CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

I read with great interest the article on the School Mathematics Project by Dr. Cundy in *The Mathematical Gazette* for February, 1963. You may be interested to learn of our experiences here since we started to use D. E. Mansfield's text last September on the new 11 + intake. The result after two terms is most gratifying.

In most years the youngsters come here having already decided "I can't do sums," or the opposite. In fact they are already sorted into those who think they can and those who think they can't and we have found that the sense of inferiority of the latter increases as the time goes on. With this book the transition from primary to grammar school brings a complete change into mathematics from "sums." This in itself is a stimulus and the fact that on this new work—new to ALL—they start level means that the worst is as well off as the best. Incidentally we have found that nearly all the old routine has had to be done but has just happened as something necessary to achieve some new and exciting end. It has therefore been done unnoticed and painlessly.

From my own point of view and that of my colleagues the approach has provided a most exciting and rewarding two terms of teaching. The reactions of the boys have been most stimulating for obviously they have found the sense of exploration just as exciting. It has been nearly impossible to plan a lesson except in very broad outline for the children themselves produce ideas which are valuable sometimes and at all times evidence of thought. After all this is one of our objects in teaching. Farey series and the drawing of lattices for instance produce the question "why don't they measure angles this way?" and one was or could easily be involved in tangents if one wished. When I did in an odd moment the "1089" problem with them one day a whole lot came back later on when I had forgotten all about it: "look sir, it works in every scale."

Everything has been worth it from this enthusiasm and interest which holds both the top and the bottom of the 11 + lists.

Yours faithfully

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To the Editor of the *Mathematical Gazette*

DEAR SIR,

A letter some time ago from Mr. Wheeler (*Gazette* 42, 197–200) pointed out that the notation $\sin^{-1}$, unlike its rival arcsin, is consistent with modern usage where manipulations such as $TT^{-1}x = T^{-1}Tx = x$ are commonplace. He joined his advocacy of the notation $\sin^{-1} x$ to a plea for a "*consistent* policy by examiners to reserve the symbol with the lower case initial letter for the principal value of the inverse function,

https://doi.org/10.1017/S0025557200072004 Published online by Cambridge University Press

and not to use it indiscriminately for either the principal or general value, as happens with at least one Examining Board."

I agree with Mr. Wheeler that where possible we should choose our elementary notation with an eye on that used in more advanced work, but I do not think he goes far enough. We should do well to follow the established convention in topology that, if f is a function, we write $f^{-1}(x)$ for the *class* of those α making $f(\alpha) = x$, a class which may be empty, have just one member or have more than one member. With this is associated the convention that it is only when for each x the class has just one member that we call $f^{-1}(x)$ a function of x. We must of course state unambiguously what sets (of numbers or other entities) α and x may be chosen from. If this convention is applied to the sine defined on the real numbers, $\sin^{-1} 2$ denotes the empty class, $\sin^{-1} \frac{1}{2}$ the class $\{n\pi + (-1)^n\pi/6\}$ and we must deny to $\sin^{-1} x$ the description "function". So far we have done no more than replace the vague term "general value" with the more precise "class of numbers", but it will be observed that $\sin^{-1}$, not Sin^{-1}, becomes the natural symbol for the general value.

When $-1 \leqq x \leqq 1$, the class $\sin^{-1} x$ is of the form $\{n\pi + (-1)^n\alpha\}$, $= S_\alpha$ say, for some α. (As will be observed, this notation implies that $S_\alpha = S_{\pi-\alpha} = S_{2\pi+\alpha} = \ldots$.) If we wish to take the principal value of $\sin^{-1} x$, we set up the *rule* for choosing one member α_0 from S_α that $-\frac{1}{2}\pi \leqq \alpha_0 \leqq \frac{1}{2}\pi$. It seems to me that, rather than use a convention on small and capital letters, we should say our rule defines a function from the set of the classes S_α onto the set of the numbers in $-\frac{1}{2}\pi \leqq \alpha_0 \leqq \frac{1}{2}\pi$, and for this function I propose for your readers' consideration the notation

$$\text{pr}\, S_\alpha.$$

If this is accepted, we write

$$\tfrac{1}{2}\pi = \text{pr}\,(\sin^{-1} 1), \quad -\tfrac{1}{4}\pi = \text{pr}\,(\sin^{-1}(-1/\sqrt{2}))$$

and so on. If we omit the brackets we can regard $\text{pr}\sin^{-1}$ as a genuine numerical function from $[-1, 1]$ onto $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$; this would be a suitable description for beginners which could easily be modified later in their careers.

