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# Holomorphic Functions of Exponential Growth on Abelian Coverings of a Projective Manifold

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**Abstract.** Let $M$ be a projective manifold, $p: M_G \longrightarrow M$ a regular covering over $M$ with a free Abelian transformation group $G$. We describe the holomorphic functions on $M_G$ of an exponential growth with respect to the distance defined by a metric pulled back from $M$. As a corollary, we obtain Cartwright and Liouville-type theorems for such functions. Our approach brings together the $L_2$ cohomology technique for holomorphic vector bundles on complete Kähler manifolds and the geometric properties of projective manifolds.

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**Key words.** holomorphic function, $L_2$ cohomology, regular covering with an Abelian transformation group, positive vector bundle.

## 1. Introduction

**1.1.** Recently there has been substantial progress in the study of the harmonic functions of polynomial growth on complete Riemannian manifolds (see, in particular, [CM, Gu, Ka, L, LZ, Li, LySu] for the results and for further references). As a corollary one also obtains a description of the holomorphic functions of polynomial growth on nilpotent coverings of compact Kähler manifolds (see also [Br]). On the other hand, very little is known about the existence and behaviour of slowly growing harmonic (respectively, holomorphic) functions on covering spaces of compact Riemannian (respectively, Kähler) manifolds. The methods in the above-cited papers do not seem to be sufficient for applications to the general situation. This paper is devoted to the study of slowly growing holomorphic functions on Abelian coverings of projective manifolds. Our approach is based on the $L_2$ cohomology technique for holomorphic vector bundles on complete Kähler manifolds and on the geometric properties of projective manifolds and differs from the methods of the above-mentioned papers.

In order to formulate the results of the paper, we consider a projective manifold $M$ and its regular covering $p: M_G \longrightarrow M$ with a free Abelian transformation group $G$. Denote by $r$ the distance from a fixed point in $M_G$ defined by a metric pulled

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back from $M$. We study the holomorphic functions $f$ on $M_G$ satisfying (for some $\varepsilon > 0$)

$$|f(z)| \leqslant c\mathrm{e}^{\varepsilon r^2(z)}, \qquad (z \in M_G). \tag{1.1}$$

Recall that the covering space $M_G$ can be described as follows: Let $\omega_1, \dots, \omega_n$ be a basis of holomorphic 1-forms on $M$ and $A\colon M \longrightarrow \mathbb{CT}^n$ be the Albanese map of $M$ associated with this basis. By definition,

$$A(z) = \left(\int_{z_0}^{z} \omega_1, \dots, \int_{z_0}^{z} \omega_n\right)$$

for a fixed $z_0 \in M$. Consider a free Abelian quotient group $G$ of the fundamental group $\pi_1(\mathbb{CT}^n) \cong \mathbb{Z}^{2n}$. Let $t\colon T_G \longrightarrow \mathbb{CT}^n$ be the regular covering over a torus with the transformation group $G$. We can think of $T_G$ as a locally trivial fibre bundle over $\mathbb{CT}^n$ with discrete fibres. Then $M_G = A^*T_G$ is the pullback of $T_G$ to $M$. By definition, the fundamental group of $M_G$ is $H := A_*^{-1}(\mathrm{Ker}\,\pi) \subset \pi_1(M)$, where $\pi\colon \mathbb{Z}^{2n} \longrightarrow G$ denotes the quotient map. By the covering homotopy theorem there is a proper holomorphic map $A_G\colon M_G \longrightarrow T_G$ that covers $A$ and such that $\widetilde{M}_G := A_G(M_G) \subset T_G$ is a covering of complex variety $A(M) \subset \mathbb{CT}^n$.

Our main result shows that if $f$ satisfies (1.1), then there is a uniquely defined holomorphic function $g$ on $T_G$ with a similar growth condition such that $f = A_G^*(g)$. To its formulation, we let $\phi$ be a smooth nonnegative function on $T_G$ and $\tilde{\phi} = A_G^*(\phi)$. Consider the Hilbert space $\mathcal{H}_{\tilde{\phi}}(M_G)$ of holomorphic functions $f$ on $M_G$ with the norm

$$|f| := \int_{M_G} |f|^2 \mathrm{e}^{-\tilde{\phi}} \mathrm{d}V.$$

Here $\mathrm{d}V$ is the pullback of the volume form on $M$ defined by a Kähler metric. Similarly we introduce the Hilbert space $\mathcal{H}_\phi(T_G)$ of holomorphic functions $f$ on $T_G$ with the norm

$$|f| := \int_{T_G} |f|^2 \mathrm{e}^{-\phi} \mathrm{d}\widetilde{V},$$

where $\mathrm{d}\widetilde{V}$ is the pullback of the standard volume form on $\mathbb{CT}^n$. Let $\{\mathrm{d}z_1, \dots \mathrm{d}z_n\}$ be the basis of holomorphic 1-forms on $\mathbb{CT}^n$ such that $A^*(\mathrm{d}z_i) = \omega_i$ for $i = 1, \dots, n$. Using the same symbol, we denote the pullback of these forms to $T_G$. Let $\mathcal{L}(\phi) = \sum_{i,j} a_{ij}(z, \bar{z}) \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$ be the Levi form of $\phi$. We set $|\mathcal{L}(\phi)| := \sup_{i,j,z \in T_G} |a_{ij}(z)|$. Assume that there is a constant $c > 0$ such that

$$|\phi(x) - \phi(y)| \leqslant c d(x, y), \tag{1.2}$$

where $d(.,.)$ is the distance on $T_G$ defined by the pullback of the flat metric on $\mathbb{CT}^n$.

