathscr{L}, s) = \mathcal{O}_X \oplus \mathscr{L}^{-1} \oplus \cdots \oplus \mathscr{L}^{-n+1},
$$

where, for $i, j < n$ and $i + j > n$, the multiplication $\mathscr{L}^{-i} \otimes \mathscr{L}^{-j} \to \mathscr{L}^{-i-j+n}$ is performed by the formula $a \otimes b \mapsto abs$. We define $\nu : Y = Y_{\mathscr{L},s} = \mathbf{Spec}\,\mathscr{R}(X, \mathscr{L}, s) \to X$. Note we did not assume that s was non-zero or even locally a non-zero divisor.

Now let us work locally on an affine chart U trivializing $\mathscr{L}$, where $U = \mathrm{Spec}\, R \subseteq X$ with corresponding $\nu^{-1}(U) = \mathrm{Spec}\, S$. Fix an isomorphism $\mathscr{L}|_U \cong \mathcal{O}_U$ and write

$$
S = R \oplus Rt \oplus Rt^2 \oplus \cdots \oplus Rt^{n-1},
$$

where t is a dummy variable used to help keep track of the degree. The choice of the section $s|_U \in H^0(X, \mathscr{L}^n)$ is the same as the choice of map $\mathcal{O}_X \to \mathscr{L}^n$ (send 1 to the section s).

If s is chosen to be general and $\mathscr{L}^n$ is very ample, then this map is an inclusion, but it is not injective in general. Working locally where we have fixed $\mathscr{L}|_U = \mathcal{O}_U$, we also have implicitly chosen $\mathscr{L}^n = \mathcal{O}_U$ and hence we have just specified that $t^n = s|_U \in \Gamma(U, \mathscr{L}^n) = \Gamma(U, \mathcal{O}_U)$. In other words,

$$S = R[t]/\langle t^n - s \rangle.$$

In particular, it follows that ν is a finite map.

LEMMA 2.5. *The map ν is functorial with respect to taking closed subschemes of X. In particular, if $Z \subseteq X$ is a closed subscheme and $\mathscr{L}$ and s are as above, then we have the following commutative diagram.*

$$
\begin{array}{ccccc}
Y_{\mathscr{L},s} & = = = = & \mathbf{Spec}\,\mathscr{R}(X, \mathscr{L}, s) & \xrightarrow{\;\nu_X\;} & X \\
& & \Big\uparrow & & \Big\uparrow \\
W_{\mathscr{L}|_Z, s|_Z} & = = = = & \mathbf{Spec}\,\mathscr{R}(Z, \mathscr{L} \otimes \mathcal{O}_Z, s|_Z) & \xrightarrow[\;\nu_Z\;]{} & Z
\end{array}
$$

Furthermore, $W_{\mathscr{L}|_Z, s|_Z} = \pi^{-1}(Z)$ is the scheme-theoretic preimage of Z.

Proof. The first statement follows since $\mathscr{R}(X, \mathscr{L}, s) = \mathcal{O}_X \oplus \mathscr{L}^{-1} \oplus \cdots \oplus \mathscr{L}^{-n+1}$ surjects onto $\mathscr{R}(Z, \mathscr{L}|_Z, s|_Z) = \mathcal{O}_Z \oplus \mathscr{L}|_Z^{-1} \oplus \cdots \oplus \mathscr{L}|_Z^{-n+1}$ in the obvious way. For the second statement, we simply notice that

$$\mathscr{R}(X, \mathscr{L}, s) \otimes_{\mathcal{O}_X} \mathcal{O}_Z = \mathscr{R}(Z, \mathscr{L}|_Z, s|_Z). \qquad \square$$

LEMMA 2.6. *Suppose that we are working in Setting 2.4 and that X is projective and also reduced (respectively normal), $\mathscr{L}^n$ is globally generated and s is chosen to be general. Then $Y = Y_{\mathscr{L},s}$ is also reduced (respectively normal).*

Proof. To show that Y is reduced, we will show that it is S_1 and R_0. We work locally on an open affine chart $U = \operatorname{Spec} R \subseteq X$.

To show that it is R_0, it suffices to consider the case where $R = K$ is a field. Because s was picked generally, we may assume that the image of s in K is non-zero (we identify s with its image). Then we need to show that $K[t]/\langle t^n - s \rangle$ is a product of fields. But this is obvious since we are working in characteristic 0.

Next, we show that it is S_1. Indeed, if $(R, \mathfrak{m})$ is a local ring of depth $\geqslant 1$, then obviously $S = R[t]/\langle t^n - s \rangle$ has depth $\geqslant 1$ as well since it is a free R-module. Finally, the depth condition is preserved after localizing at the maximal ideals of the semilocal ring of S.

This proves the reduced case. The normal case is well known but we sketch it. The R_1 condition follows from [KM98, Lemma 2.51] and the fact that s is chosen generally, utilizing Bertini's theorem. The S_2 condition follows in the same way that S_1 followed above. $\qquad \square$

2.3 F-pure and F-injective singularities

Du Bois singularities are conjectured to be the characteristic-0 analog of F-injective singularities [Sch09, BST13]. In this short subsection we collect some definitions about singularities in positive characteristic. Our main focuses are F-pure and F-injective singularities.

A local ring $(R, \mathfrak{m})$ is called *F-pure* if the Frobenius endomorphism $F \colon R \to R$ is pure.[2] Under mild conditions, for example when R is F-finite, which means that the Frobenius map

---

[2] A map of R-modules $N \to N'$ is pure if for every R-module M the map $N \otimes_R M \to N' \otimes_R M$ is injective. This implies that $N \to N'$ is injective, and is weaker than the condition that $0 \to N \to N'$ be split.

$R \xrightarrow{F} R$ is a finite map, R being F-pure is equivalent to the condition that the Frobenius endomorphism $R \xrightarrow{F} R$ is split [HR76, Corollary 5.3]. The Frobenius endomorphism on R induces a natural Frobenius action on each local cohomology module $H^i_{\mathfrak{m}}(R)$ and we say that a local ring is *F-injective* if this natural Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for every i [Fed83]. This holds if R is F-pure [HR76, Lemma 2.2]. For some other basic properties of F-pure and F-injective singularities, see [HR76, Fed83, EH08].

3. The Cohen–Macaulay property for families of Du Bois pairs

We begin with a lemma, which we assume is well known to experts; indeed, it is explicitly stated in [Kol13, Corollary 6.9]. However, because many of the standard references also include implicit reducedness hypotheses, we include a careful proof that deduces it from the reduced case. Of course, one could also go back to first principles but we believe the path we take below is quicker.

LEMMA 3.1. *Let X be a projective scheme over $\mathbb{C}$ and let $Z \subseteq X$ be a closed subscheme (X, Z are not necessarily reduced). Then the natural map*

$$H^i(X, \mathscr{I}_Z) \to \mathbb{H}^i(X_{\mathrm{red}}, \underline{\Omega}^0_{X_{\mathrm{red}}, Z_{\mathrm{red}}}) \cong \mathbb{H}^i(X, \underline{\Omega}^0_{X, Z})$$

is surjective for every $i \in \mathbb{Z}$.

Proof. Note that if the result holds for $\mathbb{C}$, it certainly also holds for other fields of characteristic 0. The isomorphism in the statement of the lemma follows from the definition. For the surjectivity, we consider the following commutative diagram, where we let $U = X \setminus Z$. Note that the rows are not exact.

$$
\begin{array}{ccccc}
H^i_c(U_{\mathrm{red}}, \mathbb{C}) & \longrightarrow & H^i(X_{\mathrm{red}}, \mathscr{I}_{Z_{\mathrm{red}}}) & \longrightarrow & \mathbb{H}^i(X_{\mathrm{red}}, \underline{\Omega}^0_{X_{\mathrm{red}}, Z_{\mathrm{red}}}) \\
\cong \uparrow & & \uparrow & & \cong \uparrow \\
H^i_c(U, \mathbb{C}) & \longrightarrow & H^i(X, \mathscr{I}_Z) & \longrightarrow & \mathbb{H}^i(X, \underline{\Omega}^0_{X, Z})
\end{array}
$$

The composite map in the top horizontal line is a surjection by [KS16b, Lemma 2.17] or [Kov11, Corollary 4.2]. The vertical isomorphism on the left holds because the constant sheaf $\mathbb{C}$ does not see the non-reduced structure. Likewise for the vertical isomorphism on the right, where $\underline{\Omega}^0_{X, Z}$ does not see the non-reduced structure. The diagram then shows that $H^i(X, \mathscr{I}_Z) \to \mathbb{H}^i(X_{\mathrm{red}}, \underline{\Omega}^0_{X_{\mathrm{red}}, Z_{\mathrm{red}}})$ is a surjection. $\square$

Next, we prove the key injectivity lemma for possibly non-reduced pairs. The proof is essentially the same as in [KS16b] or [KS16a]. We reproduce it here carefully because we need it in the non-reduced setting.

LEMMA 3.2. *Let X be a scheme of essentially finite type over $\mathbb{C}$ and $Z \subseteq X$ a closed subscheme (X and Z are not necessarily reduced). Then the natural map*

$$h^j(\underline{\omega}^{\bullet}_{X, Z}) \hookrightarrow h^j(\mathbf{R}\,\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathscr{I}_Z, \omega^{\bullet}_X))$$

is injective for every $j \in \mathbb{Z}$, where $\underline{\omega}^{\bullet}_{X, Z} = \mathbf{R}\,\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\underline{\Omega}^0_{X, Z}, \omega^{\bullet}_X)$.

