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COMPARISON THEOREMS FOR THE EIGENVALUES OF THE LAPLACIAN IN THE UNIT BALL IN R^N

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ABSTRACT. We obtain inequalities relating the eigenvalues of the Dirichlet and the Neumann problems for the Laplacian in the unit ball in R^n.

In this paper we will compare the eigenvalues of radial symmetric eigenfunctions of the Laplacian in the unit ball $B(0,1)$ in R^n. More specifically we compare the eigenvalues of the following two problems:

$$\begin{cases} \Delta\phi + \lambda\rho\phi = 0 & \text{in } B(0,1) \\ \phi\,|_{\partial B(0,1)} = 0 & \phi \text{ is radial} \end{cases} \tag{1}$$

$$\begin{cases} \Delta\psi + \mu\rho\psi = 0 & \text{in } B(0,1) \\ \frac{\partial\psi}{\partial N}\,|_{\partial B(0,1)} = 0 & \psi \text{ is radial} \end{cases} \tag{2}$$

or in polar coordinates,

$$\begin{cases} (r^{n-1}\phi')' + \lambda r^{n-1}\rho\phi = 0 & \text{in } (0,1) \\ \phi'(0) = \phi(1) = 0 \end{cases} \tag{3}$$

$$\begin{cases} (r^{n-1}\psi')' + \mu r^{n-1}\rho\psi = 0 & \text{in } (0,1) \\ \psi'(0) = \psi(1) = 0 \end{cases} \tag{4}$$

where $\rho(x) = \rho(|x|) > 0$ for $x \in B(0,1)$, $n \in Z^+$. It is well-known that the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$, $\{\mu_i\}_{i=1}^{\infty}$ of (3), (4) are discrete with $0 < \lambda_1 < \lambda_2 < \cdots \longrightarrow \infty$, $0 < \mu_1 < \mu_2 < \cdots \longrightarrow \infty$ as $n \longrightarrow \infty$ and that we have the inequality $\mu_k \leq \lambda_k$ for $k = 1, 2, \ldots$.

In a recent paper of C. Bandle and G. Philippin [1], they proved that

$$\mu_k \leq \lambda_k - 2\lambda_1 \tag{5}$$

for $k = 2, 3, \ldots$ where λ_k, μ_k are the kth eigenvalue of (1) and (2) when $n = 1$ and $\rho(x)$ is a decreasing function [0,1]. We shall show that (5) remains valid for eigenvalues of (3) and (4) when $n \in Z^+$ and $\rho(x) = \rho(|x|)$ is a decreasing function of $|x|$ on $[0, 1]$.

The proof is a modification of the proof [1], [5]. We will assume without loss of generality that $\phi_1 > 0$ in $r \in (0,1)$ throughout the paper and we will also assume $\rho \in C^2([0,1])$ in the following three lemmas.

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LEMMA 1. *Let ϕ_k be the kth eigenfunction of (3). Let $v_k = \phi_k/\phi_1$. Then v_k satisfies the equation*

$$(r^{n-1}v'\phi_1^2)' + \nu r^{n-1}\rho\phi_1\phi_k = 0 \text{ in } (0,1), \quad v'(0) = v'(1) = 0 \tag{6}$$

where $\nu = \lambda_k - \lambda_1$.

PROOF. A straightforward computation shows that v_k satisfies (6) with $\nu = \lambda_k - \lambda_1$. By [2], $\phi_1(0) > 0$. So

$$v_k'(0) = \lim_{r\to 0^+} \frac{\phi_1\phi_k' - \phi_1'\phi_k}{\phi_1^2} = 0$$

Since $\phi_1(r) = \phi_1'(1)(r-1) + O\big((r-1)^2\big)$ and $\phi_1(r) > 0$ for $r \in (0,1)$, $\phi_1'(1) \neq 0$. From (3) ϕ_k satisfies