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THEOREM 1.1. *There is a constant $C = C(M, A) > 0$ such that if $|\mathcal{L}(\phi)| < C$ and $\phi$ satisfies (1.2), then $A_G^*$ maps $\mathcal{H}_\phi(T_G)$ isomorphically onto $\mathcal{H}_{\tilde{\phi}}(M_G)$.*

Assume now that, instead of (1.2), $\phi$ satisfies: for any $\varepsilon > 0$, $x, y \in T_G$ with $d(x, y) \leqslant t$, there is a function $c(\varepsilon, t) > 0$ increasing in $t$ such that

$$\phi(x) \leqslant (1+\varepsilon)\phi(y) + c(\varepsilon, t)\ . \tag{1.3}$$

THEOREM 1.2. *Let $C$ be as in Theorem 1.1, $|\mathcal{L}(\phi)| < C' < C$ and $\phi$ satisfies (1.3). There is a constant $\tilde{\varepsilon}(C') > 0$ such that for any $f \in \mathcal{H}_{\tilde{\phi}}(M_G)$, there exists a unique $\hat{f} \in \cap_{\varepsilon < \tilde{\varepsilon}(C')} \mathcal{H}_{(1+\varepsilon)\phi}(T_G)$ satisfying $A_G^*(\hat{f}) = f$ and $|\hat{f}| \leqslant c(\varepsilon)|f|$. Here we regard $\hat{f}$ as an element of $\mathcal{H}_{(1+\varepsilon)\phi}(T_G)$.*

In the following examples, $M_G$ is a regular covering over $M$ with the maximal free Abelian transformation group $G$ (so $T_G = \mathbb{C}^n$).

EXAMPLES. (1) Let $\phi(z) = k\log(p + |z|^2)$ on $\mathbb{C}^n$, where $|z|$ is the Euclidean norm of the vector $z \in \mathbb{C}^n$ and $p > 0$ is so big that $|\mathcal{L}(\phi)| < C$. (Such $p$ exists because $\mathcal{L}(\log|z|) \to 0$ when $|z| \to \infty$.) Then $\mathcal{H}_\phi(\mathbb{C}^n)$ is isomorphic to the space of holomorphic polynomials of degree $\leqslant k - n - 1$. Therefore every holomorphic function on $M_G$ of the corresponding polynomial growth is the pullback by $A_G$ of a uniquely defined holomorphic polynomial on $\mathbb{C}^n$. This gives another proof for the projective manifolds of the main result of [Br].

(2) Let $\phi(z) = 2\sigma\sqrt{p + |z|^2}$ on $\mathbb{C}^n$, where $p$ is such that $|\mathcal{L}(\phi)| < C$. Then $\mathcal{H}_\phi(\mathbb{C}^n)$ consists of entire functions of the exponential type $< \sigma$. Now Theorem 1.1 describes holomorphic functions $f$ on $M_G$ satisfying $|f(z)| < ce^{\sigma' r(z)}$, $z \in M_G$, $\sigma' < \sigma$.

(3) Let $\phi(z) = 2\sigma|z|^2$ on $\mathbb{C}^n$ with $2\sigma < C$. Then the assumptions of Theorem 1.2 are fulfilled and the theorem describes holomorphic functions $f$ on $M_G$ satisfying $|f(z)| < ce^{\sigma' r^2(z)}$, $z \in M_G$, $\sigma' < \sigma$.

(4) Assume that $C$ is a compact complex curve of genus $g \geqslant 1$. Then $C_G$ can be thought of as a submanifold in $\mathbb{C}^g$. Applying Theorem 1.2, we obtain the following Cartwright type theorem:

THEOREM. *There is a positive number $\sigma = \sigma(C_G)$ such that any holomorphic function $f$ on $\mathbb{C}^g$ satisfying $|f(z)| \leqslant ce^{\sigma'|z|^2}$, $0 < \sigma' < \sigma$, $z \in \mathbb{C}^g$, and $f|_{C_G} = 0$, identically equals 0.*

**1.2.** The classical Liouville theorem asserts that every bounded holomorphic function on $\mathbb{C}^n$ is a constant. Based on Theorem 1.1, we prove Liouville-type theorems for holomorphic functions of slow growth on Abelian coverings over a projective manifold.

Let $\Gamma \subset H_1(M, \mathbb{Z}) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)]$ be the maximal free Abelian subgroup of the homology group of $M$. Further, let $\Omega^1(M)$ be the space of holomorphic 1-forms on $M$. Any $\omega \in \Omega^1(M)$ determines a complex-valued linear functional on $\Gamma$ by integration. For a subgroup $H \subset \Gamma$, denote by $\Lambda(H)$ the minimal complex subspace of holomorphic 1-forms vanishing on $H$. Assume also that the quotient group $G = \Gamma/H$ is torsion-free and $M_G$ is the regular covering over $M$ with the transformation group $G$.

THEOREM 1.3. *Let $H$ be such that $\Lambda(H) = \Omega^1(M)$. Then any holomorphic on $M_G$ function $f$ satisfying for any $\varepsilon > 0$, $|f(z)| \leqslant c(\varepsilon)e^{\varepsilon r(z)}$ $(z \in M_G)$ is a constant.*

*Remark* 1.4. It can be conjectured that the results of this paper are also true for Abelian coverings of an arbitrary compact Kähler manifold.

## 2. Preliminaries

### 2.1. $L_2$ COHOMOLOGY THEORY

In the proof of our main results, we use the $L_2$ cohomology technique for holomorphic vector bundles on complete Kähler manifolds. We start by reviewing some results of $L_2$ cohomology (see, e.g., Lárusson [La] for more details and further references).

Let $X$ be a complex manifold of dimension $n$ with a Hermitian metric and $E$ be a holomorphic vector bundle over $X$ with a Hermitian metric. Let $L_2^{p,q}(X, E)$ be the space of $E$-valued $(p, q)$-forms on $X$ with the $L_2$ norm, and let $W_2^{p,q}(X, E)$ be the subspace of forms $\eta$ such that $\bar\partial\eta$ is $L_2$. The forms $\eta$ may be taken to be either smooth or just measurable, in which case $\bar\partial\eta$ is understood in the distributional sense. The cohomology of the resulting $L_2$ Dolbeault complex $(W_2^{\cdot,\cdot}, \bar\partial)$ is the $L_2$-cohomology

$$H_{(2)}^{p,q}(X, E) = Z_2^{p,q}(X, E)/B_2^{p,q}(X, E),$$

where $Z_2^{p,q}(X, E)$ and $B_2^{p,q}(X, E)$ are the spaces of $\bar\partial$-closed and $\bar\partial$-exact forms in $L_2^{p,q}(X, E)$, respectively. Let $E^*$ be the dual bundle of $E$ with the dual metric. In our proofs we use the following result discovered by Lárusson [La]:

PROPOSITION 2.1. *Let $E$ be a Hermitian vector bundle with curvature $\Theta$ on a complex manifold $X$ of dimension $n \geqslant 2$ with a complete Kähler form $\omega$. If $\Theta \geqslant \varepsilon\omega$ for some $\varepsilon > 0$ in the sense of Nakano, then $H_{(2)}^{0,q}(X, E^*) = 0$, for $q < n$.*

*Remark* 2.2. Let $E$ satisfy conditions of Proposition 2.1. Consider the linear map $\bar\partial$: $W_2^{0,0}(X, E^*) \longrightarrow Z_2^{0,1}(X, E^*)$ and introduce the norm in $W_2^{0,0}(X, E^*)$ by

$$|f| := |f|_2 + |\bar\partial f|_2, \quad f \in W_2^{0,0}(X, E^*)\ .$$

According to Proposition 2.1, for $q = 1$ and $q = 0$, there is a linear map $s$: $Z_2^{0,1}(X, E^*) \longrightarrow W_2^{0,0}(X, E^*)$ such that $s \circ \bar\partial = \mathrm{id}$ and $\bar\partial \circ s = \mathrm{id}$. Then, by the Banach theorem, $\bar\partial$ is open and $s = (\bar\partial)^{-1}$.

## 2.2. $\bar\partial$-METHOD

Let $i\colon X \hookrightarrow Y$ be a complex compact submanifold of codimension 1 of an $n$-dimensional compact Kähler manifold $Y$, $n \geqslant 2$, with a Kähler form $\omega$. Assume that the induced homomorphism $i_*\colon H_1(X, \mathbb{R}) \longrightarrow H_1(Y, \mathbb{R})$ is surjective. Let $G$ be a free Abelian quotient group of $\pi_1(Y)$. Consider the regular covering $Y_G$ over $Y$ with the transformation group $G$. From the assumption for $i_*$ it follows that there are a regular covering $X_G$ over $X$ with the transformation group $G$ (the pullback of $Y_G$ by $i$) and the holomorphic embedding $i_G\colon X_G \hookrightarrow Y_G$ that covers $i$. Divisor $X \subset Y$ determines a holomorphic line bundle $L$ over $Y$ and a holomorphic section $s\colon Y \longrightarrow L$ with a simple zero along $X$. Further, for every $p \in X$, there is a coordinate neighbourhood $(U, z)$ centered at $p$ and a holomorphic frame e for $L$ on $U$ such that $s = z_1\mathrm{e}$ on $U$. Let $h$ be a Hermitian metric on $L$ and $\nabla$ be the canonical connection with curvature $\Theta$ constructed by $h$. Using the same letters, we denote the pullback of $L$, $h$, $s$ and $\Theta$ to $Y_G$. Note also that if $\phi$ is a smooth function on $Y_G$, then the weighted metric $\mathrm{e}^{\phi}h$ on $L$ has a curvature $\Theta' = -\mathcal{L}(\phi) + \Theta$.

Let $U_0$ be the pullback of the complement of a closed neighbourhood of $X \subset Y$ and $U_1, \dots, U_N$ be the pullbacks of shrunk coordinate polydisks covering a larger neighbourhood of $X$. Also pull back a smooth partition of unity $(\xi_i)$ subordinate to $(U_i)$. Let $f$ be a holomorphic function on $X_G$ such that $f^2\mathrm{e}^{-\phi}$ is integrable on $X_G$. For $i \geqslant 1$, extend $f$ to a holomorphic function $f_i$ on $U_i$ which is constant on each line $\{z_2, \dots, z_n$ constant$\}$. Let $f_0 = 0$ on $U_0$. Since $f_i = f = f_j$ on $X_G$ and $X_G$ is smooth, we can define a holomorphic section of the dual bundle $L^*$ on $U_{ij} = U_i \cap U_j$ by the formula $u_{ij} = (f_i - f_j)s^{-1}$ . Then $v_i = \sum_i u_{ij}\xi_j$ is a smooth section of $L^*$ on $U_j$ and $v_i - v_j = u_{ij}$. Hence, $\bar\partial v_i = \bar\partial v_j$ on $U_{ij}$, so we get a $\bar\partial$-closed, $L^*$-valued (0,1)-form $\eta$ on $Y_G$ defined as $\bar\partial v_i$ on $U_i$. Assume that $\phi$ satisfies (1.2) or (1.3), where $d$ is the distance on $Y_G$ defined by the pullback of a metric on $Y$. Denote by $|f|$ the weighted $L_2$-norm of $f$ with the weight $\mathrm{e}^{-\phi}$.

LEMMA 2.3. (1) *If $\phi$ satisfies (1.2), then $\eta \in L_2^{0,1}(Y_G, L^*)$ for $L$ equipped with the metric $\mathrm{e}^{\phi}h$ and $|\eta| \leqslant C(X, Y, h, \phi)|f|$ in the corresponding $L_2$-norms.*

(2) *If $\phi$ satisfies (1.3), then $\eta \in L_2^{0,1}(Y_G, L^*)$ for $L$ equipped with the metric $\mathrm{e}^{(1+\varepsilon)\phi}h$, $\varepsilon > 0$, and $|\eta| \leqslant C(X, Y, h, \phi, \varepsilon)|f|$.*

*Proof.* We prove (2). The proof of (1) goes along the same lines (see also the arguments in [La, Th. 3.1]).

We have to show that $|\eta|^2\mathrm{e}^{-(1+\varepsilon)\phi}$ is integrable on $Y_G$. On $U_0$, $s$ is bounded away from 0 and

$$\eta = \bar\partial v_0 = -\sum_j f_j s^{-1}\bar\partial\xi_j \ ,$$

so $|\eta|^2 \leqslant c\sum_j |f_j|^2$, where $c$ depends only on $X$, $Y, h$. Further,

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$$\int_{U_j} |f_j|^2 \mathrm{e}^{-(1+\varepsilon)\phi} \omega^n \leqslant c'(\varepsilon, X, Y, \phi) \int_{X_G \cap U_j} |f|^2 \mathrm{e}^{-\phi} \omega^{n-1}$$

because $\phi$ satisfies (1.3). Since $f^2 \mathrm{e}^{-\phi}$ is integrable on $X_G$, so is $|\eta|^2 \mathrm{e}^{-(1+\varepsilon)\phi}$ on $U_0$. For $i \geqslant 1$,

$$\eta = \bar{\partial} v_i = \sum_j (f_i - f_j) s^{-1} \bar{\partial} \xi_j$$

on $U_i$ and it remains to show that

$$\sum_{i,j \geqslant 1} \int_{U_{ij}} |f_i - f_j|^2 |s|^{-2} \mathrm{e}^{-(1+\varepsilon)\phi} \omega^n < \infty \ . \tag{2.1}$$