Proof. The question is local and compatible with restricting to an open subset; hence, we may assume that X is projective with ample line bundle $\mathscr{L}$. Let $s \in H^0(X, \mathscr{L}^n)$ be a general global section for some $n \gg 0$ and let $\eta \colon Y = \mathbf{Spec} \bigoplus_{i=0}^{n-1} \mathscr{L}^{-i} \to X$ be the nth cyclic cover corresponding to s. Set $W = \eta^{-1}(Z)$. Then, for $n \gg 0$ and s general, the restriction of η to W is the cyclic cover $W = \mathbf{Spec} \bigoplus_{i=0}^{n-1} \mathscr{L}|_Z^{-i} \to Z$ by Lemma 2.5. Likewise, η also induces the corresponding cyclic covers of the closed subschemes $Y_{\mathrm{red}} \to X_{\mathrm{red}}$ and $W_{\mathrm{red}} \to Z_{\mathrm{red}}$ by Lemmas 2.5 and 2.6. We have $\eta_* \mathscr{I}_W = \bigoplus_{i=0}^{n-1} \mathscr{I}_Z \otimes \mathscr{L}^{-i}$ and

$$\eta_* \underline{\Omega}^0_{Y,W} \cong \eta_* \underline{\Omega}^0_{Y_{\mathrm{red}}, W_{\mathrm{red}}} \cong \bigoplus_{i=0}^{n-1} \underline{\Omega}^0_{X_{\mathrm{red}}, Z_{\mathrm{red}}} \otimes \mathscr{L}^{-i} \cong \bigoplus_{i=0}^{n-1} \underline{\Omega}^0_{X,Z} \otimes \mathscr{L}^{-i},$$

where the second isomorphism is [KS16b, Lemma 3.1] (see also [KS16a, Lemma 3.1]). Since Lemma 3.1 implies that $H^j(Y, \mathscr{I}_W) \twoheadrightarrow \mathbb{H}^j(Y, \underline{\Omega}^0_{Y,W})$ is surjective for every $j \in \mathbb{Z}$, we know that $H^j(X, \mathscr{I}_Z \otimes \mathscr{L}^{-i}) \twoheadrightarrow \mathbb{H}^j(X, \underline{\Omega}^0_{X,Z} \otimes \mathscr{L}^{-i})$ is surjective for every $i \geqslant 0$ and $j \in \mathbb{Z}$.

Applying Grothendieck–Serre duality, we obtain an injection

$$\mathbb{H}^j(X, \underline{\omega}^\bullet_{X,Z} \otimes \mathscr{L}^i) \hookrightarrow \mathbb{H}^j(X, \mathbf{R}\,\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{I}_Z, \omega^\bullet_X) \otimes \mathscr{L}^i)$$

for all $i \geqslant 0$ and $j \in \mathbb{Z}$. Since $\mathscr{L}$ is ample, for $i \gg 0$ the spectral sequence computing the above hypercohomology degenerates. Hence, for $i \gg 0$, we get

$$H^0(X, h^j(\underline{\omega}^\bullet_{X,Z}) \otimes \mathscr{L}^i) \hookrightarrow H^0(X, h^j(\mathbf{R}\,\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{I}_Z, \omega^\bullet_X)) \otimes \mathscr{L}^i).$$

But again since $\mathscr{L}$ is ample, the above injection for $i \gg 0$ implies the injection $h^j(\underline{\omega}_{X,Z}) \hookrightarrow h^j(\mathbf{R}\,\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{I}_Z, \omega^\bullet_X))$. $\qquad\square$

Next, we prove the key lemma stated in the introduction (and we do the generalized pair version). It follows immediately from the above injectivity, Lemma 3.2. For simplicity, we will use $(S, S/J)$ to denote the pair $(\mathrm{Spec}\, S, \mathrm{Spec}\, S/J)$.

LEMMA 3.3. *Let $(S, \mathfrak{m})$ be a local ring essentially of finite type over $\mathbb{C}$ and let $J \subseteq S$ be an ideal. Suppose further that $S' = S/N$, where $N \subseteq S$ is an ideal contained in the nilradical (for instance, N could be the nilradical and then $S' = S_{\mathrm{red}}$). Suppose that $(S', S'/JS')$ is a Du Bois pair. Then the natural map $H^i_{\mathfrak{m}}(J) \to H^i_{\mathfrak{m}}(JS')$ is surjective for every i. In particular, if S_{red} is Du Bois, then $H^i_{\mathfrak{m}}(S) \to H^i_{\mathfrak{m}}(S_{\mathrm{red}})$ is surjective for every i.*

Proof. We consider the following commutative diagram.

$$
\begin{array}{ccc}
H^i_{\mathfrak{m}}(J) & \longrightarrow & H^i_{\mathfrak{m}}(JS') \\
\downarrow & & \downarrow{\scriptstyle\cong} \\
\mathbb{H}^i_{\mathfrak{m}}(\underline{\Omega}^0_{S,S/J}) & \overset{\cong}{\longrightarrow} & \mathbb{H}^i_{\mathfrak{m}}(\underline{\Omega}^0_{S',S'/JS'})
\end{array}
$$

Here the left vertical map is surjective by the Matlis dual of Lemma 3.2 applied to $X = \mathrm{Spec}\, S$ and $Z = \mathrm{Spec}\, S/J$, the right vertical map is an isomorphism because $(S', S'/JS')$ is a Du Bois pair and the bottom map is an isomorphism because $S_{\mathrm{red}} = S'_{\mathrm{red}}$. Chasing the diagram shows that $H^i_{\mathfrak{m}}(J) \to H^i_{\mathfrak{m}}(JS')$ is surjective. The last sentence is the case that J is the unit ideal (i.e., the non-pair version). $\qquad\square$

Remark 3.4. The characteristic $p > 0$ analog of the above lemma (in the case $J = S$, i.e., the non-pair version) holds if S_{red} is F-pure. We give a short argument here. Without loss of generality, we may assume that S and S_{red} are complete, so $S = A/I$ and $S_{\mathrm{red}} = A/\sqrt{I}$, where A is a complete regular local ring. We may pick $e \gg 0$ such that $(\sqrt{I})^{[p^e]} \subseteq I$. We have a composite map

$$H^i_{\mathfrak{m}}(A/(\sqrt{I})^{[p^e]}) \to H^i_{\mathfrak{m}}(A/I) \cong H^i_{\mathfrak{m}}(S) \xrightarrow{\phi} H^i_{\mathfrak{m}}(A/\sqrt{I}) \cong H^i_{\mathfrak{m}}(S_{\mathrm{red}}).$$

We know from [Lyu06, Lemma 2.2] that the image of the composite map is equal to $\mathrm{span}_A \langle f^e(H^i_{\mathfrak{m}}(S_{\mathrm{red}})) \rangle$, where f^e denotes the natural eth Frobenius action on $H^i_{\mathfrak{m}}(S_{\mathrm{red}})$. In particular, the Frobenius action on $H^i_{\mathfrak{m}}(S_{\mathrm{red}})/\mathrm{im}\,\phi$ is nilpotent. However, since $\mathrm{im}\,\phi$ is an F-stable submodule of $H^i_{\mathfrak{m}}(S_{\mathrm{red}})$ and S_{red} is F-pure, [Ma14, Theorem 3.7] shows that the Frobenius acts injectively on $H^i_{\mathfrak{m}}(S_{\mathrm{red}})/\mathrm{im}\,\phi$. Hence, we must have $H^i_{\mathfrak{m}}(S_{\mathrm{red}}) = \mathrm{im}\,\phi$, that is, $H^i_{\mathfrak{m}}(S) \to H^i_{\mathfrak{m}}(S_{\mathrm{red}})$ is surjective.

We next give an example showing that the analog of Lemma 3.3 for F-injectivity fails in general. The example is a modification of [EH08, Example 2.16].

Example 3.5. Let $K = k(u, v)$, where k is an algebraically closed field of characteristic $p > 0$, and let $L = K[z]/(z^{2p} + uz^p + v)$ be as in [EH08, Example 2.16]. Now let $R = K + (x, y)L[[x, y]]$ with $\mathfrak{m} = (x, y)L[[x, y]]$. Then it is easy to see that $(R, \mathfrak{m})$ is a local ring of dimension 2 and we have a short exact sequence:

$$0 \to R \to L[[x, y]] \to L/K \to 0.$$

The long exact sequence of local cohomology gives

$$H^1_{\mathfrak{m}}(R) \cong L/K, \quad H^2_{\mathfrak{m}}(R) \cong H^2_{\mathfrak{m}}(L[[x, y]]).$$

Moreover, one can check that the Frobenius action on $H^1_{\mathfrak{m}}(R)$ is exactly the Frobenius action on L/K. The Frobenius action on L/K is injective since $L^p \cap K = K^p$, and the Frobenius action on $H^2_{\mathfrak{m}}(L[[x, y]])$ is injective because $L[[x, y]]$ is regular. Hence, the Frobenius actions on both $H^1_{\mathfrak{m}}(R)$ and $H^2_{\mathfrak{m}}(R)$ are injective. This proves that R is F-injective.

Write $R = A/I$ for a regular local ring $(A, \mathfrak{m})$. One checks that the Frobenius action $F : H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(R)$ is not surjective up to K-span (and hence not surjective up to R-span, since the residue field of R is K) because $L \neq L^p[K]$ by our choice of K and L (see [EH08, Example 2.16] for a detailed computation on this). But now [Lyu06, Lemma 2.2] shows that $H^1_{\mathfrak{m}}(A/I^{[p]}) \to H^1_{\mathfrak{m}}(A/I)$ is not surjective. Therefore, we can take $S = A/I^{[p]}$ with $S_{\mathrm{red}} = R$ that is F-injective, but $H^1_{\mathfrak{m}}(S) \to H^1_{\mathfrak{m}}(S_{\mathrm{red}})$ is not surjective.

In view of Remark 3.4 and Example 3.5 and the relation between F-injective and Du Bois singularities, it is tempting to try to define a more restrictive variant of F-injective singularities for local rings. In particular, if $(R, \mathfrak{m})$ is a local ring such that R_{red} has these more restrictive F-injective singularities, then it should follow that $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R_{\mathrm{red}})$ surjects for all i.

THEOREM 3.6. *Suppose that (X, Z) is a reduced pair and that $H \subseteq X$ is a Cartier divisor such that H does not contain any irreducible components of either X or Z. If $(H, Z \cap H)$ is a Du Bois pair, then for all i and all points $\eta \in H \subseteq X$, the following segment of the long exact sequence:*

$$0 \to H^i_\eta(\mathscr{I}_{Z,\eta} \cdot \mathcal{O}_{H,\eta}) \hookrightarrow H^{i+1}_\eta(\mathscr{I}_{Z,\eta} \cdot \mathcal{O}_{X,\eta}(-H)) \twoheadrightarrow H^{i+1}_\eta(\mathscr{I}_{Z,\eta}) \to 0$$

is exact for all i. Dually, in the special case that $\mathscr{I}_Z = \mathcal{O}_X$, we can also phrase this as saying that

$$0 \to h^{-i}(\omega_X^\bullet) \to h^{-i}(\omega_X^\bullet(H)) \to h^{-i+1}(\omega_H^\bullet) \to 0$$

is exact for all i.