$$\phi'' = -\Big(\frac{n-1}{r}\phi' + \lambda_k\rho\phi\Big) \tag{7}$$

So by l'Hôpital's rule

$$\begin{aligned}
v_k'(1) &= \lim_{r\to 1^-} v_k'(r) = \lim_{r\to 1^-} \frac{\phi_1\phi_k' - \phi_1'\phi_k}{\phi_1^2} \\
&= \frac{1}{2\phi_1'(1)} \lim_{r\to 1^-} \frac{\phi_1\phi_k'' - \phi_1''\phi_k}{\phi_1} \\
&= \frac{1}{2\phi_1'(1)} \lim_{r\to 1^-} \frac{-\phi_1(\frac{n-1}{r}\phi_k' + \lambda_k\rho\phi_k) + \phi_k(\frac{n-1}{r}\phi_1' + \lambda_1\rho\phi_1)}{\phi_1} \\
&= \frac{n-1}{2\phi_1'(1)} \lim_{r\to 1^-} \frac{\phi_k\phi_1' - \phi_k'\phi_1}{\phi_1} \\
&= \frac{n-1}{2\phi_1'(1)^2} \lim_{r\to 1^-} \Big\{-\phi_k\Big(\frac{n-1}{r}\phi_1' + \lambda_1\rho\phi_1\Big) + \phi_1\Big(\frac{n-1}{r}\phi_k' + \lambda_k\rho\phi_k\Big)\Big\} \\
&= 0
\end{aligned}$$

LEMMA 2. *Let v_k be as in Lemma 1. Then $w_{k-1} = r^{n-1}v_k'\phi_1$ satisfies the equation*

$$\Big(\frac{w'}{r^{n-1}\rho}\Big)' - \frac{w}{r^{n-1}}\Big\{\frac{2\phi_1'^2}{\rho\phi_1^2} - \frac{\phi_1' r^{n-1}}{\phi_1}\Big(\frac{1}{r^{n-1}\rho}\Big)'\Big\} + \nu\frac{w}{r^{n-1}} = 0 \text{ in } (0,1)$$

where $\nu = \lambda_k - 2\lambda_1$ with $w_{k-1}(0) = w_{k-1}(1) = 0$ and w_{k-1} has a zero of order n at $r = 0$.

PROOF. Since $v_k'(0) = v_k'(1) = 0$, $w_{k-1}(0) = w_{k-1}(1) = 0$ and w_{k-1} has a zero of order n at $r = 0$. By (6), w_{k-1} satisfies

$$(w\phi_1)' + \nu r^{n-1}\rho\phi_1\phi_k = 0 \Leftrightarrow \frac{w'}{r^{n-1}\rho}\phi_1 + \frac{w}{r^{2n-2}\rho}(r^{n-1}\phi_1') + \nu\phi_1\phi_k = 0 \tag{8}$$

Differentiating with respect to r, we get

$$\Big(\frac{w'}{r^{n-1}\rho}\Big)'\phi_1 + 2\frac{w'\phi_1'}{r^{n-1}\rho} + \frac{w(r^{n-1}\phi_1')'}{r^{2n-2}\rho} + r^{n-1}w\phi_1'\Big(\frac{1}{r^{2n-2}\rho}\Big)' + \nu(\phi_1\phi_k' + \phi_1'\phi_k) = 0 \tag{9}$$

By substituting into (9) the expression for w' from (8) and simplifying the resulting equation, the lemma follows.

LEMMA 3. *Let ν be a positive constant. If u is a solution of*

$$\left(\frac{u'}{r^{n-1}\rho}\right)' + \nu\frac{u}{r^{n-1}} = 0 \text{ in } (0,1), \quad u(0) = u(1) = 0 \tag{10}$$

and u has a zero of order n at $r = 0$, then $\psi = u'/(r^{n-1}\rho)$ is a solution of (4). Conversely, if ψ is a solution of (4), then

$$u(r) = \int_0^r t^{n-1}\rho(t)\psi(t)\,dt$$

is a solution of (10) and u has a zero of order n at $r = 0$.