For $x \in U_{ij}$, $i, j \geqslant 1$, there are $x_i \in X_G \cap U_i$ and $x_j \in X_G \cap U_j$ such that $f_i(x) = f(x_i)$, $f_j(x) = f(x_j)$ and $d(x_i, x_j) \leqslant c(h, X, Y)|s(x)|$. So,

$$|f_i(x) - f_j(x)||s(x)|^{-1} \leqslant c'(X, Y, d) \sup |\mathrm{d}f|,$$

where supremum is taken over $X_G \cap (U_i \cup U_j)$. By the Cauchy inequalities and since $\phi$ satisfies (1.3),

$$\begin{aligned}
&\int_{U_{ij}} |f_i - f_j|^2 |s|^{-2} \mathrm{e}^{-(1+\varepsilon)\phi} \omega^n \\
&\quad \leqslant c'(X, Y, d) \int_{U_{ij}} \sup |\mathrm{d}f|^2 \mathrm{e}^{-(1+\varepsilon)\phi} \omega^n \\
&\quad \leqslant c''(X, Y, d, \varepsilon) \int_{X_G \cap (V_i \cup V_j)} |f|^2 \mathrm{e}^{-\phi} \omega^{n-1},
\end{aligned}$$

where $V_i \supset U_i$, $V_j \supset U_j$ are pullbacks of larger polydisks. Since $f^2 \mathrm{e}^{-\phi}$ is integrable on $X_G$, (2.1) follows.

The lemma is proved. □

Assume now that under the conditions of Lemma 2.3 there is a smooth section $w$ of $L^*$ such that $\bar{\partial} w = \eta$ and $|w|^2 \mathrm{e}^{-\phi}$ (respectively, $|w|^2 \mathrm{e}^{-(1+\varepsilon)\phi}$) is integrable. Let $u_i = v_i - w$, then $u_i$ is a holomorphic section of $L^*$ on $U_i$ and $u_i - u_j = u_{ij}$, so

$$f_i - u_i \otimes s = f_j - u_j \otimes s \quad \text{on } U_{ij}\,.$$

Hence, we obtain a holomorphic extension $F$ of $f$ to $Y$ by setting

$$F = f_i - u_i \otimes s = f_i + w \otimes s - \sum_j (f_i - f_j) \xi_j \quad \text{on } U_i.$$

The term $w \otimes s$ is $L_2$ with respect to $\mathrm{e}^{-\phi}$ (respectively, $\mathrm{e}^{-(1+\varepsilon)\phi}$) by construction of $w$ and since $s$ is bounded. The other two terms on the right-hand side can be shown to be $L_2$ with respect to $e^{-\phi}$ (respectively, $e^{-(1+\varepsilon)\phi}$) by arguments similar to those used for $\eta$ above. Hence, $F^2e^{-\phi}$ (respectively, $F^2e^{-(1+\varepsilon)\phi}$) is integrable.

### 2.3. SYMMETRIC PRODUCTS OF CURVES

Let $\Gamma$ be a complex compact curve of genus $g \geqslant 1$, $\Gamma^{\times g}$ and $S\Gamma^{\times g}$ be the direct and the symmetric products of $g$-copies of $\Gamma$. Then the manifold $S\Gamma^{\times g}$ is the quotient of $\Gamma^{\times g}$ by the action of the permutation group $S_g$. Therefore, there exists a finite holomorphic surjective map $\pi\colon \Gamma^{\times g} \longrightarrow S\Gamma^{\times g}$. Further, $S\Gamma^{\times g}$ is birational isomorphic to $\mathbb{CT}^g$ (denote this isomorphism by $j$). Let $(p, \dots, p) \in \Gamma^{\times g}$ be a fixed point. Denote by $\Gamma^k$, $k \leqslant g$, submanifold $\{(p, \dots, p, z_1, \dots, z_k)|\ z_1, \dots, z_k \in \Gamma\} \subset \Gamma^{\times g}$.

LEMMA 2.4. *For any $k$, the image $\pi(\Gamma^k)$ is a complex submanifold of $S\Gamma^{\times g}$.*

*Proof.* For a point $y = (p, \dots, p, z_1, \dots, z_k) \in \Gamma^k$ consider its orbit $o(y) := S_g(y)$. By definition, $\pi$ maps $o(y)$ to $\pi(y)$ and intersection $o(y) \cap \Gamma^k = \{(p, \dots, p, S_k(z))\}$; here $z = (z_1, \dots, z_k)$ and $S_k$ is the permutation group acting on the set of $k$ elements. The quotient by the action of $S_k$ is manifold $X_k := (p, \dots, p, S\Gamma^{\times k})$. So we have a holomorphic injective mapping $\pi_k\colon X_k \longrightarrow S\Gamma^{\times g}$ whose image coincides with $\pi(\Gamma^k)$. Now let $y_0$ be local coordinates in a neighbourhood $U_0$ of $p \in \Gamma$ and $y_i$, $1 \leqslant i \leqslant k$, be local coordinates in a neighbourhood $U_i$ of $z_i \in \Gamma$ such that $y_0(p) = 0$, $y_i(U_i) \cap y_j(U_j) = \emptyset$ for $z_i \neq z_j$ and $y_i = y_j$ in $U_i = U_j$ for $z_i = z_j$. Denote by $\sigma_1, \dots, \sigma_g$ the elementary symmetric functions from $g$ variables. For

$$(z_1, \dots, z_g) \in U_0 \times \dots \times U_0 \times U_1 \times \dots \times U_k \subset \Gamma^{\times g}$$

set

$$u_i(z) = y_0(z_i), \quad 1 \leqslant i \leqslant g-k, \quad \text{and} \quad u_i(z) = y_i(z_{g-k+i}), \quad 1 \leqslant i \leqslant k.$$