Proof. Localizing at η, we may assume that $X = \operatorname{Spec} R$, $Z = \operatorname{Spec} R/I$, $H = \operatorname{div}(x)$ with $(R, \mathfrak{m})$ a local ring and $\eta = \{\mathfrak{m}\}$. Moreover, the hypotheses imply that x is a non-zero divisor on both R and R/I. It is enough to show that the segment of the long exact sequence

$$0 \to H_{\mathfrak{m}}^i(I/xI) \to H_{\mathfrak{m}}^{i+1}(I) \xrightarrow{\cdot x} H_{\mathfrak{m}}^{i+1}(I) \to 0 \tag{3.6.1}$$

induced by $0 \to I \xrightarrow{\cdot x} I \to I/xI \to 0$ is exact.

Consider $S = R/x^n R$ and $J = I(R/x^n R)$. The pair $(S' = R/xR, S'/IS') = (H, Z \cap H)$ is Du Bois by hypothesis (note that we do not assume that H is reduced). Thus, Lemma 3.3 implies that

$$H_{\mathfrak{m}}^i(I(R/x^n R)) \twoheadrightarrow H_{\mathfrak{m}}^i(I(R/xR)) \tag{3.6.2}$$

is surjective for every i, n. Since x is a non-zero divisor on R/I, $x^n I = I \cap x^n R$. Thus,

$$I(R/x^n R) = \frac{I + x^n R}{x^n R} \cong \frac{I}{I \cap x^n R} = I/x^n I.$$

Hence, (3.6.2) shows that

$$H_{\mathfrak{m}}^i(I/x^n I) \twoheadrightarrow H_{\mathfrak{m}}^i(I/xI)$$

is surjective for every i, n. The long exact sequence of local cohomology induced by $0 \to I/x^{n-1}I \xrightarrow{\cdot x} I/x^n I \to I/xI \to 0$ tells us that

$$H_{\mathfrak{m}}^i(I/x^{n-1}I) \xrightarrow{\cdot x} H_{\mathfrak{m}}^i(I/x^n I)$$

is injective for every i, n. Hence, after taking a direct limit, we obtain that

$$\phi_i : H_{\mathfrak{m}}^i(I/xI) \to \varinjlim_n H_{\mathfrak{m}}^i(I/x^n I) \cong H_{\mathfrak{m}}^i\left(\varinjlim_n (I/x^n I)\right) \cong H_{\mathfrak{m}}^i(H_x^1(I)) \cong H_{\mathfrak{m}}^{i+1}(I)$$

is injective for every i. Here the last two isomorphisms come from the fact that x is a non-zero divisor on I (since it is a non-zero divisor on R) and a simple computation using the local cohomology spectral sequence.

CLAIM 3.7. *The ϕ_i are exactly the connection maps in the long exact sequence of local cohomology induced by $0 \to I \xrightarrow{\cdot x} I \to I/xI \to 0$:*

$$\cdots \to H_{\mathfrak{m}}^i(I) \to H_{\mathfrak{m}}^i(I/xI) \xrightarrow{\phi_i} H_{\mathfrak{m}}^{i+1}(I) \xrightarrow{\cdot x} H_{\mathfrak{m}}^{i+1}(I) \to \cdots.$$

This claim immediately produces (3.6.1) because we proved that ϕ_i is injective for every i. Thus, it remains to prove the claim.

Proof of claim. Observe that, by definition, ϕ_i is the natural map in the long exact sequence of local cohomology

$$\cdots \to H_{\mathfrak{m}}^i(I/xI) \xrightarrow{\phi_i} H_{\mathfrak{m}}^i(I_x/I) \xrightarrow{\cdot x} H_{\mathfrak{m}}^i(I_x/I) \to \cdots,$$

which is induced by $0 \to I/xI \to I_x/I \xrightarrow{\cdot x} I_x/I \to 0$ (note that x is a non-zero divisor on I and $H_x^1(I) \cong I_x/I$). However, it is easy to see that the multiplication by x map $H_{\mathfrak{m}}^i(I_x/I) \xrightarrow{\cdot x} H_{\mathfrak{m}}^i(I_x/I)$ can be identified with the multiplication by x map $H_{\mathfrak{m}}^{i+1}(I) \xrightarrow{\cdot x} H_{\mathfrak{m}}^{i+1}(I)$ because we have a natural identification $H_{\mathfrak{m}}^i(I_x/I) \cong H_{\mathfrak{m}}^i(H_x^1(I)) \cong H_{\mathfrak{m}}^{i+1}(I)$. This finishes the proof of the claim and thus also the local cohomology statement. □

The global statement can be checked locally, where it is simply a special case of the local dual of the local cohomology statement. □

The following is the main theorem of the section; it answers a question of Kovács and the second author [KS16b, Question 5.7] and is a generalization to pairs of a characteristic-0 analog of [HMS14, Corollary A.5].

THEOREM 3.8. *Suppose that (X, Z) is a reduced pair and that $H \subseteq X$ is a Cartier divisor such that H does not contain any irreducible components of either X or Z. If $(H, Z \cap H)$ is a Du Bois pair and $\mathscr{I}_Z|_{X \setminus H}$ is S_k, then $\mathscr{I}_Z$ is S_k.*

Proof. Suppose that $\mathscr{I}_Z$ is not S_k; choose an irreducible component Y of the non-S_k locus of $\mathscr{I}_Z$. Since $\mathscr{I}_Z|_{X \setminus H}$ is S_k, we know that $Y \subseteq H$. Let η be the generic point of Y. Theorem 3.6 tells us that

$$H_\eta^i(\mathscr{I}_Z \cdot \mathcal{O}_X(-H)) \twoheadrightarrow H_\eta^i(\mathscr{I}_Z)$$

is surjective for every i. Note that if we localize at η and use the same notation as in the proof of Theorem 3.6, this surjection is the multiplication by x map $H_{\mathfrak{m}}^i(I) \xrightarrow{\cdot x} H_{\mathfrak{m}}^i(I)$.

However, after we localize at η, I is S_k on the punctured spectrum and $H_{\mathfrak{m}}^i(I)$ has finite length for every $i < k$. In particular, for $h \gg 0$, x^h annihilates $H_{\mathfrak{m}}^i(I)$ for every $i < k$. Therefore, for $i < k$, the multiplication by x map $H_{\mathfrak{m}}^i(I) \xrightarrow{\cdot x} H_{\mathfrak{m}}^i(I)$ cannot be surjective unless $H_{\mathfrak{m}}^i(I) = 0$, which means that I is S_k as an R-module. But this contradicts our choice of Y (because we pick Y as an irreducible component of the non-S_k locus of $\mathscr{I}_Z$). Therefore, $\mathscr{I}_Z$ is S_k. □

COROLLARY 3.9. *Let X be a reduced scheme and let $H \subseteq X$ be a Cartier divisor. If H is Du Bois and $X \setminus H$ is Cohen–Macaulay, then X is Cohen–Macaulay and hence so is H.*

As mentioned in the Introduction, these results should be viewed as a generalization of [KK10, Corollaries 1.3 and 7.13] and [KS16b, Theorem 5.5]. In those results, one considered a flat projective family $X \to B$ with Du Bois fibers such that the general fiber is S_k. It was then shown that the special fiber is S_k. Let us explain this result in a simple case. Suppose that B is a smooth curve, $Z = 0$, the special fiber is a Cartier divisor H and $X \setminus H$ is Cohen–Macaulay. Then we are trying to show that H is Cohen–Macaulay, which is of course the same as showing that X is Cohen–Macaulay, which is exactly what our result proves. The point is that we had to make no projectivity hypothesis for our family.

This result also implies [KS16b, Conjecture 7.9], which we state next. We leave out the definition of a rational pair here (this is a rational pair in the sense of Kollár and Kovács: a generalization of rational singularities analogous to divisorially log terminal singularities; see [Kol13]); the interested reader can look it up, or simply assume that $D = 0$.

COROLLARY 3.10. *Suppose that (X, D) is a pair with D a reduced Weil divisor. Further suppose that H is a Cartier divisor on X not having any components in common with D such that $(H, D \cap H)$ is Du Bois and such that $(X \setminus H, D \setminus H)$ is a rational pair in the sense of Kollár and Kovács. Then (X, D) is a rational pair in the sense of Kollár and Kovács.*

Proof. We follow [KS16b, Proof of Theorem 7.1]. It goes through word for word once one observes that $\mathcal{O}_X(-D)$ is Cohen–Macaulay, which follows immediately from Theorem 3.8. □

This corollary is also an analog of [FW89, Proposition 2.13] and [HMS14, Corollary A.4].

4. Applications to local cohomology

In this section we give many applications of Lemma 3.3 to local cohomology; several of these applications are quite easy to prove. However, they provide strong and surprising characteristic-0 analogs and generalizations of results in [HR76, SW05, SW07, Var13, DN16] as well as answer their questions. Moreover, our results can give a generalization of the classical Kodaira vanishing theorem.

4.1 Injectivity of Ext and a generalization of the Kodaira vanishing theorem

Our first application gives a solution to [EMS00, Question 6.2] on the injectivity of Ext in characteristic 0, which parallels its characteristic $p > 0$ analog [SW07, Theorem 1.3].

PROPOSITION 4.1. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring essentially of finite type over $\mathbb{C}$, and let $I \subseteq R$ be an ideal such that R/I has Du Bois singularities. Then the natural map $\operatorname{Ext}_R^j(R/I, R) \to H_I^j(R)$ is injective for every j.*

Proof. By Lemma 3.3 applied to $S = R/I^t$ and applying local duality, we know that

$$\operatorname{Ext}_R^j(R/I, R) \to \operatorname{Ext}_R^j(R/I^t, R)$$

is injective for every j and every $t > 0$. Now taking a direct limit, we obtain the desired injectivity $\operatorname{Ext}_R^j(R/I, R) \to H_I^j(R)$. □

Remark 4.2. Theorem 1.3 of [SW07] proves the same injectivity result in characteristic $p > 0$ when R/I is F-pure (and when R is regular). Perhaps it is worth pointing out that Proposition 4.1 fails in general if we replace Du Bois by F-injective: Example 3.5 is a counterexample, because there we have that $S = A/I$ is F-injective but $H_\mathfrak{m}^1(A/I^{[p]}) \to H_\mathfrak{m}^1(A/I)$ is not surjective. Hence, applying local duality shows that

$$\operatorname{Ext}_A^{\dim A-1}(A/I, A) \to \operatorname{Ext}_A^{\dim A-1}(A/I^{[p]}, A)$$

is not injective, so neither is $\operatorname{Ext}_A^{\dim A-1}(A/I, A) \to H_I^{\dim A-1}(A)$.

An immediate corollary of Proposition 4.1 is the following, which is a characteristic-0 analog of [DN16, Theorem 7.3] (this also answers [DN16, Question 7.5], which is motivated by Stillman's conjecture).