PROOF. Suppose first that u is a solution of (10) and has a zero of order n at $r = 0$. Let $\psi = u'/(r^{n-1}\rho)$. Then by (10) we have

$$\begin{aligned} \psi' + \nu\frac{u}{r^{n-1}} = 0 &\Rightarrow r^{n-1}\psi' + \nu u = 0 \\ &\Rightarrow (r^{n-1}\psi')' + \nu u' = 0 \\ &\Rightarrow r^{n-1}\psi' + \nu r^{n-1}\rho\psi = 0 \end{aligned}$$

By (11) we have $\psi'(1) = \lim_{r\to 1^-}\left(-\nu u(r)/r^{n-1}\right) = 0$. Since u has a zero of order n at $r = 0$, $\lim_{r\to 0^+} u(r)/r^{n-1} = 0$. Hence $\psi'(0) = 0$ by (11).

Conversely, suppose ψ satisfies (4). Let

$$u(r) = \int_0^r t^{n-1}\rho(t)\psi(t)\,dt$$

Then $u(0) = 0$, $u'(r) = r^{n-1}\rho(r)\psi(r) \Rightarrow \psi = u'/r^{n-1}\rho$. Substituting into (4) we get

$$(r^{n-1}\psi')' + \nu u'(r) = 0$$

Integrating with respect to r,

$$r^{n-1}\psi' + \nu u(r) = 0 \Rightarrow \left(\frac{u'}{r^{n-1}\rho}\right)' + \nu\frac{u}{r^{n-1}} = 0 \tag{12}$$

By (12), $\nu u(r) = r^{n-1}\psi'(r)$. Hence $u(1) = 0$ and u has a zero of order n at $r = 0$.

Before stating the main theorem, we need to recall the Sturm-Liouville Theorem ([2], [4]):

STURM-LIOUVILLE THEOREM. *Let $f \in C([0,1])$, $f > 0$. If ν_k and $\tilde{\nu}_k$ are the kth eigenvalues of the equations*

$$\begin{cases} (\frac{w'}{r^{n-1}\rho})' - \frac{w}{r^{n-1}}f(r) + \nu\frac{w}{r^{n-1}} = 0 \text{ in } (0,1) \\ w(0) = w(1) = 0, \\ w \text{ has a zero of order } n \text{ at } r=0 \end{cases}$$

and

$$\begin{cases} (\frac{u'}{r^{n-1}\rho})' + \nu\frac{u}{r^{n-1}} = 0 \text{ in } (0,1), u(0) = u(1) = 0 \\ u \text{ has a zero of order } n \text{ at } r = 0 \end{cases}$$

respectively. Then $\nu_k \geq \tilde{\nu}_k$.

We are now ready to state the main theorem:

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THEOREM 1. *If $\rho(r) > 0$ for $r \in (0,1)$ and $r^{n-1}\rho(r)$ is a decreasing function in $(0,1)$, then $\lambda_k - 2\lambda_1 \geq \mu_k$ for $k = 2, 3, \ldots$.*

PROOF. Since λ_k, μ_k depend continuously on ρ [2], we may assume without loss of generality that $\rho \in C^2([0,1])$. Then $(1/r^{n-1}\rho)' \geq 0$. So

$$\left\{\frac{2\phi_1'^2}{\rho\phi_1^2} - \frac{\phi_1' r^{n-1}}{\phi_1}\left(\frac{1}{r^{n-1}\rho}\right)'\right\} \geq 0 \text{ for } r \in (0,1)$$

since $\phi_1 > 0$ in $(0,1) \Rightarrow \phi_1' < 0$ in $(0,1)$ by [3]. By combining Lemmas 1, 2, 3 and using the Sturm-Liouville theorem, the result follows.

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