Using the theorem on symmetric polynomials, the mapping

$$f\colon (w_1, \dots, w_g) \longmapsto (\sigma_1(u(w)), \dots, \sigma_g(u(w)))$$

determines a local coordinate system on $\pi(U_0 \times \dots \times U_0 \times U_1 \times \dots \times U_k) \subset S\Gamma^{\times g}$ (see [GH, Ch. 2, p. 259]). Then the image of restriction $f|_{\pi(\Gamma_k)}$ belongs to $\mathbb{C}^k \subset \mathbb{C}^g$. For the same reason, $f \circ \pi_k$ determines a local coordinate system in the corresponding neighbourhood on $X_k$. This shows that $\pi_k$ is a biholomorphic embedding. Thus, we proved that $\pi(\Gamma_k)$ is smooth. $\square$

### 2.4. NORM ESTIMATES

Let $M$ and $N$ be compact Riemannian manifolds and $f\colon M \longrightarrow N$ be a smooth surjective map. Assume that $f_*\colon \pi_1(M) \longrightarrow \pi_1(N)$ is a surjection. Let $G$ be a quotient group of $\pi_1(N)$ and $N_G$, $M_G$ regular coverings with the transformation group $G$ over $N$ and $M$, respectively, such that $M_G = f^*N_G$. Then there is a map $f_G\colon M_G \longrightarrow N_G$ that covers $f$. We consider $M_G$ and $N_G$ in the metrics pulled back from $M$ and

$N$, respectively. Further, if $E^p(K)$ is the space of $p$-forms on a Riemannian manifold $K$ denoted by $|\cdot|_x$, the norm in the vector space $E^p(K)|_x (\cong \wedge^p T_x^*)$, $x \in K$, constructed by the metric dual to the Riemannian one.

LEMMA 2.5. *Let $\omega$ be a bounded differential $p$-form on $N_G$, i.e.,* $\sup_{x\in N_G} |\omega|_x < \infty$. *Then there is $C = C(f,p) > 0$ such that $|f_G^*(\omega)|_x \leqslant C|\omega|_{f_G(x)}$ .*

*Proof.* Let us write $\omega$ in local orthogonal coordinates lifted from $N$. Then the compactness arguments show that the statement follows easily from a similar statement for elements of the orthogonal basis. We leave the details to the reader. □

## 3. Proofs

We prove Theorem 1.2 only. The proof of Theorem 1.1 is similar and can be obtained by removing $\varepsilon$ in the arguments below.

**3.1.** We start by proving Theorems 1.1 and 1.2 for curves.

*Proof of Theorem 1.2 for curves.* Assume that the Albanese map $A: \Gamma \longrightarrow \mathbb{CT}^g$ is defined with respect to a basic point $p \in \Gamma$. For $X_i := \pi(\Gamma^i) \subset S\Gamma^{\times g}$ consider the flag of submanifolds $X_1 \subset \cdots \subset X_g = S\Gamma^{\times g}$ (see the definitions in Section 2.3). The Jacobi map $j: S\Gamma^{\times g} \longrightarrow \mathbb{CT}^g$ maps, by definition, $X_1$ biholomorphically to $A(\Gamma)$ (which we identify with $\Gamma$). Moreover, the fundamental group $\pi_1(S\Gamma^{\times g})$ is isomorphic (under $j_*$) to $\pi_1(\mathbb{CT}^g) = \mathbb{Z}^{2g}$ and embedding $X_i \subset S\Gamma^{\times g}$ induces a surjective homomorphism of fundamental groups. Thus, if $G$ is a quotient group of $\pi_1(\mathbb{CT}^g)$, one can construct regular coverings $X_{iG}$ over $X_i$, $i = 1, \ldots, g$, with transformation group $G$ such that $X_{1G} \subset \cdots \subset X_{gG}$ is a flag of complex submanifolds covering the flag $X_1 \subset \cdots \subset X_g$ and there is a proper surjective map with connected fibres $j_G: X_{gG} \longrightarrow T_G$ that covers $j$.

For any function $f \in \mathcal{H}_\phi(\Gamma_G)$, consider its pullback $f_1 := j_G^*(f)$ on $X_{1G}$. Then according to Lemma 2.5, $f_1$ belongs to the space $\mathcal{H}_{j_G^*(\phi)}(X_{1G})$ determined with respect to the pullback of the volume form of $X_1$. Moreover, $j_G^*(\phi)$ satisfies condition (1.3) (respectively, (1.2)) for the distance $d'$ defined by the pullback of a Kähler metric on $X_g$. It follows from the inequality that

$$d(j_G(x), j_G(y)) \leqslant C(j_G)d'(x,y) \quad (x, y \in X_{gG}).$$

Now for a sufficiently small $\varepsilon > 0$, we prove that $f_1$ admits an extension $f_2 \in \mathcal{H}_{(1+\varepsilon)j_G^*(\phi)}(X_{2G})$ satisfying the conditions of Theorem 1.2; $f_2$ admits a similar extension $f_3 \in \mathcal{H}_{(1+2\varepsilon)j_G^*(\phi)}(X_{3G})$, etc. Finally, we obtain an extension $f_g \in \mathcal{H}_{(1+(g-1)\varepsilon)j_G^*(\phi)}(X_{gG})$ of $f_1$. Clearly, $f_g$ is constant on fibres of $j_G$, and thus determines a function $f' \in \mathcal{H}_{(1+(g-1)\varepsilon)\phi}(T_G)$, that extends $f$. Our arguments will guarantee its uniqueness and fulfillment of the required norm estimates. This will finish the proof.

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We use inductive arguments. Assume that we have the required extension $f_k \in \mathcal{H}_{(1+(k-1)\varepsilon)j_G^*(\phi)}(X_{kG})$ of $f_{k-1}$. Construct now extension $f_{k+1} \in \mathcal{H}_{(1+k\varepsilon)j_G^*(\phi)}(X_{(k+1)G})$.