COROLLARY 4.3. *Let R be a regular ring of essentially finite type over $\mathbb{C}$. Let $I \subseteq R$ be an ideal such that $S = R/I$ has Du Bois singularities. Then $\operatorname{pd}_R S \leqslant \nu(I)$, where $\nu(I)$ denotes the minimal number of generators of I.*

Proof. For every $j > \nu(I)$ and every maximal ideal $\mathfrak{m}$ of R, we have $H_{IR_\mathfrak{m}}^j(R_\mathfrak{m}) = 0$. Therefore, by Proposition 4.1, we have $\operatorname{Ext}_{R_\mathfrak{m}}^j(S_\mathfrak{m}, R_\mathfrak{m}) = 0$ for every $j > \nu(I)$. Since $\operatorname{pd}_{R_\mathfrak{m}} S_\mathfrak{m} < \infty$, we have

$$\operatorname{pd}_{R_\mathfrak{m}} S_\mathfrak{m} = \sup\{j \mid \operatorname{Ext}_{R_\mathfrak{m}}^j(S_\mathfrak{m}, R_\mathfrak{m}) \neq 0\} \leqslant \nu(I)^3.$$

Because this is true for every maximal ideal $\mathfrak{m}$, we have $\operatorname{pd}_R S \leqslant \nu(I)$. □

---

[3] This follows from applying the functor $\operatorname{Hom}_{R_\mathfrak{m}}(-, R_\mathfrak{m})$ to a minimal free resolution of $S_\mathfrak{m}$ and observing that the matrix defining the maps has elements contained in $\mathfrak{m}$. See also [DGI06, 2.4].

Next, we want to prove a stronger form of Proposition 4.1 in the graded case. We will first need the following criterion of Du Bois singularities for $\mathbb{N}$-graded rings. This result and related results should be well known to experts (or at least well believed by experts): for example [Ma15, Theorem 4.4], [BST13, Lemma 2.14], [KS16b, Lemma 2.12] or [GK14]. However, all these references only deal with the case that R is the section ring of a projective variety with respect to a certain ample line bundle, while here we allow arbitrary $\mathbb{N}$-graded rings which are not even normal. We could not find a reference that handles the generality that we will need and hence we write down a careful argument.

PROPOSITION 4.4. *Let* $(R, \mathfrak{m})$ *be a reduced Noetherian $\mathbb{N}$-graded $(R_0 = k)$-algebra with $\mathfrak{m}$ the unique homogeneous maximal ideal. Suppose that R_P is Du Bois for all $P \neq \mathfrak{m}$. Then we have*

$$h^i(\underline{\Omega}_R^0) \cong [H_{\mathfrak{m}}^{i+1}(R)]_{>0}$$

for every $i > 0$ and

$$h^0(\underline{\Omega}_R^0)/R \cong [H_{\mathfrak{m}}^1(R)]_{>0}.$$

In particular, R is Du Bois if and only if $[H_{\mathfrak{m}}^i(R)]_{>0} = 0$ for every $i > 0$.

Proof. Let $R^\natural$ denote the Rees algebra of R with respect to the natural filtration $R_{\geqslant t}$. That is, $R^\natural = R \oplus R_{\geqslant 1} \oplus R_{\geqslant 2} \oplus \cdots$. Let $Y = \operatorname{Proj} R^\natural$. We first claim that Y is Du Bois: Y is covered by $D_+(f)$ for homogeneous elements $f \in R_{\geqslant t}$ and $t \in \mathbb{N}$.

If $\deg f > t$, then $[R_f^\natural]_0 \cong R_f$ is Du Bois. If $\deg f = t$, then $[R_f^\natural]_0 \cong [R_f]_{\geqslant 0}$ is also Du Bois[4] (see [KSSW09, Lemma 5.4] for an analogous analysis on rational singularities).

Since R and thus $R^\natural$ are Noetherian, there exists n such that $R_{\geqslant nt} = (R_{\geqslant n})^t$ for every $t \geqslant 1$ [Bou98, ch. III, §3, Proposition 3]. Let $I = (R_{\geqslant n})$; then we immediately see that $Y \cong \operatorname{Proj}(R \oplus I \oplus I^2 \oplus \cdots)$ is the blow up of $\operatorname{Spec} R$ at I. We define the exceptional divisor to be $E \cong \operatorname{Proj}(R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots)$. We next claim that
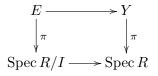
$$(R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots)_{\mathrm{red}} \cong R_0 \oplus R_n \oplus R_{2n} \oplus \cdots. \tag{4.4.1}$$

The point is that, for $\overline{x} \in I^t/I^{t+1} = R_{\geqslant nt}/R_{\geqslant n(t+1)}$, if $x \in R_{nt+a}$ for some $a > 0$, then we can pick $b > 0$ such that $ba > n$; we then have

$$x^b \in R_{bnt+ba} \subseteq R_{\geqslant n(bt+1)}.$$

But this means that $\overline{x}^b = 0$ in I^{bt}/I^{bt+1} and thus $\overline{x}$ is nilpotent in $R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$. This proves (4.4.1).

By (4.4.1), we have $E_{\mathrm{red}} \cong \operatorname{Proj} R_0 \oplus R_n \oplus R_{2n} \oplus \cdots \cong \operatorname{Proj} R$ is Du Bois (because $[R_f]_0$ is Du Bois for every homogeneous $f \in R$). We consider the following commutative diagram.

$$
\begin{array}{ccc}
E & \longrightarrow & Y \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\operatorname{Spec} R/I & \longrightarrow & \operatorname{Spec} R
\end{array}
$$

---

[4] In general, if S is a $\mathbb{Z}$-graded ring, then $S_{\geqslant 0} \cong S[z]_0$, where $\deg z = -1$. Hence, if S is Du Bois, then $S_{\geqslant 0}$, being a summand of $S[z]$, is also Du Bois [Kov99].

Since Y, E_{red} and $(\operatorname{Spec} R/I)_{\mathrm{red}} \cong \operatorname{Spec} k$ are all Du Bois by the above discussion, the exact triangle $\underline{\Omega}_R^0 \to \mathbf{R}\pi_*\underline{\Omega}_Y^0 \oplus \underline{\Omega}_{R/I}^0 \to \mathbf{R}\pi_*\underline{\Omega}_E^0 \xrightarrow{+1}$ reduces to

$$\underline{\Omega}_R^0 \to \mathbf{R}\pi_*\mathcal{O}_Y \oplus k \to \mathbf{R}\pi_*\mathcal{O}_{E_{\mathrm{red}}} \xrightarrow{+1} . \tag{4.4.2}$$

Next, we study the map $\mathbf{R}\pi_*\mathcal{O}_Y \to \mathbf{R}\pi_*\mathcal{O}_{E_{\mathrm{red}}}$ using the Čech complex. We pick $x_1,\ldots,$ $x_m \in I = (R_{\geqslant n})$ in $R \oplus I \oplus I^2 \oplus \cdots$ such that:

(a) the internal degree $\deg x_i = n$ for each x_i (in other words, $x_i \in R_n \subseteq R_{\geqslant n} = I$);

(b) the images $\overline{x}_1,\ldots,\overline{x}_m$ in $(R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots)_{\mathrm{red}} = R_0 \oplus R_n \oplus R_{2n} \oplus \cdots$ are algebra generators of $R_0 \oplus R_n \oplus R_{2n} \oplus \cdots$ over $R_0 = k$.

Note that conditions (a) and (b) together imply that the radical of $(x_1,\ldots,x_m)$ in $R \oplus I \oplus I^2 \oplus \cdots$ is the irrelevant ideal $I \oplus I^2 \oplus \cdots$. In particular, $\{D_+(x_i)\}_{1 \leqslant i \leqslant m}$ forms an affine open cover of Y. The point is that for any $y \in I^t = R_{\geqslant tn}$, $y^n \in I^{tn} = R_{\geqslant tn^2}$ as an element in $R \oplus I \oplus I^2 \oplus \cdots$ is always contained in the ideal $(x_1,\ldots,x_m)$; this is because the internal degree of y^n is divisible by n, so it can be written as a sum of monomials in x_i by (b).

The natural map $\mathcal{O}_Y \to \mathcal{O}_{E_{\mathrm{red}}}$ induces a map between the sth spot of the Čech complexes of $\mathcal{O}_Y$ and $\mathcal{O}_{E_{\mathrm{red}}}$ with respect to the affine cover $\{D_+(x_i)\}_{1 \leqslant i \leqslant m}$. The induced map on Čech complexes can be explicitly described as follows (all the direct sums in the following diagram are taken over all s-tuples $1 \leqslant i_1 < \cdots < i_s \leqslant m$):

$$
\begin{array}{ccc}
\bigoplus \mathcal{O}_Y(D_+(x_{i_1}x_{i_2}\cdots x_{i_s})) & \xrightarrow{\hspace{3cm}} & \bigoplus \mathcal{O}_{E_{\mathrm{red}}}(D_+(\overline{x}_{i_1}\overline{x}_{i_2}\cdots \overline{x}_{i_s})) \\
\cong \downarrow & & \cong \downarrow \\
\bigoplus \left\{ \dfrac{y}{(x_{i_1}x_{i_2}\cdots x_{i_s})^h} \ \middle| \ h>0, y \in I^{sh} = R_{\geqslant nsh} \right\} & \xrightarrow{\hspace{1.5cm}} & \bigoplus \left\{ \dfrac{\overline{y}}{(\overline{x}_{i_1}\overline{x}_{i_2}\cdots \overline{x}_{i_s})^h} \ \middle| \ h>0, y \in R_{nsh} \right\} \\
\cong \downarrow & & \cong \downarrow \\
\bigoplus [R_{x_{i_1}x_{i_2}\cdots x_{i_s}}]_{\geqslant 0} & \xrightarrow{\hspace{1.5cm}\phi\hspace{1.5cm}} & \bigoplus [R_{x_{i_1}x_{i_2}\cdots x_{i_s}}]_0
\end{array}
$$