If this suggestion were adopted we should similarly consider the numerically valued function $\text{prcos}^{-1} x$ from $-1 \leqq x \leqq 1$ onto $0 \leqq \alpha \leqq \pi$. It is to be observed that $\cos^{-1} x$ is of the form $\{2n\pi \pm \alpha\}$, the class C_α, say, which is different from S_α or, indeed, from S_β for any β. There is thus no contradiction in saying that pr is also to be defined for the set of the classes C_α but with $0 \leqq \text{pr}\, C_\alpha \leqq \pi$. Similarly, pr would be suitably defined for classes of other special forms, in particular for the classes

(i) $\{n\pi + \alpha\}$ to deal with $\tan^{-1}$ and $\cot^{-1}$,

(ii) $\{\pm\alpha\}$ to deal with $\cosh^{-1}$ and $\surd$, and

(iii) $\{2n\pi i + \alpha\}$ to deal with log, that is $\exp^{-1}$,

but it would not be defined for an arbitrary class.

The classes of the form (i) deserve special mention. If we think of such a class as being necessarily $\tan^{-1} x$ for some x we are tempted to prescribe $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, open at each end, as the range of pr. However, since the class

$$\{n\pi + \tfrac{1}{2}\pi\} = \cot^{-1} 0$$

is genuinely of this form it should not be excluded, and we are led to prescribe $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$ as the range of pr for such classes. It is to be observed that, if $x = \infty$ is not allowed, the functions $\mathrm{prtan}^{-1} x$ and $\mathrm{prcot}^{-1} x$ will omit the values $\frac{1}{2}\pi$ and 0 respectively from the common range just prescribed for them; it is because some authors accept the punctured range $(-\frac{1}{2}\pi, 0) \cup (0, \frac{1}{2}\pi]$ and others opt for $(0, \pi)$ that there are different conventions for $\mathrm{prcot}^{-1} x$ in standard texts. Our discussion gives a clear reason for resolving this small ambiguity in favour of $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$, whether considered punctured or not; it has the consequence that although, as equalities between classes, both

$$\cot^{-1} x = \tan^{-1}(1/x)$$

and

$$\cot^{-1} x = \tfrac{1}{2}\pi - \tan^{-1} x$$

are valid, we have, for the functions $\mathrm{prcot}^{-1} x$ and $\mathrm{prtan}^{-1} x$,

$$\mathrm{prcot}^{-1} x = \mathrm{prtan}^{-1}(1/x)$$

and

$$\mathrm{prcot}^{-1} x = \begin{cases} \tfrac{1}{2}\pi - \mathrm{prtan}^{-1} x, & x \geqq 0, \\ -\tfrac{1}{2}\pi - \mathrm{prtan}^{-1} x, & x < 0. \end{cases}$$

Yours faithfully
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1980. Laura's eyes, as she entered the room, were treated to the sight of a riotous marriage of mathematical forms and pastel tints, a nuptial delirium of Euclid and Iris. The low tables were planes tangent to and resting on metal circles; the carpet was a vast grey square, innocent of designs; the chairs were upholstered green parabolas; the huge sofa was a hollowed tan parallelepiped; the drape masking the far end of the room was a single yellow oblong; the very flower vases and ashtrays were coppery polygonal prisms; and all was lighted from concealed sources with a diffused radiance that cast nothing so irregular and uncalculated on the scene as a shadow.

[From *Aurora Dawn* by Herman Wouk. Per Miss F. Gross]

https://doi.org/10.1017/S0025557200072004 Published online by Cambridge University Press