For each $k$ consider the regular covering $Y_k$ over $\Gamma^k \subset \Gamma^{\times g}$ with the transformation group $G$. Since the map $j \circ \pi\colon \Gamma^k \longrightarrow \mathbb{CT}^g$ is invariant with respect to the action of the permutation group $S_k$ acting on $\Gamma^k(\cong \Gamma^{\times k})$ and $(p, \dots, p) \in \Gamma^{\times g}$ is a fixed point with respect to $S_k$, by the covering homotopy theorem there is a covering action of $S_k$ on $Y_k$. Moreover, there is a holomorphic map $\pi_G\colon Y_g \longrightarrow X_{gG}$ that covers $\pi\colon \Gamma^{\times g} \longrightarrow S\Gamma^{\times g}$ and is invariant with respect to the action of $S_g$. Consider the orbit $V_k = S_{k+1}(Y_k)$ in $Y_{k+1}$. Then $V_k$ covers the orbit $W_k = S_{k+1}(\Gamma^k) \subset \Gamma^{k+1}$.

LEMMA 3.1. *Divisor $W_k$ determines a positive line bundle $E_k$ over $\Gamma^{k+1}$.*

*Proof.* Assume without loss of generality that $\Gamma^{k+1} = \Gamma^{\times(k+1)}$. Let $P\colon \Gamma^{k+1} \longrightarrow \Gamma$ be the projection defined by

$$P(z_1, \dots, z_{k+1}) = z_1, \quad (z_1, \dots, z_{k+1}) \in \Gamma^{\times(k+1)} .$$

Then $P^{-1}(x) = (x, \Gamma^{\times k})$ for a fixed $x \in \Gamma$. Denote by $E_x$ a positive line bundle over $\Gamma$ defined by the divisor $\{x\}$ and by $\Theta_x$ its curvature (for a suitable Hermitian metric on $E_x$) such that $(\sqrt{-1}/2\pi)\Theta_x$ is a positive (1,1)-form. Let $e_i \in S_{k+1}$, $i = 1, \dots, k+1$, be such that $\cup_i e_i^{-1}(\Gamma^k) = S_{k+1}(\Gamma^k)$. Then, by definition, $E_k = \otimes_i e_i^*(P^* E_x)$ is a positive line bundle over $\Gamma_{k+1}$. In fact, if in local coordinates $P^*\Theta_x = a(z_1, \overline{z_1})\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1$ with $a(z_1, \overline{z_1}) > 0$, the curvature $\Theta_k$ of $E_k$ equals $\sum_{i=1}^k a(z_i, \overline{z_i})\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$. Clearly, $(\sqrt{-1}/2\pi)\Theta_k$ is positive implying that $E_k$ is positive. □

Let $h_k$ be a Hermitian metric on $E_k$ with the curvature $\Theta_k$. Using the same letters, we denote the pullback of $h_k$, $E_k$ and $\Theta_k$ to $Y_{k+1}$. Let $L_k$ be the holomorphic vector bundle on $X_{k+1}$ defined by the divisor $X_k$ and $h'_k$ a Hermitian metric on $L_k$. Using the same letters, we also denote the pullback of $L_k$ and $h'_k$ to $X_{(k+1)G}$. Below we consider $L_k$ with the weighted metric $\mathrm{e}^{(1+k\varepsilon)j_G^*(\phi)}h'_k$. By Lemma 2.3 (2) there is a linear continuous mapping

$$F_{k,\varepsilon}\colon \mathcal{H}_{(1+(k-1)\varepsilon)j_G^*(\phi)}(X_{kG}) \longrightarrow Z_2^{0,1}(X_{(k+1)G}, L_k^*).$$

Put $\eta_k = F_{k,\varepsilon}(f_k)$. Since, by definition, $\pi^{-1}(X_k) \cap \Gamma^{k+1} = W_k$, the bundle $\pi_G^* L_k$ equals $E_k$ on $\Gamma^{k+1}$. In particular, $\eta'_k = \pi_G^*(\eta_k)$ is a $\bar{\partial}$-closed (0,1)-form on $Y_{k+1}$ with values in $E_k^*$ and $\eta'_k \in L_2^{0,1}(Y_{k+1}, E_k^*)$ for $E$ equipped with the metric $\mathrm{e}^{(1+k\varepsilon)\phi'}h_k$, where $\phi' = \pi_G^*(j_G^*(\phi))$. Further, the curvature $\mathcal{R}_k$ of $E_k$ equals $-(1+k\varepsilon)\mathcal{L}(\phi') + \Theta_k$. Moreover, according to Lemma 2.5,

$$|\mathcal{L}(\phi')|_x \leqslant C(j_G \circ \pi_G, 2)||\mathcal{L}(\phi)|_{(j_G \circ \pi_G)(x)} \quad (x \in Y_g).$$

In particular, there is a positive constant $C$ (depending on $\Gamma$ only) such that

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if $\sup_{x\in T_G}|\mathcal{L}(\phi)|_x < C' < C$ and $0 < \varepsilon \leqslant 1/g$, $1 \leqslant k \leqslant g$, there is an $a = a(C') > 0$ so that $\mathcal{R}_k > a\Theta_k$.

Let $\phi$ satisfy the above condition and $\varepsilon < 1/g$. Since $\Theta_k$ is a Kähler form on $\Gamma_{k+1}$, according to Proposition 2.1 and Remark 2.2, there is a linear continuous mapping $s_k\colon Z_2^{0,1}(Y_{k+1}, E_k^*) \longrightarrow W_2^{0,0}(Y_{k+1}, E_k^*)$ which is inverse to $\bar\partial$. Then for $r_k = s_k(\eta_k')$, we have $\bar\partial r_k = \eta_k'$ and $r_k \in L_2(Y_{k+1}, E_k^*)$. Applying now arguments similar to those used in Section 2.2 (for the pullback to $Y_{k+1}$ of local extensions of $f_k$) get a holomorphic function $g_{k+1}$ on $Y_{k+1}$ that extends $\pi_G^*(f_k)$ and belongs to $\mathcal{H}_{(1+k\varepsilon)\phi'}(Y_{k+1})$.