The induced map on the second line takes the element $y/(x_{i_1}x_{i_2}\cdots x_{i_s})^n$ to $\overline{y}/(\overline{x}_{i_1}\overline{x}_{i_2}\cdots \overline{x}_{i_s})^n$, where $\overline{y}$ denotes the image of y in R_{nsh}. Hence, the same map ϕ on the third line is exactly 'taking the degree-0 part'. Therefore, we have

$$\mathbf{R}^i\pi_*\mathcal{O}_Y \cong H^i(Y, \mathcal{O}_Y) = [H_{\mathfrak{m}}^{i+1}(R)]_{\geqslant 0},$$

while

$$\mathbf{R}^i\pi_*\mathcal{O}_{E_{\mathrm{red}}} \cong H^i(E_{\mathrm{red}}, \mathcal{O}_{E_{\mathrm{red}}}) \cong [H_{\mathfrak{m}}^{i+1}(R)]_0$$

for every $i \geqslant 1$, and the map $\mathbf{R}^i\pi_*\mathcal{O}_Y \to \mathbf{R}^i\pi_*\mathcal{O}_{E_{\mathrm{red}}}$ is taking the degree-0 part. Therefore, taking cohomology of (4.4.2), we have

$$h^i(\underline{\Omega}_R^0) \cong [H_{\mathfrak{m}}^{i+1}(R)]_{>0} \quad \text{for every } i \geqslant 2. \tag{4.4.3}$$

Moreover, for $i = 0, 1$, the cohomology of (4.4.2) gives

$$0 \to h^0(\underline{\Omega}_R^0) \to H^0(Y, \mathcal{O}_Y) \oplus k \xrightarrow{\phi} H^0(E_{\mathrm{red}}, \mathcal{O}_{E_{\mathrm{red}}}) \to h^1(\underline{\Omega}_R^0) \to [H_{\mathfrak{m}}^2(R)]_{>0} \to 0. \tag{4.4.4}$$

A similar Čech complex computation as above shows that

$$H^0(Y, \mathcal{O}_Y) = \ker(\oplus[R_{x_i}]_{\geqslant 0} \to \oplus[R_{x_ix_j}]_{\geqslant 0}),$$

while $H^0(E_{\mathrm{red}}, \mathcal{O}_{E_{\mathrm{red}}}) = \ker(\oplus[R_{x_i}]_0 \to \oplus[R_{x_i x_j}]_0)$. Therefore, ϕ is surjective, which implies that

$$h^1(\underline{\Omega}^0_R) \cong [H^2_{\mathfrak{m}}(R)]_{>0}. \tag{4.4.5}$$

Taking the degree > 0 part of (4.4.4), we get an exact sequence

$$0 \to h^0(\underline{\Omega}^0_R)_{>0} \to \oplus[R_{x_i}]_{>0} \to \oplus[R_{x_i x_j}]_{>0}.$$

This implies that $h^0(\underline{\Omega}^0_R)_{>0} \cong (\Gamma(\operatorname{Spec} R \backslash \mathfrak{m}, \mathcal{O}_{\operatorname{Spec} R}))_{>0}$ and so

$$h^0(\underline{\Omega}^0_R)_{>0}/R_{>0} \cong [H^1_{\mathfrak{m}}(R)]_{>0}. \tag{4.4.6}$$

Finally, we notice that $h^0(\underline{\Omega}^0_R) \cong R^{\mathrm{sn}}$ is the seminormalization of R [Sai00, 5.2]. We know that $R^{\mathrm{sn}}_0 \subseteq R^{\mathrm{sn}}$ is reduced. But we can also view R^{sn}_0 as the quotient $R^{\mathrm{sn}}/R^{\mathrm{sn}}_{>0}$; in particular, the prime ideals of R^{sn}_0 correspond to prime ideals of R^{sn} that contain $R^{\mathrm{sn}}_{>0}$, so they all contract to $\mathfrak{m}$ in R. Since seminormalization induces a bijection on spectrum, R^{sn}_0 has a unique prime ideal. Thus, R^{sn}_0, being a reduced Artinian local ring, must be a field. Since seminormalization also induces isomorphism on residue fields, $R^{\mathrm{sn}}_0 = k$ and thus $h^0(\underline{\Omega}^0_R)_0 \cong R^{\mathrm{sn}}_0 = R_0$. Hence, (4.4.6) tells us that

$$h^0(\underline{\Omega}^0_R)/R \cong [H^1_{\mathfrak{m}}(R)]_{>0}. \tag{4.4.7}$$

Now (4.4.3), (4.4.5) and (4.4.7) together finish the proof. $\qquad\square$

Now we prove our result on injectivity of Ext. Later we will see that this theorem can be viewed as a generalization of the Kodaira vanishing theorem.

THEOREM 4.5. *Let* $(R, \mathfrak{m})$ *be a reduced Noetherian* $\mathbb{N}$-*graded* $(R_0 = \mathbb{C})$-*algebra with* $\mathfrak{m}$ *the unique homogeneous maximal ideal. Suppose that* R_P *is Du Bois for all* $P \neq \mathfrak{m}$. *Write* $R = A/I$, *where* $A = \mathbb{C}[x_1, \ldots, x_n]$ *is a polynomial ring with* $\deg x_i = d_i > 0$ *and* I *is a homogeneous ideal. Then the natural degree-preserving map* $\operatorname{Ext}^j_A(R, A) \to H^j_I(A)$ *induces an injection*

$$[\operatorname{Ext}^j_A(R, A)]_{\geqslant -d} \hookrightarrow [H^j_I(A)]_{\geqslant -d}$$

for every j, *where* $d = \sum d_i$.

Proof. We have the hypercohomology spectral sequence

$$H^p_{\mathfrak{m}}(h^q(\underline{\Omega}^0_R)) \Rightarrow \mathbb{H}^{p+q}_{\mathfrak{m}}(\underline{\Omega}^0_R).$$

Since R is Du Bois away from $V(\mathfrak{m})$, $h^q(\underline{\Omega}^0_R)$ has finite length when $q \geqslant 1$. Thus, we know that $H^p_{\mathfrak{m}}(h^q(\underline{\Omega}^0_R)) = 0$ unless $p = 0$ or $q = 0$. We also have that $H^0_{\mathfrak{m}}(h^i(\underline{\Omega}^0_R)) \cong h^i(\underline{\Omega}^0_R)$ for $i \geqslant 1$. Hence, the above spectral sequence carries the data of a long exact sequence

$$\begin{aligned}
0 \to H^1_{\mathfrak{m}}(h^0(\underline{\Omega}^0_R)) &\to \mathbb{H}^1_{\mathfrak{m}}(\underline{\Omega}^0_R) \to h^1(\underline{\Omega}^0_R) \to \\
&\to H^2_{\mathfrak{m}}(h^0(\underline{\Omega}^0_R)) \to \mathbb{H}^2_{\mathfrak{m}}(\underline{\Omega}^0_R) \to h^2(\underline{\Omega}^0_R) \to \\
&\quad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
&\to H^i_{\mathfrak{m}}(h^0(\underline{\Omega}^0_R)) \to \mathbb{H}^i_{\mathfrak{m}}(\underline{\Omega}^0_R) \to h^i(\underline{\Omega}^0_R) \to \cdots
\end{aligned} \tag{4.5.1}$$

We also have a short exact sequence $0 \to R \to h^0(\underline{\Omega}^0_R) \to [H^1_{\mathfrak{m}}(R)]_{>0} \to 0$ by Proposition 4.4. Therefore, the long exact sequence of local cohomology and the observation that $[H^1_{\mathfrak{m}}(R)]_{>0}$ has finite length tell us that

$$H^1_{\mathfrak{m}}(h^0(\underline{\Omega}^0_R)) \cong H^1_{\mathfrak{m}}(R)/[H^1_{\mathfrak{m}}(R)]_{>0}$$

and

$$H^i_{\mathfrak{m}}(h^0(\underline{\Omega}^0_R)) \cong H^i_{\mathfrak{m}}(R) \quad \text{for every } i \geqslant 2.$$

Now taking the degree $\leqslant 0$ part of (4.5.1) and again using Proposition 4.4 yields

$$[H^i_{\mathfrak{m}}(R)]_{\leqslant 0} \cong [\mathbb{H}^i_{\mathfrak{m}}(\underline{\Omega}^0_R)]_{\leqslant 0} \quad \text{for every } i. \tag{4.5.2}$$

At this point, note that by the Matlis dual of Lemma 3.2 (see the proof of Lemma 3.3 applied to $S = A/J$ for $\sqrt{J} = I$, and thus $S_{\text{red}} = A/I = R$), we always have a surjection $H^i_{\mathfrak{m}}(A/J) \twoheadrightarrow \mathbb{H}^i_{\mathfrak{m}}(\underline{\Omega}^0_R)$ for every i and $\sqrt{J} = I$. Taking the degree $\leqslant 0$ part and applying (4.5.2), we thus get that

$$[H^i_{\mathfrak{m}}(A/J)]_{\leqslant 0} \twoheadrightarrow [H^i_{\mathfrak{m}}(R)]_{\leqslant 0} \tag{4.5.3}$$

is surjective for every i and $\sqrt{J} = I$. Now taking $J = I^t$ and applying graded local duality (we refer to [BS98] for definitions and standard properties of graded canonical modules, graded injective hulls and graded local duality, but we emphasize here that the graded canonical module of A is $A(-d)$), we have that

$$[\text{Ext}^j_A(R, A(-d))]_{\geqslant 0} \hookrightarrow [\text{Ext}^j_A(A/I^t, A(-d))]_{\geqslant 0}$$

is injective for every j and t. So, after taking a direct limit and a degree shift, we have

$$[\text{Ext}^j_A(R, A)]_{\geqslant -d} \hookrightarrow [H^j_I(A)]_{\geqslant -d}$$

is injective for every j. This finishes the proof. $\qquad\square$

Remark 4.6. The dual form of Theorem 4.5 says that $[H^i_{\mathfrak{m}}(A/J)]_t \twoheadrightarrow [H^i_{\mathfrak{m}}(R)]_t$ is surjective for every i, every $t \leqslant 0$ and every $\sqrt{J} = I$; see (4.5.3). When $R = A/I$ has an isolated singularity, A is standard graded and $t = 0$, this was proved in [Var13, Proposition 3.8]. Therefore, our Theorem 4.5 greatly generalizes this result.

In general, we cannot expect that $[\text{Ext}^j_A(R, A)]_{<-d} \hookrightarrow [H^j_I(A)]_{<-d}$ is injective under the hypothesis of Theorem 4.5 (even if R is an isolated singularity). Consider the following example.