Assume also that there is another extension $g' \in \mathcal{H}_{(1+k\varepsilon)\phi'}(Y_{k+1})$ of $\pi_G^*(f_k)$. Let $s'$ be the pullback to $Y_{k+1}$ of a holomorphic section of the bundle $E_k$ on $\Gamma_{k+1}$ with a simple zero along $W_k$. (Recall that the pullback of $E_k$ is denoted by the same letter.) Then $d = (g_{k+1} - g')(s')^{-1}$ is an $L_2$ integrable holomorphic section of $E_k^*$. Here $E_k$ is taken with the weighted metric $\mathrm{e}^{(1+k\varepsilon')\phi'}h_k$, where an $\varepsilon'$ satisfies $\varepsilon < \varepsilon' < 1/g$. The arguments are similar to those used in the proof of Lemma 2.3. Therefore, according to Proposition 2.1 for $q = 0$, the function $d$ is zero. This proves the uniqueness of the extension. Since $\pi_G^*(f_k)$ is invariant with respect to the action of the permutation group $S_k$, for any $e \in S_k$, the function $e^*(g_{k+1})$ is also an extension of $\pi_G^*(f_k)$ belonging to $\mathcal{H}_{(1+k\varepsilon)\phi'}(Y_{k+1})$. Thus, the uniqueness of extension implies that $e^*(g_{k+1}) = g_{k+1}$. So there is a uniquely defined holomorphic function $f_{k+1}$ on $X_{(k+1)G}$ such that

$$\pi_G^*(f_{k+1}) = g_{k+1},\ f_{k+1} \in \mathcal{H}_{(1+k\varepsilon)j_G^*(\phi)}(X_{k+1}G)$$

and $f_{k+1}$ is an extension of $f_k$. In fact, our arguments (based on Remark 2.2) show that we constructed a linear continuous extension operator which gives us the required norm estimates. Therefore, by induction, we get a holomorphic function $f_g$ on $X_{gG}$ which belongs to $\mathcal{H}_{(1+(g-1)\varepsilon)j_G^*(\phi)}(X_{gG})$ and extends $f_1$. As was noted at the beginning of the proof, $f_g$ determines the required extension of $f$. This proves Theorem 1.2 for curves. □

*Proof of Theorem 1.2 for projective manifolds.* Let $M$ be a projective manifold of dimension $n \geqslant 2$ with a very ample line bundle $L$ and with a Kähler form $\omega$. We may think of $M$ as embedded in some projective space and of $L$ as the restriction to $M$ of the hyperplane bundle with the standard positively curved metric. Then zero loci of sections of $L$ are hyperplane sections of $M$. By Bertini's theorem, the generic linear subspace of codimension $n-1$ intersects $M$ transversely in a smooth curve $C$. By the Lefschetz hyperplane theorem, $C$ is connected and the map $\pi_1(C) \longrightarrow \pi_1(M)$ is surjective. Let $M_G$ be the regular covering over $M$ with a free Abelian transformation group $G$. Then the regular covering $C_G$ over $C$ with the same transformation group $G$ is embedded into $M_G$. Assume that $f \in \mathcal{H}_{\tilde\phi}(M_G)$ with $\tilde\phi$ satisfying (1.3). Then $g\colon= f|_{C_G}$ belongs to $\mathcal{H}_{(1+\varepsilon)\tilde\phi}(C_G)$ for any positive $\varepsilon$. Indeed, let $U_1, \dots, U_N$ be the pullbacks to $M_G$ of shrunk coordinate polydisks covering an open neighbourhood of $C \subset M$ and $V_i \supset U_i$ be the pullbacks of larger polydisks. We may assume that $C_G \cap V_i = \{z_1 = 0\}$, $i = 1, \dots, N$, for the pullback of the corresponding local coordinates. Then from the application of (1.3) and the subharmonicity of $|f|^2$ we get

$$\int_{C_G \cap U_i} |f|^2 e^{-(1+\varepsilon)\tilde{\phi}} \omega \leqslant c(M) \int_{V_i} |f|^2 e^{-\tilde{\phi}} \omega^n < \infty .$$

This implies $g \in \mathcal{H}_{(1+\varepsilon)\tilde{\phi}}(C_G)$. Let $C = M_1 \subset M_2 \subset \cdots \subset M_n = M$ be a flag of projective submanifolds of $M$, where $M_i$ is intersection of $M$ with the generic linear subspace of codimension $n-i$. Let $C_G = M_{1G} \subset \cdots \subset M_{nG} = M_G$ be the flag of the corresponding regular coverings with the transformation group $G$. Then the arguments similar to those used in Section 3.1 (see also arguments in Theorem 3.1 of [La]) show that if $L$ is very ample then $g$ admits a unique extension $f' \in \mathcal{H}_{(1+\varepsilon+\delta)\tilde{\phi}}(M_G)$ for a sufficiently small positive $\varepsilon$ and $\delta = \delta(\varepsilon)$. But clearly, in this case $f = f'$. Thus we proved that $f$ is uniquely determined by $f|_{C_G}$.

Now let $A: M \longrightarrow \mathbb{CT}^k$ be the Albanese map for $M$ defined with respect to a point $p \in C$ by integration of holomorphic 1-forms $\omega_1, \dots, \omega_k \in \Omega^1(M)$ (generating a basis there). Set $\eta_i := \omega_i|_C$ for $i = 1, \dots, k$. Then by the Lefschetz theorem $\eta_1, \dots, \eta_k$ are linearly independent in $\Omega^1(C)$. Choose 1-forms $\eta_{k+1}, \dots, \eta_s \in \Omega^1(C)$ such that $\eta_1, \dots, \eta_s$ generates a basis. Further, define the Albanese map $A': C \longrightarrow \mathbb{CT}^s$ with respect to the point $p$ by integration the forms of this basis. Then according to our construction, there is a surjective map $P: \mathbb{CT}^s \longrightarrow \mathbb{CT}^k$ whose fibres are complex tori such that $P_*: \pi_1(\mathbb{CT}^s) \longrightarrow \pi_1(\mathbb{CT}^k)$ is a surjection and $A = P \circ A'$. Denote by $T'_G$ a regular covering over $\mathbb{CT}^s$ with the transformation group $G$. Then there is a complex map $P_G: T'_G \longrightarrow T_G$ that covers $P$ whose fibres are also tori. Let $A'_G: C_G \longrightarrow T'_G$ be the map covering $A'$ and $\phi' = P^*_G(\phi)$. Note that $\phi'$ satisfies (1.3) on $T'_G$ and $(A'_G)^*(\phi') = \tilde{\phi}|_{C_G}$. Applying Theorem 1.2 for curves to the map $A'_G: C_G \longrightarrow T'_G$ and the function $\phi'$, we obtain that