Example 4.7 (Cf. [SW07, Example 3.5]). Let $R = \mathbb{C}[s^4, s^3 t, st^3, t^4]$. Then we can write $R = A/I$, where $A = \mathbb{C}[x, y.z, w]$ with standard grading (i.e., x, y, z, w all have degree one). It is straightforward to check that R is an isolated singularity with

$$H^1_{\mathfrak{m}}(R) = [H^1_{\mathfrak{m}}(R)]_{>0} \neq 0$$

(in particular, depth $R = 1$). By graded local duality, we have $[\text{Ext}^3_A(R, A)]_{<-4} \neq 0$. On the other hand, using standard vanishing theorems in [HL90, Theorem 2.9], we know that $H^3_I(A) = 0$. Therefore, the map $[\text{Ext}^3_A(R, A)]_{<-4} \to [H^3_I(A)]_{<-4}$ is not injective.

An important consequence of Theorem 4.5 (in fact, we only need the injectivity in degree $> -d$) is the following vanishing result.

THEOREM 4.8. *Let $(R, \mathfrak{m})$ be a Noetherian $\mathbb{N}$-graded $(R_0 = \mathbb{C})$-algebra with $\mathfrak{m}$ the unique homogeneous maximal ideal. Suppose that R_P is Du Bois for all $P \neq \mathfrak{m}$ and $H^i_{\mathfrak{m}}(R)$ has finite length for some i. Then $[H^i_{\mathfrak{m}}(R)]_{<0} = 0$.*

In particular, if R is equidimensional and is Cohen–Macaulay Du Bois on $\text{Spec } R - \{\mathfrak{m}\}$, then $[H^i_{\mathfrak{m}}(R)]_{<0} = 0$ for every $i < \dim R$.

Proof. Let $R = A/I$, where $A = \mathbb{C}[x_1, \ldots, x_n]$ is a (not necessarily standard graded) polynomial ring and I is a homogeneous ideal. Set $\deg x_i = d_i > 0$ and $d = \sum d_i$. The graded canonical module of A is $A(-d)$. By graded local duality,

$$\mathrm{Ext}_A^{n-i}(R, A)(-d) \cong H_{\mathfrak{m}}^i(R)^*.$$

Therefore, if $[H_{\mathfrak{m}}^i(R)]_{-j} \neq 0$ for some $j > 0$, then $[\mathrm{Ext}_A^{n-i}(R, A)]_{j-d} \neq 0$ for some $j > 0$. By Theorem 4.5, we have an injection

$$[\mathrm{Ext}_A^{n-i}(R, A)]_{j-d} \hookrightarrow [H_I^{n-i}(A)]_{j-d}.$$

Since $H_{\mathfrak{m}}^i(R)$ has finite length, $\mathrm{Ext}_A^{n-i}(R, A)$ also has finite length. Hence, the natural degree-preserving map $\mathrm{Ext}_A^{n-i}(R, A) \to H_I^{n-i}(A)$ factors through

$$\mathrm{Ext}_A^{n-i}(R, A) \to H_{\mathfrak{m}}^0 H_I^{n-i}(A) \to H_I^{n-i}(A).$$

Taking the degree-$(j - d)$ part, we thus get an injection

$$[\mathrm{Ext}_A^{n-i}(R, A)]_{j-d} \to [H_{\mathfrak{m}}^0 H_I^{n-i}(A)]_{j-d}.$$

It follows that $[H_{\mathfrak{m}}^0 H_I^{n-i}(A)]_{j-d} \neq 0$ for some $j > 0$. However, $H_{\mathfrak{m}}^0 H_I^{n-i}(A)$ is an Eulerian graded $\mathscr{D}$-module supported only at $\mathfrak{m}$. Thus, by [MZ14, Theorem 1.2],[5] the socle of $H_{\mathfrak{m}}^0 H_I^{n-i}(A)$ is concentrated in degree $-d$, so that $[H_{\mathfrak{m}}^0 H_I^{n-i}(A)]_{>-d} = 0$, which is a contradiction. □

If R is the section ring of a normal Cohen–Macaulay and Du Bois projective variety X with respect to an ample line bundle $\mathscr{L}$, then $[H_{\mathfrak{m}}^i(R)]_{<0} = 0$ is exactly the Kodaira vanishing for X (which is well known; for example see [Ma15] or [Pat15]). But Theorem 4.8 can handle more general R, i.e., R need not be a section ring of an ample line bundle. If R is normal, then any graded ring is the section ring of some $\mathbb{Q}$-divisor [Dem88] and, in that case, our results yield variants and consequences of Kawamata–Viehweg vanishing (also see [Wat81, Lemma 2.1 and Proposition 2.2]). But for general graded rings we do not know how to view them as section rings. Thus, our results (Theorems 4.5 and 4.8) should be viewed as generalizations of the Kodaira vanishing theorem for Cohen–Macaulay Du Bois projective varieties. It would also be natural to try to generalize the results of Proposition 4.4 through Theorem 4.8 to the context of Du Bois pairs. One particular obstruction is the use of the Eulerian graded $\mathscr{D}$-module at the end of the proof of Theorem 4.8.

4.2 Set-theoretic Cohen–Macaulayness

Our next application is a characteristic-0 analog of [SW05, Lemma 3.1] on set-theoretic Cohen–Macaulayness. Recall that an ideal I in a regular ring R is *set-theoretically Cohen–Macaulay* if there exists an ideal J such that $\sqrt{I} = \sqrt{J}$ and R/J is Cohen–Macaulay.

PROPOSITION 4.9. *Let $(R, \mathfrak{m})$ be a regular local ring essentially of finite type over $\mathbb{C}$ and let $I \subseteq R$ be an ideal. If R/I is Du Bois but not Cohen–Macaulay, then the ideal I is not set-theoretically Cohen–Macaulay.*

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[5] [MZ14, Theorem 1.2] assumes that $A = \mathbb{C}[x_1, \ldots, x_n]$ has standard grading, i.e., $d_i = 1$ and hence $d = \sum d_i = n$. However, the same proof can be adapted to the general case: one only needs to replace the Euler operator $\sum x_i \partial_i$ by $\sum d_i x_i \partial_i$. The reader is referred to [Put15, § 2] for a discussion on this.

Proof. Suppose that $I = \sqrt{J}$ for some J such that R/J is Cohen–Macaulay. Applying Lemma 3.3 to $S = R/J$, we find that for every $i < \dim R/I$,

$$0 = H_{\mathfrak{m}}^i(R/J) \twoheadrightarrow H_{\mathfrak{m}}^i(R/I)$$

is surjective and thus R/I is Cohen–Macaulay, which is a contradiction. □

In the graded characteristic-0 case, we have a stronger criterion for set-theoretic Cohen–Macaulayness.

COROLLARY 4.10. *Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring with possibly non-standard grading. Let I be a homogeneous ideal of R such that R/I is Du Bois on $\operatorname{Spec} R - \{\mathfrak{m}\}$ (e.g., R/I has an isolated singularity at $\{\mathfrak{m}\}$).*

Suppose that $[H_{\mathfrak{m}}^i(R/I)]_{\leqslant 0} \neq 0$ for some $i < \dim R/I$ (e.g., R/I is not Cohen–Macaulay on $\operatorname{Spec} R - \{\mathfrak{m}\}$, or $H^i(X, \mathcal{O}_X) \neq 0$ for $X = \operatorname{Proj} R/I$). Then I is not set-theoretically Cohen–Macaulay.

Proof. Suppose that $I = \sqrt{J}$ for some J such that R/J is Cohen–Macaulay. Applying the dual form of Theorem 4.5 (see Remark 4.6), we get

$$0 = [H_{\mathfrak{m}}^i(R/J)]_{\leqslant 0} \twoheadrightarrow [H_{\mathfrak{m}}^i(R/I)]_{\leqslant 0}$$

is surjective for every $i < \dim R/I$. This clearly contradicts our hypothesis. □

We point out the following example as an application.

Example 4.11. Let k be a field and let $E_k \subseteq \mathbb{P}_k^2$ be a smooth elliptic curve over k. We want to study the defining ideal of the Segre embedding $E_k \times \mathbb{P}_k^1 \subseteq \mathbb{P}_k^5$. We let $k[x_0, \ldots, x_5]/I = A/I$ be this homogeneous coordinate ring. It is well known that A/I is not Cohen–Macaulay.

(a) k has characteristic $p > 0$ and E_k is an ordinary elliptic curve. In this case it is well known that A/I is F-pure, so [SW05, Lemma 3.1] shows that I is not set-theoretically Cohen–Macaulay.

(b) k has characteristic $p > 0$ and E_k is supersingular. We want to point out that, at least when k is F-finite, I is still not set-theoretically Cohen–Macaulay. This answers a question in [SW05, Remark 3.4]. Suppose that there exists J such that $\sqrt{J} = I$ and A/J is Cohen–Macaulay. Let $e \gg 0$ be such that $I^{[p^e]} \subseteq J$. The composite of the Frobenius map on A/I with the natural surjection $A/I \xrightarrow{F} A/I^{[p^e]} \twoheadrightarrow A/J$ makes A/J a small Cohen–Macaulay algebra over A/I (note that k, and hence A, is F-finite). However, by a result of Bhatt [Bha14, Example 3.11], A/I does not have any small Cohen–Macaulay algebra, which is a contradiction.

(c) k has characteristic 0. In this case, it is easy to check using Proposition 4.4 that A/I is Du Bois. Hence, Proposition 4.9 immediately shows that I is not set-theoretically Cohen–Macaulay. This example was originally obtained in [SW05, Theorem 3.3] using reduction to characteristic $p > 0$. Thus, our Proposition 4.9 can be viewed as a vast generalization of that result.

It is worth pointing out that in Example 4.11, we know that $H_{\mathfrak{m}}^i(A/J) \twoheadrightarrow H_{\mathfrak{m}}^i(A/I)$ for every i and every $\sqrt{J} = I$ in characteristic 0, since A/I is Du Bois. In characteristic $p > 0$, we have $H_{\mathfrak{m}}^i(A/J) \twoheadrightarrow H_{\mathfrak{m}}^i(A/I)$ for every i and every $\sqrt{J} = I$ when E_k is ordinary; however, it is straightforward to check that $H_{\mathfrak{m}}^2(A/I^{[p]}) \to H_{\mathfrak{m}}^2(A/I)$ is not surjective (it is the zero map) when E_k is supersingular. Therefore, the surjective property proved in Lemma 3.3 does not pass to reduction mod $p \gg 0$.

4.3 Koszul cohomology versus local cohomology

Our last application in this section is a strengthening of the main result of [HR76]. We start by proving a general result, which is characteristic-free. The proof is inspired by [Sch82].

THEOREM 4.12. *Let R be a Noetherian $\mathbb{N}$-graded k-algebra, where $k = R_0$ is an arbitrary field. Let $\mathfrak{m}$ be the unique homogenous maximal ideal. If $H^r_{\mathfrak{m}}(R) = [H^r_{\mathfrak{m}}(R)]_0$ for every $r < n = \dim R$, then $[H^r(\underline{x}, R)]_0 \cong H^r_{\mathfrak{m}}(R)$ for every $r < n$ and every homogeneous system of parameters $\underline{x} = x_1, \ldots, x_n$, where $H^r(\underline{x}, R)$ denotes the rth Koszul cohomology of $\underline{x}$. In other words, it is not necessary to take a direct limit when computing the local cohomology.*

Proof. We fix $\underline{x} = x_1, \ldots, x_n$, a homogeneous system of parameters. Let $\deg x_i = d_i > 0$. Consider the graded Koszul complex

$$K_{\bullet} : 0 \to R(-d_1 - d_2 - \cdots - d_n) \to \cdots \to \oplus R(-d_i) \to R \to 0.$$

After we apply $\operatorname{Hom}_R(-, R)$, we obtain the graded Koszul cocomplex

$$K^{\bullet} : 0 \to R \to \oplus R(d_i) \to \cdots \to R(d_1 + d_2 + \cdots + d_n) \to 0.$$

We note that $K^{\bullet}$ lives in cohomology degree $0, 1, \ldots, n$.

Let $\omega_R^{\bullet}$ be the graded normalized dualizing complex of R; thus, $\omega_R = h^{-n}\omega_R^{\bullet}$ is the graded canonical module of R. Let $(-)^* = \operatorname{Hom}_R(-, {}^*E)$, where ${}^*E$ is the graded injective hull of k. We have a triangle

$$\omega_R[n] \to \omega_R^{\bullet} \to \tau_{>-n}\omega_R^{\bullet} \xrightarrow{+1} .$$

Applying $\mathbf{R}\operatorname{Hom}_R(K^{\bullet}, -)$, we get

$$\mathbf{R}\operatorname{Hom}_R(K^{\bullet}, \omega_R[n]) \to \mathbf{R}\operatorname{Hom}_R(K^{\bullet}, \omega_R^{\bullet}) \to \mathbf{R}\operatorname{Hom}_R(K^{\bullet}, \tau_{>-n}\omega_R^{\bullet}) \xrightarrow{+1} .$$

Applying $\operatorname{Hom}_R(-, {}^*E)$ and using graded local duality, we obtain

$$\mathbf{R}\operatorname{Hom}_R(K^{\bullet}, \tau_{>-n}\omega_R^{\bullet})^* \to K^{\bullet} \to \mathbf{R}\operatorname{Hom}_R(K^{\bullet}, \omega_R[n])^* \xrightarrow{+1} .$$

Note that $\mathbf{R}\operatorname{Hom}_R(K^{\bullet}, \omega_R[n])^*$ lives in cohomological degree $n, n+1, \ldots, 2n$; hence, we obtain a graded isomorphism in the derived category:

$$\tau^{<n}K^{\bullet} \cong \tau^{<n}\mathbf{R}\operatorname{Hom}_R(K^{\bullet}, \tau_{>-n}\omega_R^{\bullet})^*.$$

Therefore, for every $r < n$, we have

$$h^r(K^{\bullet}) \cong h^r(\mathbf{R}\operatorname{Hom}_R(K^{\bullet}, \tau_{>-n}\omega_R^{\bullet})^*).$$

At this point, notice that $H^r_{\mathfrak{m}}(R) = [H^r_{\mathfrak{m}}(R)]_0$ for every $r < n$ implies that R is Buchsbaum by [Sch82, Theorem 3.1]. This means that $\tau_{>-n}\omega_R^{\bullet}$ is quasi-isomorphic to a complex of graded k-vector spaces. Moreover, we know that all these graded vector spaces have degree 0 because $H^i_{\mathfrak{m}}(R) = [H^i_{\mathfrak{m}}(R)]_0$ for every $i < n$. In other words, we have

$$\tau_{>-n}\omega_R^{\bullet} \cong 0 \to k^{s_{n-1}} \to k^{s_{n-2}} \to \cdots \to k^{s_1} \to k^{s_0} \to 0,$$

where the complex on the right-hand side has zero differentials and k^{s_i} has internal degree 0 and sits in cohomology degree $-i$, with $s_i = \dim_k [H^i_{\mathfrak{m}}(R)]_0$.

Recall that $d_i > 0$; hence, by keeping track of the internal degrees, we see that

$$[\mathbf{R}\operatorname{Hom}_R(K^\bullet, \tau_{>-n}\omega_R^\bullet)]_0 \cong \tau_{>-n}\omega_R^\bullet.$$

Now, by graded local duality, we have

$$[H^r(\underline{x}, R)]_0 = [h^r(K^\bullet)]_0 \cong [h^r(\mathbf{R}\operatorname{Hom}_R(K^\bullet, \tau_{>-n}\omega_R^\bullet)^*)]_0 \cong h^r((\tau_{>n}\omega_R^\bullet)^*) \cong H_{\mathfrak{m}}^r(R)$$

for every $r < n$. This finishes the proof. $\qquad\square$

Now we can prove the following extension of the main result of [HR76].

COROLLARY 4.13. *Let R be a Noetherian $\mathbb{N}$-graded $(R_0 = k)$-algebra with $\mathfrak{m}$ the unique homogeneous maximal ideal. Suppose that R is equidimensional and Cohen–Macaulay on $\operatorname{Spec} R - \{\mathfrak{m}\}$. Assume one of the following:*

(a) *k has characteristic $p > 0$ and R is F-injective;*

(b) *k has characteristic 0 and R is Du Bois.*

Then $[H^r(\underline{x}, R)]_0 \cong H_{\mathfrak{m}}^r(R)$ for every $r < n = \dim R$ and every homogeneous system of parameters $\underline{x} = x_1, \ldots, x_n$, where $H^r(\underline{x}, R)$ denotes the rth Koszul cohomology of $\underline{x}$. In other words, it is not necessary to take a direct limit when computing the local cohomology.

Proof. Since R is equidimensional and Cohen–Macaulay on the punctured spectrum, we know that $H_{\mathfrak{m}}^r(R)$ has finite length for every $r < n = \dim R$. We will show that $H_{\mathfrak{m}}^r(R) = [H_{\mathfrak{m}}^r(R)]_0$ for every $r < n$ in situation (a) or (b). This will finish the proof by Theorem 4.12.

In situation (a), $H_{\mathfrak{m}}^r(R) = [H_{\mathfrak{m}}^r(R)]_0$ is obvious because $H_{\mathfrak{m}}^r(R)$ has finite length and the Frobenius acts injectively on it. In situation (b), notice that $[H_{\mathfrak{m}}^r(R)]_{<0} = 0$ by Theorem 4.8 while $[H_{\mathfrak{m}}^r(R)]_{>0} = 0$ by Theorem 4.4; hence, $H_{\mathfrak{m}}^r(R) = [H_{\mathfrak{m}}^r(R)]_0$. $\qquad\square$

Remark 4.14. In situation (b) above, if R is normal standard graded, then $H_{\mathfrak{m}}^r(R) = [H_{\mathfrak{m}}^r(R)]_0$ also follows from [Ma15, Theorem 4.5]. However, in the above proof we do not need any normal or standard graded hypothesis thanks to Proposition 4.4 and Theorem 4.8.

Remark 4.15. Corollary 4.13 was proved when R is F-pure and k is perfect in [HR76, Theorem 1.1] and, by a technical reduction to a $p > 0$ technique, it was also proved when R is of F-pure type [HR76, Theorem 4.8]. Since F-pure certainly implies F-injective and F-injective type implies Du Bois (see [Sch09]), our theorem gives a generalization of Hochster–Roberts's result, and our proof is quite different from that of [HR76].

We end this section by pointing out an example showing that in Theorem 4.12 or Corollary 4.13, it is possible that $H^r(\underline{x}, R) \neq H_{\mathfrak{m}}^r(R)$, i.e., we must take the degree-0 piece of the Koszul cohomology. This is a variant of Example 4.11.

Example 4.16. Let $R = (k[x, y, z]/(x^3 + y^3 + z^3))\#k[a, b, c]$ be the Segre product of $k[x, y, z]/(x^3 + y^3 + z^3)$ and $k[a, b, c]$, where the characteristic of k is either 0 or congruent to 1 mod 3. Therefore, R is the homogeneous coordinate ring of the Segre embedding of $X = E \times \mathbb{P}^2$ to $\mathbb{P}^8$, where $E = \operatorname{Proj}(k[x, y, z]/(x^3 + y^3 + z^3))$ is an elliptic curve. Notice that $\dim X = 3$ and $\dim R = 4$. Since R has an isolated singularity, it is Cohen–Macaulay on the punctured spectrum. It is easy to check that R is Du Bois in characteristic 0 and F-pure (and thus F-injective) in

characteristic $p > 0$, since $p \equiv 1 \bmod 3$. So, we know that $H^i_{\mathfrak{m}}(R) = [H^i_{\mathfrak{m}}(R)]_0$ for every $i < 4$, where $\mathfrak{m}$ is the unique homogeneous maximal ideal of R. Now we compute

$$H^2_{\mathfrak{m}}(R) = [H^2_{\mathfrak{m}}(R)]_0 = H^1(X, \mathcal{O}_X) = \bigoplus_{i+j=1} H^i(E, \mathcal{O}_E) \otimes_k H^j(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \cong k,$$

$$H^3_{\mathfrak{m}}(R) = [H^3_{\mathfrak{m}}(R)]_0 = H^2(X, \mathcal{O}_X) = \bigoplus_{i+j=2} H^i(E, \mathcal{O}_E) \otimes_k H^j(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0.$$

The first line shows that R is not Cohen–Macaulay (actually, we have depth $R = 2$), so for any homogeneous system of parameters $\underline{x}$ of R, we have $H^3(\underline{x}, R) \cong H_1(\underline{x}, R) \neq 0$. Hence, $H^3(\underline{x}, R) \neq H^3_{\mathfrak{m}}(R)$.

5. Deformation of dense F-injective type

In this section we use results in §3 to prove that singularities of (dense) F-injective type in characteristic 0 deform. In fact, we prove a slightly stronger result. Our motivation for studying this question is a recent result of Kovács and Schwede [KS16a] that Du Bois singularities in characteristic 0 deform.