*there is $C = C(M, A') > 0$ such that for $|\mathcal{L}(\phi')| < C' < C$ and for sufficiently small positive numbers $\varepsilon \leqslant \varepsilon(C')$, $\delta \leqslant \delta(C')$, there is a uniquely defined holomorphic function $\tilde{f} \in \mathcal{H}_{(1+\varepsilon+\delta)\phi'}(T'_G)$ satisfying $g = (A'_G)^*(\tilde{f})$ and $|\tilde{f}| \leqslant C(\varepsilon, \delta)|g|$ (in the corresponding $L_2$-norms).*

Since $P_G$ is a proper map with connected fibres, $\tilde{f}$ determines a function $h \in \mathcal{H}_{(1+\varepsilon+\delta)\phi}(T_G)$ such that $A^*_G(h)|_{C_G} = g$ and $|h| \leqslant \tilde{C}(\varepsilon, \delta)|g|$. But as we proved, $f$ is uniquely determined by $g = f|_{C_G}$ and $|g| \leqslant c(\varepsilon)|f|$. Therefore $A^*_G h = f$ and $h$ satisfies the required norm estimate. Finally, by Lemma 2.5, $|\mathcal{L}(\phi')| \leqslant c(P)|\mathcal{L}(\phi)|$ and so the above extension theorem is valid for any $B'$ satisfying $|\mathcal{L}(\phi)| < B' < C/c(P)$.

This completes the proof of Theorem 1.2 for projective manifolds. □

### 3.2. PROOF OF THEOREM 1.3

Let $M_G$ be a regular covering over $M$ with the transformation group $G$ and $A_G: M_G \longrightarrow T_G$ be the covering of the Albanese map $A: M \longrightarrow \mathbb{CT}^n$. Assume that

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$f$ is a holomorphic function on $M_G$ satisfying

$$|f(z)| \leqslant c(\varepsilon)e^{\varepsilon r(z)} \quad (z \in M_G)$$

for any $\varepsilon > 0$. Let $\phi$ be the distance from a fixed point in $T_G$ in the flat metric pulled back from $\mathbb{C}\mathbb{T}^n$ and $\tilde{\phi} = A_G^*(\phi)$. Further, denote by $\rho_G$ the distance from 0 on $G(\cong \mathbb{Z}^k)$ determined with respect to the word metric. Since by our construction, growth of $r$ and $\tilde{\phi}$ is equivalent to growth of $\rho_G$, the function $f$ belongs to $\mathcal{H}_{\varepsilon\tilde{\phi}}(M_G)$ for any $\varepsilon > 0$. We now apply Theorem 1.1. Here we assume that $|\mathcal{L}(\phi)|$ is sufficiently small replacing, if necessary, $\phi$ by a smooth function $\phi_1$ with the same growth such that $|\mathcal{L}(\phi_1)|$ is small. In fact, $\phi_1$ can be constructed as follows: Note, first, that $T_G$ is diffeomorphic to $\mathbb{T}^{2n-k} \times \mathbb{R}^k$ where the second derivatives of the diffeomorphism are bounded in the flat coordinate system on $T_G$. Then put $\phi_1(v, x) := \sqrt{p + |x|^2}$, for $(v, x) \in \mathbb{T}^{2n-k} \times \mathbb{R}^k$, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^k$ and $p$ is a sufficiently large positive number.

Further, according to Theorem 1.1 there is a uniquely defined holomorphic function $f' \in \cap_{\varepsilon>0}\mathcal{H}_{\varepsilon\phi}(T_G)$ such that $A_G^*(f') = f$. Now prove that $f'$ is a constant.

We regard the maximal free Abelian subgroup $\Gamma \subset H_1(M, \mathbb{Z})$ as a lattice in $\mathbb{C}^n$ determining $\mathbb{C}\mathbb{T}^n$ and $H \subset \Gamma$ as a sublattice such that the minimal complex vector space containing $H$ is $\mathbb{C}^n$. Consider the pullback $g$ of $f'$ to $\mathbb{C}^n$. Clearly, $g$ is invariant with respect to the action (by shifts) of $H$ and satisfies $|f(z)| \leqslant c(\varepsilon)e^{\varepsilon|z|}$ for any positive $\varepsilon$. For an element $e_1 \in H$, let $X_1$ be a minimal complex vector space containing $\{ne_1\}_{n\in\mathbb{Z}}$. For any $z \in \mathbb{C}^n$, consider the restriction $g' = g|_{z+X_1}$. We identify $z + X_1$ with $\mathbb{C}$ and $\{ne_1\}_{n\in\mathbb{Z}}$ with $\mathbb{Z}$. Then $g'$ is a holomorphic function on $\mathbb{C}$ of an arbitrary small exponential type which is constant on $\mathbb{Z}$. Therefore by Cawrtright's theorem [Ca], $g'$ is constant on $\mathbb{C}$. This implies that $g(z + v) = g(z)$ for any $z \in \mathbb{C}^n$ and $v \in X_1$. In particular, there is a holomorphic function $g_1$ on the quotient $\mathbb{C}^n/X_1 = \mathbb{C}^{n-1}$ of an arbitrary small exponential type whose pullback to $\mathbb{C}^n$ coincides with $g$. Denote by $H_1$ image of $H$ in $\mathbb{C}^n/X_1 = \mathbb{C}^{n-1}$. By definition, $g_1$ is invariant with respect to the action of $H_1$ and the minimal complex vector space containing $H_1$ is $\mathbb{C}^{n-1}$. Choose $e_2 \in H_1$ and denote by $X_2$ the minimal complex subspace containing $\{ne_2\}_{n\in\mathbb{Z}}$. Applying the very same arguments, we get $g_1(z + v) = g_1(z)$ for any $z \in \mathbb{C}^{n-1}$ and $v \in X_2$. Continuing by induction, we finally obtain that the initial function $g$ is constant.

This completes the proof of the theorem. □

Note that our arguments give a more general statement.

THEOREM 3.2. *Let $H$ be such that $\Lambda(H) = \Omega^1(M)$, then there is a $\sigma = \sigma(M)$ such that any holomorphic on $M_G$ function $f$ satisfying*

$$|f(z)| \leqslant ce^{\sigma' r(z)} \quad (0 < \sigma' < \sigma,\ z \in M_G)$$

*is a constant.*

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