By the main result of [Sch09], singularities of dense F-injective type are always Du Bois. It is conjectured [BST13, Conjecture 4.1] that Du Bois singularities should be equivalent to singularities of dense F-injective type. This conjecture is equivalent to the weak ordinarity conjecture of Mustaţă and Srinivas [MS11]. If this conjecture is true, then singularities of dense F-injective type deform because Du Bois singularities deform [KS16a]. However, the weak ordinarity conjecture is wide open.

Setup 5.1 (Reduction to characteristic $p > 0$). We recall briefly reduction to characteristic $p > 0$. For more details in our setting, see [HH06, § 2.1] and [Sch09, § 6].

Suppose that $(R, \mathfrak{m})$ is essentially of finite type over $\mathbb{C}$ (or another field of characteristic 0, which we will also call $\mathbb{C}$), so that $(R, \mathfrak{m})$ is a homomorphic image of T_P, where $T = \mathbb{C}[x_1, \ldots, x_t]$ and $P \subseteq T$ is a prime ideal, so that $R = (T/J)_P$. Given a finite collection of finitely generated R-modules M_i (and finitely many maps between them), we may assume that each $M_i = (M_i')_P$ for a finitely generated T-module M_i' annihilated by J. Suppose that $E \to \operatorname{Spec} T/J$ is the reduced preimage of T/J in a log resolution of $(\operatorname{Spec} T, \operatorname{Spec} T/J)$. We also keep track of $E \to \operatorname{Spec} T/J$ in the reduction to characteristic $p > 0$ process, as well as the modules $h^i(\underline{\Omega}^0_{T/J}) = \mathbf{R}^i \pi_* \mathcal{O}_E$. Assume that x is the image of $h(x_1, \ldots, x_t) \in T$ (note that if x has a problematic denominator, we can replace x by another element that generates the same ideal and does not have a denominator).

We pick a finitely generated regular $\mathbb{Z}$-algebra $A \subseteq \mathbb{C}$ such that the coefficients of the generators of P, the coefficients of h, the coefficients of J and the coefficients of a presentation of the M_i' are contained in A. Form $T_A = A[x_1, \ldots, x_t]$ and let $P_A = P \cap T_A, J_A = J \cap T_A$ and observe that $P_A \otimes_A \mathbb{C} = P, J_A \otimes_A \mathbb{C} = J$ by construction. Note that by generic flatness, we can replace A by $A[b^{-1}]$ and so shrink $\operatorname{Spec} A$ if necessary to assume that R_A is a flat A-module and x is a non-zero divisor on R_A. Likewise form $(M_i')_A$ with the same presentation matrix of M_i' (and likewise with maps between the modules) and form $E_A \to \operatorname{Spec} T_A/J_A$, so that $(M_i')_A \otimes_A \mathbb{C} = M_i'$ and that $(\mathbf{R}^i \pi_* \mathcal{O}_{E_A}) \otimes_A \mathbb{C} = (\mathbf{R}^i \pi_* \mathcal{O}_{E_A})$. Shrinking $\operatorname{Spec} A$ yet again if necessary, we can assume that all these modules are flat over A (and that any relevant kernels and cokernels of the finitely many maps between them are also flat).

We now mod out by a maximal ideal $\mathfrak{n}$ of A with $\kappa = A/\mathfrak{n}$. In particular, we use $R_A = (T_A/J_A)_{P_A}$ etc. and $R_\kappa, T_\kappa, E_\kappa$ etc. to denote the corresponding rings, schemes and

modules over A and κ, where $\kappa = A/\mathfrak{n}$ for $\mathfrak{n}$ a maximal ideal in A. Since all relevant kernels and cokernels of the maps are flat by shrinking $\operatorname{Spec} A$, x is a non-zero divisor on R_κ for every κ.

We also need a slightly different version of the main result of [Sch09]. In that paper, it was assumed that if R is of finite type over a field of characteristic 0, and of dense F-injective type, then R has Du Bois singularities. We need a version of this result in which R is local. The reason is, that, in the notation of Theorem 5.3, if R is of finite type, and $R_p/x_p R_p$ is F-injective, then for p sufficiently divisible, we obtain that R_p is F-injective in a neighborhood of $V(x_p)$. The problem is that we do not know how to control these neighborhoods as p varies. Thus, we need the following preliminary result. In particular, we do not know whether having dense F-injective type is an open property in characteristic 0.

THEOREM 5.2. *Let $(R, \mathfrak{m})$ be a local ring essentially of finite type over $\mathbb{C}$ and suppose that R has dense F-injective type. Then R is Du Bois.*

The strategy is the same as in [Sch09]; indeed, the proof only differs in how carefully we keep track of a minimal prime of the non-Du Bois locus.

Sketch of the proof. It is easy to see that R is seminormal, so we need to show that $h^i(\underline{\Omega}_R^0) = 0$ for all $i > 0$. We use the notation of Setup 5.1. Let $Q_\mathbb{C} \subseteq P \subseteq T/J$ correspond to a minimal prime of the non-Du Bois locus of $(R, \mathfrak{m})$, so that $h^i(\underline{\Omega}_{(T/J)_{Q_\mathbb{C}}}^0) = h^i(\mathbf{R}\pi_* \mathcal{O}_E)_{Q_\mathbb{C}}$ has finite length for $i > 0$. Since $(R, \mathfrak{m})$ has dense F-injective type, it is easy to see that so does $(R_{Q_\mathbb{C}}, Q_\mathbb{C} R_{Q_\mathbb{C}})$, so we may assume that $\mathfrak{m} = Q_\mathbb{C}$. Now using Setup 5.1, reduce to some model so that $(R_\kappa, \mathfrak{m}_\kappa)$ is F-injective. The proof now follows exactly that of [Sch09, Proof 6.1], where we obtain that

$$h^i(\mathbf{R}(\pi_\kappa)_* \mathcal{O}_{E_\kappa})_{Q_\kappa} \hookrightarrow H_{Q_\kappa}^{i+1}(R_\kappa).$$

But the left-hand side is annihilated by a power of Frobenius by [Sch09, Theorem 7.1] and Frobenius acts injectively on the right by hypothesis. The result follows. $\qquad \square$

THEOREM 5.3. *Let $(R, \mathfrak{m})$ be a local ring essentially of finite type over $\mathbb{C}$ and let x be a non-zero divisor on R. Suppose that R/xR has dense F-injective type. Then, for infinitely many $p > 0$, the Frobenius action $x^{p-1}F$ on $H_{\mathfrak{m}_p}^i(R_p)$ is injective for every i, where $(R_p, \mathfrak{m}_p)$ is the reduction mod p of R. In particular, R has dense F-injective type.*

Proof. By Theorem 5.2, R/xR has Du Bois singularities. By Theorem 3.6 (taking $Z = \emptyset$), $H_{\mathfrak{m}}^i(R) \xrightarrow{\cdot x} H_{\mathfrak{m}}^i(R)$ surjects for all i. By Matlis duality, $\operatorname{Ext}_T^i(R, T) \xrightarrow{\cdot x} \operatorname{Ext}^i(R, T)$ injects for all i. Spreading this out to A, and possibly inverting an element of A, we see that $\operatorname{Ext}_{T_A}^i(R_A, T_A) \xrightarrow{\cdot x} \operatorname{Ext}_{T_A}^i(R_A, T_A)$ injects for all i (note that there are only finitely many Ext to consider). Inverting another element of A if necessary, we deduce that

$$\operatorname{Ext}_{T_\kappa}^i(R_\kappa, T_\kappa) \xrightarrow{\cdot x} \operatorname{Ext}_{T_\kappa}^i(R_\kappa, T_\kappa)$$

injects for every i and each maximal ideal $\mathfrak{n} \subseteq A$, setting $\kappa = A/\mathfrak{n}$. We abuse notation and let x denote the image of x in R_κ. Applying Matlis duality again and considering the Frobenius on the local cohomology modules, we have the following collection of short exact sequences for each i:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_{\mathfrak{m}_\kappa}^i(R_\kappa/xR_\kappa) & \longrightarrow & H_{\mathfrak{m}_\kappa}^{i+1}(R_\kappa) & \xrightarrow{\cdot x} & H_{\mathfrak{m}_\kappa}^{i+1}(R_\kappa) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle x^{p-1}F} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & H_{\mathfrak{m}_\kappa}^i(R_\kappa/xR_\kappa) & \longrightarrow & H_{\mathfrak{m}_\kappa}^{i+1}(R_\kappa) & \xrightarrow{\cdot x} & H_{\mathfrak{m}_\kappa}^{i+1}(R_\kappa) & \longrightarrow & 0,
\end{array}
$$

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where p is the characteristic of κ and F denotes the natural Frobenius action on $H^i_{\mathfrak{m}_\kappa}(R_\kappa/xR_\kappa)$ and $H^{i+1}_{\mathfrak{m}_\kappa}(R_\kappa)$.

At this point, recall that R/xR has dense F-injective type. It follows that for infinitely many $p > 0$, if the residue field κ of A has characteristic $p > 0$, then the natural Frobenius action on $H^i_{\mathfrak{m}_\kappa}(R_\kappa/xR_\kappa)$ is injective for every i. Now chasing the above diagram, if $x^{p-1}F$ is not injective, then we can pick $0 \neq y \in \mathrm{socle}(H^{i+1}_{\mathfrak{m}_\kappa}(R_\kappa)) \cap \ker(x^{p-1}F)$. Since y is in the socle of $H^{i+1}_{\mathfrak{m}_\kappa}(R_\kappa)$, it maps to zero under multiplication by x. But then $0 \neq y \in H^i_{\mathfrak{m}_\kappa}(R_\kappa/xR_\kappa)$, and chasing the diagram we find that $x^{p-1}F(y) \neq 0$, which is a contradiction.

We have established that for infinitely many p, after we do reduction to p, the Frobenius action $x^{p-1}F$ on $H^{i+1}_{\mathfrak{m}_\kappa}(R_\kappa)$ is injective for every i. This certainly implies that the natural Frobenius action F on $H^{i+1}_{\mathfrak{m}_\kappa}(R_\kappa)$ is injective for every i. Hence, R has dense F-injective type. $\quad\square$

Remark 5.4. It is still unknown whether F-injective singularities in characteristic $p > 0$ deform (and this has been open since [Fed83]). Our theorem is in support of this conjecture: it shows that this is true 'in characteristic $p \gg 0$'. For the most recent progress on deformation of F-injectivity in characteristic $p > 0$, we refer to [HMS14]. On the other hand, Theorem 5.3 provides evidence for the weak ordinarity conjecture [MS11] because of the relation between the weak ordinarity conjecture and the conjecture that Du Bois singularities have dense F-injective type [BST13].

Acknowledgements

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